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THE CAUCHY PROBLEM FOR ONE-DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS FLUID: STABILIZATION OF THE SOLUTION

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Abstract. We analyze the Cauchy problem for non-stationary 1-D flow of a compressible viscous and heat-conducting fluid, assuming that it is in the thermodynamical sense perfect and polytropic. This problem has a unique generalized solution on $\mathbb{R}\times]0,T[$ for each $T>0$. Supposing that the initial functions are small perturbations of the constants and using some a priori estimates for the solution independent of T, we prove a stabilization of the solution.

1. Introduction

In this paper we analyze the Cauchy problem for non-stationary 1-D flow of a compressible viscous and heat-conducting fluid. It is assumed that the fluid is thermodynamically perfect and polytropic. The same model has been mentioned in [1], where a global-in-time existence theorem for generalized solution is given without a rigorous proof. Here, we approach to our problem as the special case of the Cauchy problem for a micropolar fluid that is considered in [5] and [6]. Therefore we know that this problem has a unique generalized solution on $\mathbb{R}\times]0,T[$, for each $T>0$ and that the mass density and temperature are strictly positive.

Assuming that the initial functions are small perturbations of the constants, we first derive a priori estimates for a solution independent of T and then analyze the behavior of the solution as $T\to\infty$. We use some ideas of Ya. I. Kanel' ([4]) applied to the case of stabilization of Hölder continuous solution for the same model.

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2. Statement of the problem and the main result

Let ρ, v and θ denote, respectively, the mass density, velocity, and temperature of the fluid in the Lagrangean description. Supposing that in [5] the microrotation is equal to zero, we obtain the problem considered as follows:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta), \tag{2.2}$$

$$\frac{\partial \theta}{\partial t} = -K\rho\theta \frac{\partial v}{\partial x} + \rho\left(\frac{\partial v}{\partial x}\right)^2 + D\frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) \tag{2.3}$$

in $\mathbb{R} \times \mathbb{R}^+$, where K and D are positive constants. The equations (2.1)-(2.3) are, respectively, local forms of the conservation laws for the mass, momentum and energy[1]. We take the following non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x), \tag{2.4}$$

$$v(x,0) = v_0(x), \tag{2.5}$$

$$\theta(x,0) = \theta_0(x) \tag{2.6}$$

for $x \in \mathbb{R}$, where ρ_0, v_0 and θ_0 are given functions. We assume that there exist the constants $m, M \in \mathbb{R}^+$, such that

$$m \le \rho_0(x) \le M, \quad m \le \theta_0(x) \le M, \quad x \in \mathbb{R}. \tag{2.7}$$

In the papers [5] and [6] it was proved that for

$$\rho_0 - 1, v_0, \theta_0 - 1 \in \mathrm{H}^1(\mathbb{R}) \tag{2.8}$$

the problem (2.1)-(2.6) has, for each $T \in \mathbb{R}^+$, a unique generalized solution

$$(x,t) \mapsto (\rho, v, \theta)(x,t), \quad (x,t) \in \Pi = \mathbb{R}\times]0,T[, \tag{2.9}$$

with the properties:

$$\rho - 1 \in \mathrm{L}^\infty\left(0,T;\mathrm{H}^1(\mathbb{R})\right) \cap \mathrm{H}^1(\Pi), \tag{2.10}$$

$$v, \theta - 1 \in \mathrm{L}^\infty\left(0,T;\mathrm{H}^1(\mathbb{R})\right) \cap \mathrm{H}^1(\Pi) \cap \mathrm{L}^2\left(0,T;\mathrm{H}^2(\mathbb{R})\right). \tag{2.11}$$

Using the results from [5] and [1] we can easily conclude that

$$\theta, \rho > 0 \ \text{ in } \ \Pi. \tag{2.12}$$

[1]Derivation of the equations (2.1)-(2.3) from the Eulerian description is given in [1], pp. 31-42.

We denote by $\mathrm{B}^k(\mathbb{R})$, $k \in \mathbb{N}_0$, the Banach space

$$\mathrm{B}^k(\mathbb{R}) = \left\{ u \in \mathrm{C}^k(\mathbb{R}) : \lim_{|x|\to\infty} |D^n u(x)| = 0,\ 0 \le n \le k \right\} \tag{2.13}$$

where D^n is n-th derivative. The norm of the space $\mathrm{B}^k(\mathbb{R})$ is defined by

$$\|u\|_{\mathrm{B}^k(\mathbb{R})} = \sup_{n\le k} \left\{ \sup_{x\in\mathbb{R}} |D^n u(x)| \right\}. \tag{2.14}$$

From Sobolev's embedding theorem ([2, Chapter IV]) and the theory of vector-valued distributions ([3, pp. 467-480]) one can conclude that from (2.10) and (2.11) follows

$$\rho - 1 \in \mathrm{L}^\infty\left(0,T;\mathrm{B}^0(\mathbb{R})\right) \cap \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{R})\right), \tag{2.15}$$

$$v, \theta - 1 \in \mathrm{L}^2\left(0,T;\mathrm{B}^1(\mathbb{R})\right) \cap \mathrm{C}\left([0,T];\mathrm{H}^1(\mathbb{R})\right) \cap \mathrm{L}^\infty\left(0,T;\mathrm{B}^0(\mathbb{R})\right) \tag{2.16}$$

and hence

$$v, \theta - 1 \in \mathrm{C}\left([0,T];\mathrm{B}^0(\mathbb{R})\right), \quad \rho \in \mathrm{L}^\infty(\Pi). \tag{2.17}$$

From (2.7) and (2.8) it is easy to see that there exist the constants E_1, E_2, E_3, $M_1 \in \mathbb{R}^+$, $M_1 > 1$, such that

$$\frac{1}{2}\int_{\mathbb{R}} v_0^2 dx + K\int_{\mathbb{R}} \left(\frac{1}{\rho_0} - \ln\frac{1}{\rho_0} - 1\right) dx + \int_{\mathbb{R}} (\theta_0 - \ln\theta_0 - 1)\, dx = E_1, \tag{2.18}$$

$$\frac{1}{2}\int_{\mathbb{R}} \left((v_0')^2 + (\theta_0')^2\right) dx = E_2, \tag{2.19}$$

$$\frac{1}{2}\int_{\mathbb{R}} \frac{(\rho_0')^2}{\rho_0^2} dx + \int_{\mathbb{R}} v_0' \ln\frac{1}{\rho_0} dx \le E_3, \tag{2.20}$$

$$\sup_{|x|<\infty} \theta_0(x) < M_1. \tag{2.21}$$

Suppose that the quantities $\underline{\eta}$ and $\overline{\eta}$ are such that $\underline{\eta} < 0 < \overline{\eta}$ and

$$\int_{\underline{\eta}}^0 \sqrt{e^\eta - 1 - \eta}\, d\eta = \int_0^{\overline{\eta}} \sqrt{e^\eta - 1 - \eta}\, d\eta = E_5, \tag{2.22}$$

where

$$E_5 = 2\sqrt{\frac{E_1 E_4}{K}}, \quad E_4 = 2\mu E_1\left(1 + M_1 + \frac{E_3}{E_1}\right), \quad \mu = \max\left\{\frac{K}{2D}, 1\right\}. \tag{2.23}$$

Let

$$\underline{u} = \exp\underline{\eta}, \quad \overline{u} = \exp\overline{\eta}. \tag{2.24}$$

The aim of this work is to prove the following theorem.

THEOREM 2.1. *Suppose that the initial functions satisfy (2.7), (2.8) and the following conditions:*

$$E_1 \int_{\mathbb{R}} (v_0')^2 \, dx < \left(\frac{D}{16} \frac{\underline{u}}{\overline{u}} \right)^2, \tag{2.25}$$

(2.26)

$$2E_1 \left(E_1 \frac{\overline{u}}{\underline{u}} (1 + M_1) M_1 \left((16E_4)^2 \frac{\overline{u}}{\underline{u}} + \frac{K^2}{D} M_1 \right) + 2KM_1E_4 + E_2 \right)$$
$$< \min \left\{ \left(\frac{D}{16} \frac{\underline{u}}{\overline{u}} \right)^2, \left(\int_1^{M_1} \sqrt{s - 1 - \ln s} ds \right)^2, \left(\int_0^1 \sqrt{s - 1 - \ln s} ds \right)^2 \right\}$$

then

$$\rho(x,t) \to 1, \;\; v(x,t) \to 0, \;\; \theta(x,t) \to 1, \;\; \textit{when} \;\; t \to \infty, \tag{2.27}$$

uniformly with respect to all $x \in \mathbb{R}$.

REMARK 2.2. Conditions (2.25) and (2.26) mean that E_1, E_2 and E_3 are sufficiently small. In other words the initial functions are small perturbations of the constants.

In the proof of Theorem 2.1. we apply some ideas of [4], where a stabilization of the solution that is Hölder continuous was proved for the same model.

3. A PRORI ESTIMATES FOR ρ, v AND θ

Considering stabilization problem, one has to prove some a priori estimates for the solution independent of T, which is the main difficulty. Some of our considerations are similar to those of [4]. First we derive the energy equation for the solution of problem (2.1)-(2.3) under the conditions indicated above and we estimate the function ρ^{-1}.

LEMMA 3.1. *For each* $t > 0$ *we have*

$$\frac{1}{2} \int_{\mathbb{R}} v^2 dx + \int_{\mathbb{R}} (\theta - \ln \theta - 1) \, dx + K \int_{\mathbb{R}} \left(\frac{1}{\rho} - \ln \frac{1}{\rho} - 1 \right) dx$$
$$+ \int_0^t \int_{\mathbb{R}} \left(\frac{\rho}{\theta} \left(\frac{\partial v}{\partial x} \right)^2 + D \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 \right) dx d\tau = E_1, \tag{3.1}$$

where E_1 *is defined by (2.18).*

PROOF. Multiplying (2.1), (2.2) and (2.3), respectively, by $K\rho^{-1}(1 - \rho^{-1})$, v and $1 - \theta^{-1}$, integrating by parts over $\mathbb{R}$ and over $]0, t[$ and taking into account (2.15) and (2.16), after addition of the obtained equations we easily get equality (3.1) independently of t. ∎

LEMMA 3.2. *For each $t > 0$ exist the strictly positive quantities $\underline{u}_1 = \underline{u}_1\left(\overline{\theta}(t)\right)$ and $\overline{u}_1 = \overline{u}_1\left(\overline{\theta}(t)\right)$ such that*

$$\underline{u}_1 \leq \rho^{-1}(x,\tau) \leq \overline{u}_1, \quad (x,\tau) \in \mathbb{R}\times]0,t[, \tag{3.2}$$

where

$$\overline{\theta}(t) = \sup_{(x,\tau)\in\mathbb{R}\times]0,t[} \theta(x,\tau). \tag{3.3}$$

PROOF. We multiply (2.2) by $\frac{\partial}{\partial x}\ln\left(\frac{1}{\rho}\right)$, integrate over $\mathbb{R}$ and $]0,t[$ and use the following equalities, which follow using (2.1) and (2.15)-(2.17):

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial x}\left(\ln\frac{1}{\rho}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(\ln\frac{1}{\rho}\right)\right)^2, \tag{3.4}$$

(3.5)

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\frac{\partial v}{\partial t}\frac{\partial}{\partial x}\left(\ln\frac{1}{\rho}\right)dxd\tau &= -\int_0^t\int_{\mathbb{R}}\frac{\partial^2 v}{\partial x\partial t}\ln\frac{1}{\rho}dxd\tau\\ &= -\int_{\mathbb{R}}\frac{\partial v}{\partial x}\ln\frac{1}{\rho}dx\bigg|_0^t + \int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial v}{\partial x}\right)^2 dxd\tau,\end{aligned}$$

$$\begin{aligned}-\int_{\mathbb{R}}\frac{\partial v}{\partial x}\ln\frac{1}{\rho}dx\bigg|_0^t &= -\int_{\mathbb{R}}\frac{\partial}{\partial x}v\ln\frac{1}{\rho}dx + \int_{\mathbb{R}}v_0'\ln\frac{1}{\rho_0}dx\\ &= \int_{\mathbb{R}}v\frac{\partial}{\partial x}\left(\ln\frac{1}{\rho}\right)dx + \int_{\mathbb{R}}v_0'\ln\frac{1}{\rho_0}dx,\end{aligned} \tag{3.6}$$

$$\frac{\partial}{\partial x}(\rho\theta) = \frac{\partial\theta}{\partial x}\rho - \theta\rho^2\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right). \tag{3.7}$$

We also take into account the following inequalities obtained by the Young's inequality

$$\rho^2\frac{\partial\theta}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) \leq \frac{1}{2}\frac{\rho}{\theta}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{1}{2}\theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2, \tag{3.8}$$

$$v\frac{\partial}{\partial x}\ln\left(\frac{1}{\rho}\right) = v\rho\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) \leq \frac{1}{4}\rho^2\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 + v^2. \tag{3.9}$$

After simple transformations we get

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{R}}\rho^2\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx + \frac{K}{2}\int_0^t\int_{\mathbb{R}}\theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dxd\tau\\ &\leq \int_{\mathbb{R}}v^2dx + \frac{K}{2}\int_0^t\int_{\mathbb{R}}\frac{\rho}{\theta}\left(\frac{\partial\theta}{\partial x}\right)^2 dxd\tau + \int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial v}{\partial x}\right)^2 dxd\tau\\ &\quad + \int_{\mathbb{R}}v_0'\ln\left(\frac{1}{\rho_0}\right)dx + \frac{1}{2}\int_{\mathbb{R}}\frac{1}{\rho_0^2}(\rho_0')^2 dx\end{aligned} \tag{3.10}$$

for each $t>0$. Using (3.1) and (2.20) from (3.10) we obtain

(3.11)
$$\frac{1}{4}\int_{\mathbb{R}}\rho^{2}\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^{2}dx+\frac{K}{2}\int_{0}^{t}\int_{\mathbb{R}}\theta\rho^{3}\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^{2}dxd\tau\leq K_{1}\left(\overline{\theta}\left(t\right)\right),$$

where

$$K_{1}\left(\overline{\theta}\left(t\right)\right)=2\mu E_{1}\left(1+\overline{\theta}\left(t\right)+\frac{E_{3}}{E_{1}}\right) \tag{3.12}$$

and μ, E_1 and E_3 are defined by (2.23), (2.18) and (2.20).

Now we define the increasing function ψ by

$$\psi\left(\eta\right)=\int_{0}^{\eta}\sqrt{e^{\xi}-1-\xi}d\xi. \tag{3.13}$$

One can conclude the following

$$\left|\psi\left(\ln\frac{1}{\rho}\right)\right|=\left|\int_{0}^{\ln\frac{1}{\rho}}\psi'\left(\xi\right)d\xi\right|\leq\left|\int_{\mathbb{R}}\psi'\left(\ln\frac{1}{\rho}\right)\rho\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)dx\right|. \tag{3.14}$$

Using (3.1), (3.11) and the Hölder's inequality we get

$$\begin{aligned}\left|\psi\left(\ln\frac{1}{\rho}\right)\right|&\leq\left(\int_{\mathbb{R}}\left(\frac{1}{\rho}-1-\ln\frac{1}{\rho}\right)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\rho^{2}\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^{2}dx\right)^{\frac{1}{2}}\\&\leq K_{2}\left(\overline{\theta}\left(t\right)\right),\end{aligned} \tag{3.15}$$

where

$$K_{2}\left(\overline{\theta}\left(t\right)\right)=2E_{1}\left(\frac{2\mu}{K}\left(1+\overline{\theta}\left(t\right)+\frac{E_{3}}{E_{1}}\right)\right)^{\frac{1}{2}}. \tag{3.16}$$

We can also easily conclude that there exist the quantities $\underline{\eta}_1=\underline{\eta}_1\left(\overline{\theta}\left(t\right)\right)<0$ and $\overline{\eta}_1=\overline{\eta}_1\left(\overline{\theta}\left(t\right)\right)>0$, such that

$$\int_{\underline{\eta}_1}^{0}\sqrt{e^{\eta}-1-\eta}d\eta=\int_{0}^{\overline{\eta}_1}\sqrt{e^{\eta}-1-\eta}d\eta=K_{2}\left(\overline{\theta}\left(t\right)\right), \tag{3.17}$$

where $K_2\left(\overline{\theta}\left(t\right)\right)$ is defined by (3.16). Comparing (3.15) and (3.17) we conclude that

$$\underline{u}_1=\exp\underline{\eta}_1\leq\rho^{-1}\left(x,\tau\right)\leq\exp\overline{\eta}_1=\overline{u}_1 \tag{3.18}$$

for $(x,\tau)\in\mathbb{R}\times]0,t[$. $\square$

Now we find some estimates for the derivatives of the functions v and θ.

Lemma 3.3. *For each $t>0$ we have*

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\\
&\quad+\frac{8\overline{u}_1}{\left(D\underline{u}_1\right)^2}\int_0^t\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\right)^2\left(K_4\left(\overline{\theta}\left(\tau\right)\right)-\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\right)d\tau\\
&\quad+\frac{D}{8}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\leq K_3\left(\overline{\theta}\left(t\right)\right),
\end{aligned}\tag{3.19}$$

where

$$K_4\left(\overline{\theta}\left(t\right)\right)=\left(\frac{D\underline{u}_1}{16\overline{u}_1\sqrt{E_1}}\right)^2,\tag{3.20}$$

$$\begin{aligned}
K_3\left(\overline{\theta}\left(t\right)\right)=&\,E_1\frac{\overline{u}_1}{\underline{u}_1}\left(1+\overline{\theta}\left(t\right)\right)\overline{\theta}\left(t\right)\left(\left(16K_1\left(\overline{\theta}\left(t\right)\right)\right)^2\frac{\overline{u}_1}{\underline{u}_1}+\frac{K^2}{D}\overline{\theta}\left(t\right)\right)\\
&+2K\overline{\theta}\left(t\right)K_1\left(\overline{\theta}\left(t\right)\right)+E_2.
\end{aligned}\tag{3.21}$$

Proof. Multiplying equations (2.2) and (2.3), respectively, by $-\dfrac{\partial^2 v}{\partial x^2}$ and $-\dfrac{\partial^2\theta}{\partial x^2}$, integrating over $\mathbb{R}\times]0,t[$ and using the following equality

$$-\int_0^t\int_{\mathbb{R}}\frac{\partial v}{\partial t}\frac{\partial^2 v}{\partial x^2}dxd\tau=\frac{1}{2}\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\Bigg|_0^t\tag{3.22}$$

that is satisfied for the function θ as well, after addition of the obtained equalities we find that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\Bigg|_0^t+\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dxd\tau\\
&\quad+D\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&=-\int_0^t\int_{\mathbb{R}}\frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau+K\int_0^t\int_{\mathbb{R}}\rho\frac{\partial\theta}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau\\
&\quad+K\int_0^t\int_{\mathbb{R}}\theta\frac{\partial\rho}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau+K\int_0^t\int_{\mathbb{R}}\rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dxd\tau\\
&\quad-\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dxd\tau-D\int_0^t\int_{\mathbb{R}}\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dxd\tau.
\end{aligned}\tag{3.23}$$

Using (3.18), (3.11) and the following inequality

$$\left(\frac{\partial v}{\partial x}\right)^2 \leq 2\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)^{\frac{1}{2}}, \tag{3.24}$$

that holds for the function $\dfrac{\partial \theta}{\partial x}$ as well, and applying the Young's inequality with a sufficiently small parameter on the right-hand side of (3.23) we come to the estimates as follows:

(3.25)

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}\frac{\partial \rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau\right| \\
&\leq \int_0^t\int_{\mathbb{R}}\frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)^2 dxd\tau + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau \\
&\leq \frac{2\overline{u}_1^{\frac{1}{2}}}{\underline{u}_1}\int_0^t\left(\int_{\mathbb{R}}\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2dx\right)\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dx\right)^{\frac{1}{2}}d\tau \\
&\quad + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau \\
&\leq \frac{5}{16}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau + \left(16K_1\left(\overline{\theta}\left(t\right)\right)\right)^2\frac{\overline{u}_1}{\underline{u}_1^2}\int_0^t\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dxd\tau \\
&\leq \frac{5}{16}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau \\
&\quad + \left(16K_1\left(\overline{\theta}\left(t\right)\right)\frac{\overline{u}_1}{\underline{u}_1}\right)^2\overline{\theta}\left(t\right)\int_0^t\int_{\mathbb{R}}\frac{\rho}{\theta}\left(\frac{\partial v}{\partial x}\right)^2 dxd\tau,
\end{aligned}$$

$$\begin{aligned}
&\left|K\int_0^t\int_{\mathbb{R}}\rho\frac{\partial \theta}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau\right| \\
&\leq K^2\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial \theta}{\partial x}\right)^2 dxd\tau + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau \\
&\leq \frac{K^2\overline{u}_1\overline{\theta}^2\left(t\right)}{\underline{u}_1}\int_0^t\int_{\mathbb{R}}\frac{\rho}{\theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 dxd\tau + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\left|K\int_0^t\int_{\mathbb{R}}\theta\frac{\partial \rho}{\partial x}\frac{\partial^2 v}{\partial x^2}dxd\tau\right| \\
&\leq K^2\int_0^t\int_{\mathbb{R}}\frac{\theta^2}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 dxd\tau + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau \\
&\leq K^2\overline{\theta}\left(t\right)\int_0^t\int_{\mathbb{R}}\theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dxd\tau + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau,
\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
&\left|K\int_0^t\int_{\mathbb{R}}\rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dxd\tau\right|\\
&\le\frac{K^2}{D}\int_0^t\int_{\mathbb{R}}\theta^2\rho\left(\frac{\partial v}{\partial x}\right)^2dxd\tau+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&\le\frac{K^2\overline{u}_1\overline{\theta}^3(t)}{D\underline{u}_1}\int_0^t\int_{\mathbb{R}}\frac{\rho}{\theta}\left(\frac{\partial v}{\partial x}\right)^2dxd\tau+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau,
\end{aligned}\tag{3.28}
$$

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dxd\tau\right|\\
&\le\frac{1}{D}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial v}{\partial x}\right)^4dxd\tau+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&\le\frac{2\overline{u}_1^{\frac12}}{D\underline{u}_1}\int_0^t\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\right)^{\frac32}\left(\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dx\right)^{\frac12}d\tau\\
&\quad+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&\le\frac{8\overline{u}_1}{D^2\underline{u}_1^2}\int_0^t\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2dx\right)^3d\tau+\frac{1}{8}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dxd\tau\\
&\quad+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau,
\end{aligned}\tag{3.29}
$$

$$
\begin{aligned}
&\left|D\int_0^t\int_{\mathbb{R}}\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dxd\tau\right|\\
&\le D\int_0^t\int_{\mathbb{R}}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial\theta}{\partial x}\right)^2dxd\tau+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&\le\frac{2D\overline{u}_1^{\frac12}}{\underline{u}_1}\int_0^t\left(\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dx\right)^{\frac12}\left(\int_{\mathbb{R}}\left(\frac{\partial\theta}{\partial x}\right)^2dx\right)^{\frac12}\int_{\mathbb{R}}\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2dxd\tau\\
&\quad+\frac{D}{4}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau\\
&\le\frac{128DK_1^2\left(\overline{\theta}(t)\right)\overline{\theta}^2(t)\,\overline{u}_1^2}{\underline{u}_1^2}\int_0^t\int_{\mathbb{R}}\frac{\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2dxd\tau\\
&\quad+\frac{3D}{8}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dxd\tau.
\end{aligned}\tag{3.30}
$$

Taking into account (3.25)-(3.30), (3.1), (3.11) and (2.19) from (3.23) follows

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right]dx+\frac{1}{16}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxd\tau\\
&\quad+\frac{D}{8}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dxd\tau \qquad (3.31)\\
&\leq\frac{8\overline{u}_1}{D^2\underline{u}_1^2}\int_0^t\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^3 d\tau+K_3\left(\overline{\theta}\left(t\right)\right),
\end{aligned}$$

where $K_3\left(\overline{\theta}\left(t\right)\right)$ is defined by (3.21). We also get another important inequality by estimating the integral $\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$. We have

$$\begin{aligned}
\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx&=-\int_{\mathbb{R}}v\frac{\partial^2 v}{\partial x^2}dx\\
&\leq \overline{u}_1^{\frac{1}{2}}\left(\int_{\mathbb{R}}v^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{\frac{1}{2}} \qquad (3.32)\\
&\leq\left(2E_1\overline{u}_1\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}}\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\geq\left(2E_1\overline{u}_1\right)^{-1}\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2. \qquad (3.33)$$

Inserting (3.33) into (3.31) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right]dx\\
&+\frac{8\overline{u}_1}{D^2\underline{u}_1^2}\int_0^t\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2\left[\left(\frac{D\underline{u}_1}{16\overline{u}_1E_1^{\frac{1}{2}}}\right)^2-\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right]d\tau \qquad (3.34)\\
&+\frac{D}{8}\int_0^t\int_{\mathbb{R}}\rho\left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dxd\tau\leq K_3\left(\overline{\theta}\left(t\right)\right)
\end{aligned}$$

and (3.18) is satisfied. □

Similarly as in [4], in the continuation we use the above results, as well as the conditions of Theorem 2.1. We derive the estimates for the solution (ρ, v, θ) of problem (2.1)-(2.8) defined by (2.9)-(2.12) in the domain $\Pi = \mathbb{R}\times]0,T[$, for arbitrary $T>0$.

Taking into account assumption (2.21) and the fact that $\theta \in C\left(\overline{\Pi}\right)$ (see (2.17)) we have the following alternatives: either

$$\sup_{(x,t)\in\Pi} \theta\left(x,t\right) = \overline{\theta}\left(T\right) \leq M_1, \tag{3.35}$$

or there exists $t_1,\ 0 < t_1 < T$, such that

$$\overline{\theta}\left(t\right) < M_1 \ \text{ for } \ 0 \leq t < t_1, \quad \overline{\theta}\left(t_1\right) = M_{1.} \tag{3.36}$$

Now we assume that (3.36) is satisfied and we will show later, that because of the choice of the constants E_1, E_2, E_3 and M_1 (the conditions of Theorem 2.1), (3.36) is impossible.

Because $K_2\left(\overline{\theta}\left(t\right)\right)$, defined by (3.16), increases with increasing $\overline{\theta}\left(t\right)$ we can easily conclude that

$$K_2\left(\overline{\theta}\left(t\right)\right) < K_2\left(M_1\right) \ \text{ for } \ 0 \leq t < t_1 \tag{3.37}$$

and $K_2\left(M_1\right) = E_5$. Therefore we have

$$\underline{u} < \underline{u}_1\left(\overline{\theta}\left(t\right)\right), \quad \overline{u} > \overline{u}_1\left(\overline{\theta}\left(t\right)\right) \tag{3.38}$$

where $\underline{u}$, $\overline{u}$ and $\underline{u}_1\left(\overline{\theta}\left(t\right)\right)$, $\overline{u}_1\left(\overline{\theta}\left(t\right)\right)$ are defined by (2.22)-(2.23) and (3.17)-(3.18), respectively. The quantity $K_4\left(\overline{\theta}\left(t\right)\right)$, defined by (3.20), decreases with increasing $\overline{\theta}\left(t\right)$ and for $\overline{\theta}\left(t_1\right) = M_1$ it becomes

$$K_4\left(M_1\right) = \left(\frac{D\underline{u}}{16\overline{u}E_1^{\frac{1}{2}}}\right)^2. \tag{3.39}$$

Taking into account the assumption (2.25) of Theorem 2.1 and the following inclusion (See (2.16))

$$\frac{\partial v}{\partial x} \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{R}\right)\right) \tag{3.40}$$

we have again two alternatives: either

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 \left(x,t\right) dx \leq K_4\left(M_1\right) \ \text{ for } \ t \in [0,t_1], \tag{3.41}$$

or there exists $t_2,\ \ 0 < t_2 < t_1$, such that

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 \left(x,t\right) dx < K_4\left(M_1\right) \ \text{ for } \ 0 \leq t < t_2, \tag{3.42}$$

and

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 \left(x,t_2\right) dx = K_4\left(M_1\right) \ \text{ for } \ t_2 < t_1. \tag{3.43}$$

Now, we assume that (3.42)-(3.43) are satisfied. Then we have

$$\overline{\theta}\left(t\right) < M_1, \quad K_4\left(M_1\right) < K_4\left(\overline{\theta}\left(t\right)\right) \ \text{ for } \ t \in [0,t_2]. \tag{3.44}$$

Taking into account (3.44), from (3.19), for $t = t_2$, we obtain

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 (x,t)\, dx \le 2K_3\left(\overline{\theta}(t_2)\right), \quad 0 \le t \le t_2. \tag{3.45}$$

Since $K_3\left(\overline{\theta}(t)\right)$, defined by (3.21), increases with the increase of $\overline{\theta}(t)$, it holds

$$K_3\left(\overline{\theta}(t)\right) \le K_3(M_1), \quad t \in [0, t_2]. \tag{3.46}$$

Using condition (2.26) we get

$$2K_3\left(\overline{\theta}(t)\right) < K_4(M_1), \quad t \in [0, t_2], \tag{3.47}$$

and conclude that

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 (x,t_2)\, dx < K_4(M_1). \tag{3.48}$$

This inequality contradicts (3.43). Consequently, the only case possible is when

$$t_2 = t_1 \tag{3.49}$$

and then (3.41) is satisfied.

Using (3.36) and (3.41) from (3.19) we can easily obtain that

$$\int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t)\, dx < 2K_3(M_1) \quad \text{for } 0 < t \le t_1. \tag{3.50}$$

Now, we introduce the function Ψ by

$$\Psi(\theta(x,t)) = \int_1^{\theta(x,t)} \sqrt{s - 1 - \ln s}\, ds. \tag{3.51}$$

From (2.17) follows that $\theta(x,t) \to 1$ as $|x| \to \infty$ and hence

$$\Psi(\theta(x,t)) \to 0 \quad \text{as } |x| \to \infty. \tag{3.52}$$

Consequently,

$$\begin{aligned}
\psi(\theta(x,t)) \le |\psi(\theta(x,t))| &= \left|\int_1^{\theta(x,t)} \frac{d}{ds}\psi(s)\, ds\right| \\
&= \left|\int_{-\infty}^{x} \sqrt{\theta(x,t) - 1 - \ln\theta(x,t)}\, \frac{\partial\theta(x,t)}{\partial x} dx\right| \\
&\le \left(\int_{\mathbb{R}} (\theta(x,t) - 1 - \ln\theta(x,t))\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\frac{\partial\theta}{\partial x}\right)^2 (x,t)\, dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.53}$$

Taking into account (3.36), (3.50) and (3.1) from (3.53) we get

$$\max_{0 \le \theta(x,t) \le M_1} \psi(\theta(x,t)) = \psi\left(\overline{\theta}(t_1)\right) = \psi(M_1) \le (2K_3(M_1)E_1)^{\frac{1}{2}}, \tag{3.54}$$

or

$$\int_1^{M_1} \sqrt{s-1-\ln s}ds - \left(2K_3\left(M_1\right)E_1\right)^{\frac{1}{2}} \le 0. \tag{3.55}$$

Since this inequality contradicts (2.26), it remains to assume that $t_1 = T$. Hence we have

Lemma 3.4. *For each $T > 0$ we have*

$$\theta\left(x,t\right) \le M_1, \ \ (x,t) \in \Pi, \tag{3.56}$$

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 (x,t)\,dx \le K_4\left(M_1\right), \ \ 0 \le t \le T, \tag{3.57}$$

$$\int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t)\,dx \le 2K_3\left(M_1\right), \ \ 0 \le t \le T. \tag{3.58}$$

Proof. These conclusions follow from (3.36), (3.41) and (3.50) directly. □

Lemma 3.5. *The following inequalities hold true:*

$$0 < \underline{u} \le \frac{1}{\rho\left(x,t\right)} \le \overline{u}, \ \ (x,t) \in \Pi, \tag{3.59}$$

$$\sup_{(x,t)\in\Pi} \left|v\left(x,t\right)\right| \le \sqrt{8E_1K_4\left(M_1\right)}, \tag{3.60}$$

$$\theta\left(x,t\right) \ge h > 0, \ \ (x,t) \in \Pi \tag{3.61}$$

where $\underline{u}$ and $\overline{u}$ are defined by (2.22)-(2.24) and a constant h depends only on the data of problem (2.1)-(2.8).

Proof. Because the quantity $\underline{u}_1\left(\overline{\theta}\left(t\right)\right)$ in Lemma 3.2 decreases with increasing $\overline{\theta}\left(t\right)$, while $\overline{u}_1\left(\overline{\theta}\left(t\right)\right)$ increases, it follows from (3.2) and (3.56) that (3.59) is satisfied.

Using the inequality

$$v^2 = 2\int_{-\infty}^{x} v\frac{\partial v}{\partial x}dx \le 2\left(\int_{\mathbb{R}} v^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \tag{3.62}$$

and estimations (3.1) and (3.57) we get immediately (3.60). From (3.53), (3.56), (3.58) and (3.1) we have for $\theta\left(x,t\right) \le 1$ that the following holds

$$\int_{\theta(x,t)}^{1} \sqrt{s-1-\ln s}ds \le \left(2K_3\left(M_1\right)E_1\right)^{\frac{1}{2}} < \int_0^1 \sqrt{s-1-\ln s}ds \tag{3.63}$$

because of (2.26). Hence we conclude that there exists the constant $h > 0$ such that $\theta\left(x,t\right) \ge h$. □

LEMMA 3.6. *For each $T > 0$ we have*

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 dx d\tau \leq K_5, \tag{3.64}$$

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 dx d\tau \leq K_6, \tag{3.65}$$

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x}\right)^2 dx d\tau \leq K_7, \tag{3.66}$$

$$\int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x}\right)^2 dx \leq K_8, \ \ t \in [0, T], \tag{3.67}$$

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dx d\tau \leq K_9, \tag{3.68}$$

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx d\tau \leq K_{10}, \tag{3.69}$$

where the constants K_5, K_6, K_7, K_8, K_9, $K_{10} \in \mathbb{R}^+$ are independent of T.

PROOF. Taking into account (3.59) and (3.61) from (3.1), (3.11), (3.19) and (3.31) we get easily (3.64)-(3.69). ∎

4. PROOF OF THEOREM 2.1

In the following we use the results of Section 3. It is important to remark that all the estimates obtained above are preserved in the domain $\Pi = \mathbb{R} \times]0, T[$ for each $T > 0$.

The conclusions of Theorem 2.1 are immediate consequences of the following lemmas.

LEMMA 4.1. *The following relations hold true:*

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 (x,t)\, dx \to 0, \quad \int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t)\, dx \to 0, \tag{4.1}$$

when $t \to \infty$.

PROOF. Let $\varepsilon > 0$ be arbitrary. With the help of (3.64), (3.65) and (3.66) we conclude that there exists $t_0 > 0$ such that

$$\int_{t_0}^t \int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 dx d\tau < \varepsilon, \int_{t_0}^t \int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 dx d\tau < \varepsilon, \int_{t_0}^t \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x}\right)^2 dx d\tau < \varepsilon, \tag{4.2}$$

for $t > t_0$, and

$$\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x}\right)^2 (x, t_0)\, dx < \varepsilon, \quad \int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x}\right)^2 (x, t_0)\, dx < \varepsilon. \tag{4.3}$$

Similarly to (3.19), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2\right] dx \\
&\quad + \frac{8\overline{u}}{(D\underline{u})^2}\int_{t_0}^{t}\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2 \left[K_4\left(\overline{\theta}\left(\tau\right)\right) - \int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx\right] d\tau \\
&\quad + \frac{D}{8}\int_{t_0}^{t}\int_{\mathbb{R}} \rho \left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dx d\tau \\
&\leq \frac{1}{2}\int_{\mathbb{R}} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2\right](x, t_0)\, dx + K^2\int_{t_0}^{t}\int_{\mathbb{R}}\rho\left(\frac{\partial \theta}{\partial x}\right)^2 dx d\tau \\
&\quad + \left(16K_1\left(\overline{\theta}\left(t\right)\right)\right)^2 \frac{\overline{u}}{\underline{u}^2}\int_{t_0}^{t}\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx d\tau \\
&\quad + K^2\int_{t_0}^{t}\int_{\mathbb{R}}\frac{\theta^2}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 dx d\tau + \frac{K^2}{D}\int_{t_0}^{t}\int_{\mathbb{R}}\theta^2\rho\left(\frac{\partial v}{\partial x}\right)^2 dx d\tau \\
&\quad + \frac{128DK_1^2\left(\overline{\theta}\left(t\right)\right)\overline{u}}{\underline{u}^2}\int_{t_0}^{t}\int_{\mathbb{R}}\left(\frac{\partial \theta}{\partial x}\right)^2 dx d\tau.
\end{aligned} \tag{4.4}$$

Taking into account (3.56), (3.59) and (4.2)-(4.3) from (4.4) we obtain

$$\int_{\mathbb{R}} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2\right] dx \leq K_{11}\varepsilon \;\text{ for }\; t > t_0, \tag{4.5}$$

where K_{11} depends only on the data of our problem and does not depend on t_0. Hence relations (4.1) hold. ∎

Lemma 4.2. *We have*

$$v\left(x, t\right) \to 0, \quad \theta\left(x, t\right) \to 1 \tag{4.6}$$

when $t \to \infty$, uniformly with respect to all $x \in \mathbb{R}$.

Proof. We have (see (3.62) and (3.53))

$$v^2\left(x, t\right) \leq 2\left(\int_{\mathbb{R}} v^2\left(x, t\right) dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(\frac{\partial v}{\partial x}\right)^2 (x, t)\, dx\right)^{\frac{1}{2}}, \tag{4.7}$$

$$\left|\psi\left(\theta\left(x, t\right)\right)\right| \leq \left(\int_{\mathbb{R}}\left(\theta\left(x, t\right) - 1 - \ln\theta\left(x, t\right)\right) dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(\frac{\partial \theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}. \tag{4.8}$$

Taking into account (3.1) from (4.7) and (4.8) we get

$$v^2(x,t) \leq 2(2E_1)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\frac{\partial v}{\partial x} \right)^2 (x,t)\, dx \right)^{\frac{1}{2}}, \tag{4.9}$$

$$|\psi(\theta(x,t))| \leq E_1^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\frac{\partial \theta}{\partial x} \right)^2 dx \right)^{\frac{1}{2}}. \tag{4.10}$$

Using (4.1) and property (3.52) of the function ψ we can easily obtain that (4.6) holds. ◻

LEMMA 4.3. *We have*

$$\int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 (x,t)\, dx \to 0 \tag{4.11}$$

when $t \to \infty$.

PROOF. From (3.66) and (3.69) we conclude that for $\varepsilon > 0$ exists $t_0 > 0$ such that

$$\int_{t_0}^{t} \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 dx d\tau < \varepsilon, \int_{t_0}^{t} \int_{\mathbb{R}} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx d\tau < \varepsilon, \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 (x,t_0)\, dx < \varepsilon \tag{4.12}$$

for $t > t_0$. Now, from (2.1) we get the equality

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \left(\frac{1}{\rho} \right) \right) = \frac{\partial^2 v}{\partial x^2}. \tag{4.13}$$

Multiplying (4.13) by $\frac{\partial}{\partial x} \left(\frac{1}{\rho} \right)$ and integrating over $\mathbb{R}$ and $]t_0, t[$ we obtain

$$\frac{1}{2} \int_{\mathbb{R}} \frac{1}{\rho^4} \left(\frac{\partial \rho}{\partial x} \right)^2 dx = - \int_{t_0}^{t} \int_{\mathbb{R}} \frac{1}{\rho^2} \frac{\partial \rho}{\partial x} \frac{\partial^2 v}{\partial x^2} dx d\tau + \frac{1}{2} \int_{\mathbb{R}} \frac{1}{\rho^4} \left(\frac{\partial \rho}{\partial x} \right)^2 (x,t_0)\, dx. \tag{4.14}$$

Using the Young's inequality and (3.59) from (4.14) we find out

$$\begin{aligned} \frac{\underline{u}^4}{2} \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 dx \leq \overline{u}^2 \int_{t_0}^{t} \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 dx d\tau + \overline{u}^2 \int_{t_0}^{t} \int_{\mathbb{R}} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx d\tau \\ + \frac{\underline{u}^4}{2} \int_{\mathbb{R}} \left(\frac{\partial \rho}{\partial x} \right)^2 (x,t_0)\, dx. \end{aligned} \tag{4.15}$$

With the help of (4.12) from (4.15) we get (4.11). ◻

LEMMA 4.4. *We have*

$$\rho(x,t) \to 1 \tag{4.16}$$

when $t \to \infty$, *uniformly with respect to* $x \in \mathbb{R}$.

PROOF. Similarly as for the function $\psi(\theta)$, we have

$$\psi\left(\frac{1}{\rho}\right) = \int_1^{\frac{1}{\rho}} \sqrt{s-1-\ln s}\,ds \le \left(\int_{\mathbb{R}}\left(\frac{1}{\rho}-1-\ln\frac{1}{\rho}\right)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}. \tag{4.17}$$

Taking into account (3.1) from (4.17) follows

$$\psi\left(\frac{1}{\rho}\right) \le (KE_1)^{\frac{1}{2}}\,\overline{u}\left(\int_{\mathbb{R}}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \tag{4.18}$$

and with the help of (4.11) we conclude (4.16). □

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