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APPROXIMATING COMMON SOLUTIONS OF VARIATIONAL INEQUALITIES BY ITERATIVE ALGORITHMS WITH APPLICATIONS

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Abstract. In this paper, we introduce an iterative scheme for a general variational inequality. Strong convergence theorems of common solutions of two variational inequalities are established in a uniformly convex and 2-uniformly smooth Banach space. As applications, we, still in Banach spaces, consider the convex feasibility problem.

1. Introduction and Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ a nonlinear mapping. Recall the following definitions:

(1) The mapping A is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) A is said to be *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(3) A is said to be *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

The α-inverse-strongly monotone mapping is also called α-cocoercive mapping.

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Recall that the classical variational inequality, denoted by $VI(C,A)$, is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \qquad \forall v \in C. \tag{1.1}$$

It is well known that for given $z \in H$ and $u \in C$ satisfy the inequality

$$\langle u-z, v-u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$, where P_C denotes the metric projection from H onto C. From the above, we see that $u \in C$ is a solution to the problem (1.1) if and only if u satisfies the following equation:

$$u = P_C(u - \rho Au), \tag{1.2}$$

where $\rho > 0$ is a constant. This implies that the problem (1.1) and the problem (1.2) are equivalent. This alternative formula is very important from the numerical analysis point of view. Many authors studied iterative methods for the problem (1.1) provided that A has some monotonicity.

Recently, Aoyama, Iiduka and Takahashi ([1]) introduced and analyzed a general variational inequality which can be viewed as a Banach version of the variational inequality (1.1).

Let C be a nonempty closed convex subset of a Banach space E and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;
(2) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;
(3) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex, see [19].

A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U_E$. The norm of E is said to be Fréchet differentiable if, for any $x \in U_E$, the limit is attained uniformly for all $y \in U_E$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. Note that

(1) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.
(2) All Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.
(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is sunny provided $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $Q(x) + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

PROPOSITION 1.1. *[16] Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *Q is sunny and nonexpansive;*
(2) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, *for all $x, y \in E$;*
(3) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, *for all $x, y \in E$.*

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(T)$ to denote the set of fixed points of T.

PROPOSITION 1.2. *[11] Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([3],[15]). More precisely, take $t \in$

$(0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.3}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1-t)Tx_t. \tag{1.4}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder ([3]) proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich ([15]) extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Reich ([15]) showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

THEOREM 1.3. *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D; that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \le 0, \quad \forall u \in C, y \in D.$$

Let $A : C \to E$ be a nonlinear mapping. Recall the following definitions:

(1) The mapping A is said to be *accretive* if

$$\langle Ax - Ay, J(x - y)\rangle \ge 0, \quad \forall x, y \in C.$$

(2) A is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x - y)\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(3) A is said to be *α-inverse-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x - y)\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Recently, Aoyama, Iiduka and Takahashi ([1]) first considered the following variational inequality in a smooth Banach space E. Let C be a nonempty closed convex subset of E and A an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v - u)\rangle \ge 0, \quad \forall v \in C. \tag{1.5}$$

In this paper, we use $BVI(C, A)$ to denote the set of solutions of the variational inequality (1.5).

Aoyama et al. ([1]) proved that the variational inequality (1.5) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (1.5) if and only if $u \in C$ satisfies the equation

$$u = Q_C(u - \lambda Au), \tag{1.6}$$

where $\lambda > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C, see [1] for more details.

Aoyama et al. ([1]) considered the variational inequality (1.5) and obtained a weak theorem in a uniformly convex and 2-uniformly smooth Banach space. To be more precise, they proved the following result.

THEOREM 1.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be an α-inverse strongly-accretive operator of C into E with $BVI(C, A) \neq \emptyset$, where $BVI(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0,\ x \in C\}$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:*

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n)$$

converges weakly to some element z of $BVI(C, A)$, where K is the 2-uniformly smoothness constant of E.

Very recently, Cho, Yao and Zhou ([5]) considered a new iterative algorithm for approximating a solution to the variational inequality (1.5) in a Banach space. To be more precise, they considered the following iterative process

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n Ax_n), \quad n \geq 0,$$

where $u \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are control sequences in $(0, 1)$, Q_C is a sunny nonexpansive retraction from E onto its nonempty closed and convex subset C and A is an α-inverse-strongly accretive operator of C into E such that $BVI(C, A) \neq \emptyset$. They obtained a strong convergence theorem under some restrictions imposed on the control sequences.

Motivated by Aoyama et al. [1], Cho et al. [5], Ceng and Yao [6], Hao [9], Iiduka and Takahashi [10], Qin and Su [13], Qin et al. [14] and Yao and Yao [22], we study the variational inequality (1.5). To be more precise, we introduce a general iterative algorithm to approximation a common solution to two variational inequalities. Note that no Banach space is q-uniformly smooth for $q > 2$; see [20] for more details. We prove the strong convergence of the purposed iterative scheme in uniformly convex and 2-uniformly smooth Banach spaces.

In order to prove our main results, we need the following lemmas.

LEMMA 1.5. *[21] Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad \forall n \geq 1,$$

where γ_n *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

LEMMA 1.6. *[4] Let* C *be a closed convex subset of a strictly convex Banach space* E. *Let* $T_m : C \to C$ *be a nonexpansive mappings for each* $1 \leq m \leq r$, *where* r *is some integer. Suppose that* $\cap_{m=1}^{r} F(T_m)$ *is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{m=1}^{r} \lambda_n = 1$. *Then the mapping* $S : C \to C$ *defined by*

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x, \quad \forall x \in C$$

is well defined, nonexpansive and $F(S) = \cap_{m=1}^{r} F(T_m)$ *holds.*

LEMMA 1.7. *[20] Let* E *be a real* 2*-uniformly smooth Banach space with the best smooth constant* K. *Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

LEMMA 1.8. *[17] Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. MAIN RESULTS

THEOREM 2.1. *Let* E *be a uniformly convex and* 2*-uniformly smooth Banach space with the best smooth constant* K, C *a nonempty closed convex subset of* E *and* Q_C *a sunny nonexpansive retraction from* E *onto* C. *Let* $A : C \to E$ *be an* α*-inverse-strongly accretive mapping and* $B : C \to E$ *a* β*-inverse-strongly accretive mapping, respectively. Assume that* $VI = BVI(C,A) \cap BVI(C,B) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence defined in the following manner*

$$(\Upsilon) \qquad \begin{cases} x_0 = u \in C, \\ y_n = \delta_n Q_C(x_n - \rho B x_n) + (1-\delta_n) Q_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, \alpha/K^2]$ *and* $\rho \in (0, \beta/K^2]$. *Assume that the following restrictions imposed on the control sequences are satisfied:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$.

Then the sequence $\{x_n\}$ *generated in* (Υ) *converges strongly to* $q = Q_{VI}u$, *where* Q_{VI} *is the unique sunny nonexpansive retraction from* C *onto* VI.

Proof of Theorem 2.1. The proof is divided into four steps.

Step 1. Show that the sequence $\{x_n\}$ is bounded.

First, we prove that the mappings $Q_C(I - \rho B)$ and $Q_C(I - \lambda A)$ are nonexpansive. Indeed, for any $x, y \in C$, it follows from Lemma 1.7 that

$$\begin{aligned}
&\|Q_C(I - \lambda A)x - Q_C(I - \lambda A)y\|^2 \\
&\leq \|(x - y) - \lambda(Ax - Ay)\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\langle Ax - Ay, J(x - y)\rangle + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

This shows that $Q_C(I - \lambda A)$ is nonexpansive, so is $Q_C(I - \rho B)$. Since $BVI(C, A) = F(Q_C(I - R_1A))$ and $BVI(C, B) = F(Q_C(I - R_2B))$ for any constants $R_1, R_2 > 0$. That is, $VI = BVI(C, A) \cap BVI(C, B)$ is closed and convex. For any $p \in VI$, we have

$$\begin{aligned}
\|y_n - p\| &= \|\delta_n[Q_C(x_n - \rho Bx_n) - p] + (1 - \delta_n)[Q_C(x_n - \lambda Ax_n) - p]\| \\
&\leq \delta_n\|Q_C(x_n - \rho Bx_n) - p\| + (1 - \delta_n)\|Q_C(x_n - \lambda Ax_n) - p\| \\
&\leq \delta_n\|x_n - p\| + (1 - \delta_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \gamma_n(y_n - p)\| \\
&\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|.
\end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \leq \|u - p\|,$$

which gives that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. Show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.1}$$

Putting $u_n = Q_C(x_n - \lambda Ax_n)$ and $v_n = Q_C(x_n - \rho Bx_n)$ for each $n \geq 0$, we have

$$\begin{aligned} & y_{n+1} - y_n \\ & \quad = \delta_{n+1}v_{n+1} + (1-\delta_{n+1})u_{n+1} - [\delta_n v_n + (1-\delta_n)u_n] \\ & \quad = \delta_{n+1}(v_{n+1} - v_n) + (\delta_{n+1} - \delta_n)(v_n - u_n) + (1-\delta_{n+1})(u_{n+1} - u_n). \end{aligned}$$

It follows that

$$\begin{aligned} & \|y_{n+1} - y_n\| \\ & \leq \delta_{n+1}\|v_{n+1} - v_n\| + |\delta_{n+1} - \delta_n|\|v_n - u_n\| + (1-\delta_{n+1})\|u_{n+1} - u_n\| \\ & \leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1 + (1-\delta_{n+1})\|x_{n+1} - x_n\| \\ & = \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1, \end{aligned} \tag{2.2}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup_{n\geq 0}\{\|v_n - u_n\|\}$. Putting

$$e_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}, \quad \forall n \geq 0,$$

we have

$$x_{n+1} = (1-\beta_n)e_n + \beta_n x_n, \quad \forall n \geq 0. \tag{2.3}$$

Now, we compute $\|e_{n+1} - e_n\|$. From

$$\begin{aligned} e_{n+1} - e_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}u + (1-\alpha_{n+1}-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n-\beta_n)y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(u - y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(u-y_n)}{1-\beta_n} + y_{n+1} - y_n, \end{aligned}$$

we have

$$\|e_{n+1} - e_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| + \|y_{n+1} - y_n\|. \tag{2.4}$$

Substituting (2.2) into (2.4), we arrive at

$$\begin{aligned} & \|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| \\ & \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| + M_1|\delta_{n+1} - \delta_n|. \end{aligned}$$

From the conditions (b) and (c), we get that

$$\limsup_{n\to\infty}(\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 1.8 that

$$\lim_{n\to\infty}\|e_n - x_n\| = 0.$$

From (2.3), we see that

$$x_{n+1} - x_n = (1-\beta_n)(e_n - x_n).$$

It follows that (2.1) holds.

Step 3. Show that

$$\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \le 0, \tag{2.5}$$

where $q = Q_{VI}u$. Define a mapping $M : C \to C$ by

$$Mx = \delta Q_C(x - \rho Bx) + (1-\delta)Q_C(x - \lambda Ax), \quad \forall x \in C.$$

From Lemma 1.6, we have that M is nonexpansive such that

$$F(M) = F(Q_C(I-\rho B)) \cap F(Q_C(I-\lambda A)) = BVI(C,B) \cap BVI(C,A) = VI.$$

Note that

$$\begin{aligned} y_n - Mx_n &= \delta_n v_n + (1-\delta_n)u_n - \delta v_n - (1-\delta)u_n \\ &= (\delta_n - \delta)(v_n - u_n). \end{aligned}$$

From the condition (d), we arrive at

$$\lim_{n\to\infty} \|y_n - Mx_n\| = 0. \tag{2.6}$$

On the other hand, we have that

$$\begin{aligned} &\|x_n - Mx_n\| \\ &= \|x_n - x_{n+1} + x_{n+1} - y_n + y_n - Mx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Mx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + \beta_n\|x_n - y_n\| + \|y_n - Mx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + \beta_n\|x_n - Mx_n\| + (\beta_n + 1)\|Mx_n - y_n\|. \end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - Mx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + (\beta_n + 1)\|Mx_n - y_n\|.$$

From the conditions (b), (c), (2.1) and (2.6), we obtain that

$$\lim_{n\to\infty} \|x_n - Mx_n\| = 0. \tag{2.7}$$

Let z_t be the fixed point of the contraction $z \mapsto tu + (1-t)Mz$, where $t \in (0,1)$. That is,

$$z_t = tu + (1-t)Mz_t.$$

It follows that

$$\|z_t - x_n\| = \|(1-t)(Mz_t - x_n) + t(u - x_n)\|.$$

On the other hand, for any $t \in (0,1)$, we see that

$$\begin{aligned}\|z_t - x_n\|^2 &= (1-t)\langle Mz_t - x_n, J(z_t - x_n)\rangle + t\langle u - x_n, J(z_t - x_n)\rangle \\ &= (1-t)\big(\langle Mz_t - Mx_n, J(z_t - x_n)\rangle + \langle Mx_n - x_n, J(z_t - x_n)\rangle\big) \\ &\quad + t\langle u - z_t, J(z_t - x_n)\rangle + t\langle z_t - x_n, J(z_t - x_n)\rangle \\ &\le (1-t)\big(\|z_t - x_n\|^2 + \|Mx_n - x_n\|\|z_t - x_n\|\big) \\ &\quad + t\langle u - z_t, J(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\ &\le \|z_t - x_n\|^2 + \|Mx_n - x_n\|\|z_t - x_n\| + t\langle u - z_t, J(z_t - x_n)\rangle.\end{aligned}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \le \frac{1}{t}\|Mx_n - x_n\|\|z_t - x_n\|, \quad \forall t \in (0,1).$$

In view of (2.7), we see that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le 0, \quad \forall t \in (0,1). \tag{2.8}$$

Letting $t \to 0$ in (2.8), we have that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1, for $t \in (0,\delta_1)$, such that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le \frac{\epsilon}{2}. \tag{2.9}$$

On the other hand, we see that $P_{F(M)}u = \lim_{t\to 0} z_t$ and $F(M) = VI$. It follows that $z_t \to q = P_{VI}u$ as $t \to 0$. There exists $\delta_2 > 0$, for $t \in (0,\delta_2)$, such that

$$\begin{aligned}&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\le |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\ &\quad + |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\le |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \\ &\le \|u - q\|\|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| < \frac{\epsilon}{2}.\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, we have for each $t \in (0,\delta)$ that

$$\langle u - q, J(x_n - q)\rangle \le \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

This implies that

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (2.9) that

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we see that (2.5) holds.

Step 4. Show that $x_n \to q$ as $n \to \infty$.

Notice that

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n y_n - q, J(x_{n+1}-q)\rangle \\ &= \alpha_n\langle u-q, J(x_{n+1}-q)\rangle + \beta_n\langle x_n - q, J(x_{n+1}-q)\rangle \\ &\quad + \gamma_n\langle y_n - q, J(x_{n+1}-q)\rangle \\ &\le \alpha_n\langle u-q, J(x_{n+1}-q)\rangle + \beta_n\|x_n - z\|\|x_{n+1}-q\| \\ &\quad + \gamma_n\|y_n - q\|\|x_{n+1}-q\| \\ &\le \alpha_n\langle u-q, J(x_{n+1}-q)\rangle + (1-\alpha_n)\|x_n-q\|\|x_{n+1}-q\| \\ &\le \alpha_n\langle u-q, J(x_{n+1}-q)\rangle + \frac{1-\alpha_n}{2}(\|x_n-q\|^2 + \|x_{n+1}-q\|^2).\end{aligned}$$

It follows that

$$\|x_{n+1}-q\|^2 \le (1-\alpha_n)\|x_n-q\|^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle.$$

From the condition (b), we can conclude from Lemma 1.5 the desired conclusion easily. This completes the proof. ∎

In a real Hilbert space, Theorem 2.1 is reduced to the followings.

Corollary 2.2. *Let H be a real Hilbert space, C a nonempty closed convex subset of E and P_C the metric projection from H onto C. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping, respectively. Assume that $VI = VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 = u \in C, \\ y_n = \delta_n P_C(x_n - \rho B x_n) + (1-\delta_n) P_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \ge 0, \end{cases}$$

where $\rho \in (0, 2\beta]$ and $\lambda \in (0, 2\alpha]$. If the following restrictions imposed on the control sequences are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \ge 0$;
(b) $\sum_{n=0}^\infty \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;
(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$,

then the sequence $\{x_n\}$ converges strongly to $q \in VI$, where $q = P_{VI}u$.

Further, if $\lambda = \rho$ and $A = B$, then Corollary 2.2 is reduced to the following.

Corollary 2.3. *Let H be a real Hilbert space, C a nonempty closed convex subset of E and P_C the metric projection from H onto C. Let $A : C \to$*

H be an α-inverse-strongly monotone mapping. Assume that $VI(C,A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 = u \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(x_n - \lambda A x_n), \quad \forall n \geq 0 \end{cases}$$

$\lambda \in (0, 2\alpha]$. If the following restrictions imposed on the control sequences are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ converges strongly to $q \in VI(C,A)$, where $q = P_{VI}u$.

3. Applications

Recently, many authors consider the following convex feasibility problem (CFP):

$$\text{finding an } x \in \bigcap_{m=1}^{r} C_m,$$

where $r \geq 1$ is an integer and each C_m is assumed to be the fixed point set of a nonexpansive mapping T_m, $m = 1, 2, \ldots, r$. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration ([7,12]), computer tomography ([18]) and radiation therapy treatment planning ([8]). In this section, we study the CFP in the setting of Banach space.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $A_m : C \to E$ be α_m-inverse-strongly accretive mapping, where $m \in \{1, 2, \ldots, r\}$. Assume that $VI = \cap_{m=1}^{r} BVI(C, A_m) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n^m\}$ be sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 = u \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{m=1}^{r} \delta_n^m Q_C(x_n - \lambda_m A_m x_n), \quad \forall n \geq 0, \end{cases}$$

where $\lambda_m \in (0, \alpha_m/K^2]$ for each $m \in \{1, 2, \ldots, r\}$. If the following restrictions imposed on the control sequences are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{m=1}^{r} \delta_n^m = 1$, *for all* $n \geq 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;
(c) $\lim_{n\to\infty} \delta_n^m = \delta^m \in (0,1)$ *for all* $n \geq 0$;
(d) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ converges strongly to $q \in VI$, where $q = Q_{VI}u$ and Q_{VI} is the unique sunny nonexpansive retraction from C onto VI.

Let E be a Banach space and C be a nonempty closed convex subset of E. Recall that $T : C \to C$ is called a λ-strict pseudo-contraction ([2]) if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, J(x-y)\rangle \le \|x-y\|^2 - \lambda\|(I-T)x-(I-T)y\|^2, \qquad \forall x, y \in C. \tag{3.1}$$

From (3.1), we see that

$$\begin{aligned}
&\langle (I-T)x - (I-T)y, J(x-y)\rangle \\
&= \|x-y\|^2 - \langle Tx - Ty, J(x-y)\rangle \\
&\ge \|x-y\|^2 - (\|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2) \\
&= \lambda\|(I-T)x - (I-T)y\|^2.
\end{aligned}$$

This implies that $(I - T)$ is λ-inverse-strongly accretive mapping. We, therefore, have the following result.

Theorem 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C a nonempty closed convex subset of E. Let $T_A : C \to C$ be an α-strict pseudo-contraction and $T_B : C \to C$ a β-strict pseudo-contraction, respectively. Assume that $F = F(T_A) \cap F(T_B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases}
x_0 = u \in C, \\
y_n = \delta_n[(1-\rho)x_n + \rho T_B x_n] + (1-\delta_n)[(1-\lambda)x_n + \lambda T_A x_n], \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \ge 0,
\end{cases}$$

where $\lambda \in (0, \alpha/K^2]$ and $\rho \in (0, \beta/K^2]$. If the following restrictions imposed on the control sequences are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \ge 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\quad \lim_{n\to\infty} \alpha_n = 0$;
(c) $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$;
(d) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ converges strongly to $q \in F$, where $q = Q_F u$ and Q_F is the unique sunny nonexpansive retraction from C onto F.

Proof of Theorem 3.2. Putting $A = I - T_A$ and $B = I - T_B$, we have that A is α-inverse-strongly accretive and B is β-inverse-strongly accretive, respectively. We also have $F(T_A) = BVI(C, A)$ and $F(T_B) = BVI(C, B)$, respectively. Noticing that

$$Q_C(x_n - \rho B x_n) = (1-\rho)x_n + \rho T_B x_n$$

and

$$Q_C(x_n - \lambda A x_n) = (1-\lambda)x_n + \lambda_n T_A x_n,$$

we can conclude from Theorem 2.1 the desired conclusion immediately. This completes the proof. ∎

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