

New Adaptive Laws for Model Reference Adaptive Control Using Non-Quadratic Lyapunov Functions

UDK 681.513.6
IFAC 2.4.2

Preliminary communication

In the design of model reference adaptive control (MRAC) schemes, adaptive laws have been developed based on Lyapunov stability theory. Over the past few decades, it has been a common practice to use Quadratic Lyapunov Functions (QLF). In contrast to such development, using a new Non-Quadratic Lyapunov Function (NQLF), this paper presents new adaptive laws for the MRAC. These new laws have the same advantage of assurance of stability of the overall system, as the earlier adaptive laws developed using the QLF. Over and above, they have an additional advantage of improved performance: in fact, the use of NQLF improved the system output error signal converging to zero. Finally, this paper also presents simulation results supporting the arguments.

Key words: adaptive control, adaptive laws, model reference adaptive control, Lyapunov functions, Lyapunov based MRAC

1 INTRODUCTION

Since the introduction of Lyapunov stability approach to ensure stability of model reference adaptive control (MRAC) by Shackloth and Butchart [1], use of Quadratic Lyapunov Functions (QLF) to develop adaptive control laws in the design of MRAC has become a common practice. Later, many modifications to this approach have been suggested during 70's and 80's. All this development was well documented by Narendra and Annaswamy in [2]. For the disturbance-free case, adaptive control schemes with stable adaptive laws were developed by 1980. In the literature, they have been called as ideal (standard) adaptive laws. All this development used quadratic Lyapunov functions in the design of MRAC, yielding adaptive laws with the output error e_1 signal [3].

During 1990's, for the disturbance-free case, Rao and his co-workers [4, 5], introduced. The use of Non-Quadratic Lyapunov Functions (NQLF) in the design of adaptive control and identification schemes. Recently, for the disturbance present case, Donnelly and Hassan [7, 8] presented the use of Non-Quadratic Lyapunov Functions (NQLF) and new adaptive laws, which gave better convergence of the error signal, namely, error signal going to zero as time evolves.

In a similar spirit, for the disturbance-free case, this paper employs a Non-Quadratic Lyapunov Function (NQLF) and presents new adaptive laws. These new laws have an advantage of improved

performance – less oscillatory and faster decay of the error signal. The simulation results support the mathematical arguments.

This paper is organised as follows. Section II describes the first order plant and MRAC for it. Then, section III describes for the disturbance-free case the Model Reference Adaptive Control (MRAC) scheme for an n^{th} order plant with relative degree 1; later, it presents adaptive laws using QLF and NQLF. Then, section IV presents the simulation results of MRAC on a second order plant. Finally, Section V presents a conclusion.

2 A 1ST ORDER PLANT: DISTURBANCE-FREE CASE – ADAPTIVE LAW USING QLF AND NQLF

A. MRAC: First Order System

This section describes how Lyapunov stability theory is used for the design of adaptive control systems. For ease of exposition, consider a simple first order system is in this paper; the work on a higher order system will, however, be presented in future papers. First, the mathematical description of the MRAC system is given. Next, the quadratic Lyapunov function and the so-called standard adaptive law are presented. Then, the non-quadratic Lyapunov function and the new adaptive law are presented. The scheme of MRAC system is shown in Figure 1.

The plant to be controlled and the reference model are described by the differential equations:

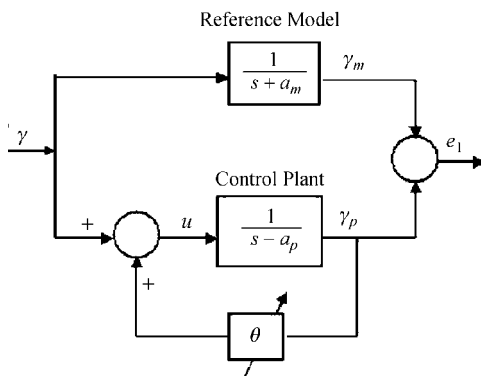


Fig. 1 Scheme of the MRAC System with First Order Plant

Plant:

$$\dot{y}_p = a_p y_p + u \quad u, y_p : \mathbb{R}^+ \rightarrow \mathbb{R} \quad (1)$$

Reference model:

$$\dot{y}_m = -a_m y_m + r \quad r, y_m : \mathbb{R}^+ \rightarrow \mathbb{R}. \quad (2)$$

a_p is the constant plant parameter that is unknown; a_p can be positive or negative. The reference model is apriori designed by choosing a_m to prescribe the desired response of the plant controller combination. The reference input r is a piecewise-continuous and bounded function of time. From Figure 1, control input u is defined by

$$u = \theta(t)y_p + r \quad (3)$$

where $\theta(t)$ is the adjustable controller parameter and v is the disturbance. Now, it can be shown as in Narendra and Annaswamy (1989) that there exists an ideal controller parameter, namely, $\theta^* = (-a_m - a_p)$ such that the error $e_1 = y_p - y_m$ satisfies the differential equation:

$$\dot{e}_1 = -a_m e_1 + \phi y_p; \quad e_1 : \mathbb{R}^+ \rightarrow \mathbb{R}$$

with controller parameter error, $\phi = \theta - \theta^*$. (4)

Then, for the control problem, the aim is to determine a bounded control input u so that all the signals remain bounded; and further as time progresses, the system error e_1 tends to zero in the disturbance-free case.

B. Lyapunov Functions and Adaptive laws

This section presents first (in) the standard adaptive laws using quadratic Lyapunov function (as available in the literature) in the absence of disturbances. Next (in and) it presents new adaptive laws using new non-quadratic Lyapunov functions – again in the absence of disturbances.

1) Standard Adaptive law using Quadratic Lyapunov Function (Q.L.F.) with e_1 signal

Consider the quadratic Lyapunov function V_1 given by

$$V_1(e_1, \phi) = \frac{1}{2} e_1^2 + \frac{1}{2} \phi^2$$

with its time derivative

$$\dot{V}_1(e_1, \phi) = e_1 [\dot{e}_1] + \phi [\dot{\phi}]. \quad (5)$$

Now, using Equation (4) and choosing the standard adaptive law as

$$\dot{\theta} = \dot{\phi} = -e_1 y_p \quad (\text{Ideal law}) \quad (6)$$

leads to the derivative of V_1 as

$$\dot{V}_1(e_1, \phi) = -a_m e_1^2 \leq 0. \quad (7)$$

From Equation (8), it can be shown that $e_1 \in L^2$. The stability analysis follows the steps given by Narendra and Annaswamy (1989). The state of the system, described in the earlier Equations (4) and (6), is uniformly stable. This feature implies that for bounded initial values, the signals $e_1(t)$, and $\phi(t)$ remain bounded for all time $t > 0$. Further, for a piecewise-continuous and bounded input $r(t)$, the right hand side of Equation (4) is bounded; therefore, the derivative of error is bounded. Now, since $e_1 \in L^2$ and the derivative of e_1 is bounded, following the Barbalat Lemma given in Narendra and Annaswamy (1989), leads to the fact that $e_1 \rightarrow 0$ as time $t \rightarrow \infty$. Finally, this feature implies that y_p asymptotically becomes y_m .

2) Non-quadratic Lyapunov function and adaptive law, using e_1^3 signal

Consider the non-quadratic Lyapunov function, used by Rao [4] (1998)

$$\dot{V}_3(e_1, \phi) = \frac{1}{4} e_1^4 + \frac{1}{2} \phi^2$$

with its time derivative

$$\dot{V}_3(e_1, \phi) = e_1^3 \dot{e}_1 + \phi \dot{\phi}. \quad (8)$$

As earlier, using Equation (4) and choosing the adaptive law as:

$$\dot{\theta} = \dot{\phi} = -e_1^3 y_p \quad (9)$$

leads to the derivative of V_3 as

$$\dot{V}_3(e, \phi) = -a_m e_1^4 \leq 0. \quad (10)$$

From Equation (12), it can be shown that $e \in L^4$. However, it is a well known fact given in Spiegel (1969) that any $L^4 \subset L^2$. Therefore, the $e \in L^2$. Thus, the stability analysis follows the steps given by Narendra and Annaswamy (1989) as given in the earlier section. Briefly, the state of the system described in the equations (4) and (6) is uniformly stab-

le. Again here, this feature implies that for bounded initial values, the signals $e_1(t)$ and $\phi(t)$ remain bounded for all time $t > 0$. Again, following the Barbalat Lemma leads to the fact that $e_1 \rightarrow 0$ as time $t \rightarrow \infty$. Finally, this feature, again, implies that x_p asymptotically becomes x_m .

3 AN n^{th} ORDER PLANT: DISTURBANCE – FREE CASE – ADAPTIVE LAWS USING QLF AND NQLF

This section shows how Lyapunov stability theory is used to the design of adaptive control systems. An n^{th} order plant is described in this section for the disturbance-free or ideal case. The mathematical description of this MRAC system is given. First, the Quadratic Lyapunov Function (QLF) and the so-called ideal/standard adaptive laws are presented for this ideal case. These are available in [2]; however, for coherence in development, they are presented here. Then, the new Non-Quadratic Lyapunov Function (NQLF) and new adaptive laws are presented.

A System Description

Following the steps and notation given in [2, 3], the system is described as given in Figure 2. Consider the adaptive control of an n^{th} order unknown plant.

Plant:

$$\dot{x}_p = A_p x_p + b_p u; \quad y_p = h_p^T x_p, \quad (11)$$

where $u: \mathbf{R}^+ \rightarrow \mathbf{R}$ is the input, $y_p: \mathbf{R}^+ \rightarrow \mathbf{R}$ is the output, and $x_p: \mathbf{R}^+ \rightarrow \mathbf{R}^n$ is the n -dimensional state vector of the plant. This plant has a transfer function, given by

$$W_p(s) = h_p^T (sI - A_p) b_p = \frac{k_p Z_p(s)}{R_p(s)}, \quad (12)$$

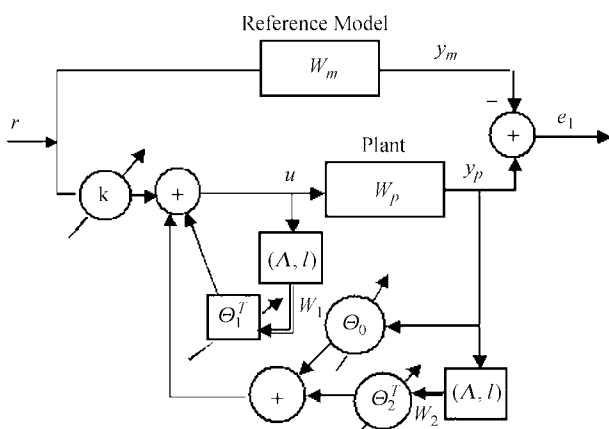


Fig. 2 The general structure of the MRAC system

where $Z_p(s)$ and $R_p(s)$ are monic coprime polynomials of degrees m and $n (> m)$, respectively, and $Z_p(s)$ is Hurwitz. The sign of k_p and values of m and n are specified, but the coefficients of $Z_p(s)$ and $R_p(s)$ unknown constants.

Reference model:

A reference model is set up with a transfer function:

$$W_m(s) = \frac{k_m Z_m(s)}{R_m(s)}; \quad \text{and} \quad \dot{x}_m = A_m x_m + b_m r, \quad (13)$$

with $y_m = h_m^T x_m$, where $r: \mathbf{R}^+ \rightarrow \mathbf{R}$ is the input, $y_m: \mathbf{R}^+ \rightarrow \mathbf{R}$ is the output, and $x_m: \mathbf{R}^+ \rightarrow \mathbf{R}^n$ is the n -dimensional state vector of the reference model; $Z_m(s)$ and $R_m(s)$ are chosen stable polynomials of degree m and n respectively. A piecewise continuous uniformly bounded reference input $r(t)$ to the reference model yields the desired output $y_m(t)$. The adaptive control problem is to determine a bounded control input u to the plant so that the output error $e_1(t)$ goes to zero together with all the other signals remain bounded.

Controller:

The adaptive controller is described by the equations:

$$\dot{w}_1 = \Lambda w_1 + l u; \quad \dot{w}_2 = \Lambda w_2 + l y_p \quad (14)$$

$$u(t) = \theta^T(t) w(t);$$

with

$$w^T = [r, w_1^T, y_p, w_2^T], \quad \text{and} \quad \theta^T = [k, \theta_1^T, \theta_0, \theta_2^T] \quad (15)$$

where θ_1, θ_2, w_1 and w_2 are $(n - 1)$ -dimensional vectors, Λ is chosen as an asymptotically stable $(n - 1) \times (n - 1)$ matrix; (Λ, l) is controllable, and θ is the $2n$ -dimensional vector. When $\theta = \theta^*$, where θ^* is a constant vector, the transfer function of the plant together with the controller matches that of the reference model shown in Figure 2, exactly. The control problem, therefore, can be restated as the determination of the adaptive law for adjusting $\theta(t)$ so that $e_1(t)$ (defined as $y_p - y_m$) goes to zero together with all other signals remain bounded.

From the above description, the plant and controller combination can be written as:

$$\dot{x} = A_{mn} x + b_{mn} (\phi^T w + k^* r); \quad y_p = h_m^T x \quad (16)$$

where

$$A_{mn} = \begin{bmatrix} A_p + b_p \theta_0^* h_p^T & b_p \theta_1^{*T} & b_p \theta_2^{*T} \\ l \theta_0^* h_p^T & \Lambda + l \theta_1^{*T} & l \theta_2^{*T} \\ l h_p^T & 0 & \Lambda \end{bmatrix};$$

$$b_{mn} = [b_p, l, 0]^T; \quad h_{mn} = [h_p^T, 0, 0]^T$$

and $x = [x_p^T, w_1^T, w_2^T]^T \in \mathbf{R}^{3n-2}$. (17)

When $\phi = \theta - \theta^* = 0$, a non-minimal representation of the reference model can be given by

$$\dot{x}_{mn} = A_{mn}x_{mn} + b_{mn}k^*r; \quad y_m = h_{mn}^T x_{mn} \quad (18)$$

where $x_{mn} \equiv [x_p^{*T}, w_1^{*T}, w_2^{*T}]^T : \mathbf{R}^+ \rightarrow \mathbf{R}^{3n-2}$; and the A_{mn} is a $(3n-2) \times (3n-2)$ matrix and h_{mn} and b_{mn} are $(3n-2)$ -dimensional vectors.

The error model:

The state error $e = x - x_{mn}$ satisfies the differential equation, with e_1 as the output error

$$\dot{e} = A_{mn}e + b_{mn}[\phi^T w]; \quad e_1 = h_{mn}^T e,$$

where $e: \mathbf{R}^+ \rightarrow \mathbf{R}^{3n-2}$; $w, \phi: \mathbf{R}^+ \rightarrow \mathbf{R}^{2n}$

$$\text{and } e_1: \mathbf{R}^+ \rightarrow \mathbf{R}. \quad (19)$$

From this error model, it is possible to develop adaptive laws for several cases [2, 3].

B Lyapunov Functions and Adaptive Laws

This section presents the QLF, NQLF and adaptive laws due to them for the MRAC system described in section A.

1) *Quadratic Lyapunov Function and adaptive laws:*

Consider the Lyapunov function V_1 [2, 3] given by

$$V_3 = e^T P_1 e + \frac{2}{|k_p|} [\phi^T \phi]. \quad (20)$$

Differentiating Equation (10) with respect to time, using Lemma 5.1 in [2] and choosing the adaptive laws for controller parameters in the vector θ in Equation (5), as

$$\dot{\phi} = \dot{\theta} = -\text{sgn}(k_p)e_1 w \quad (21)$$

leads to the $\dot{V}_1 = -e^T Q_1 e \leq 0$. (22)

Since $e_1 \in L^2$ and its derivative is bounded from Equations (18, 19), the error signal $e_1 \rightarrow 0$ as time $t \rightarrow \infty$. Thus, stability of the system is assured as in [2].

2) *Non-Quadratic Lyapunov Function and Adaptive Laws:*

Consider the new Non Quadratic Lyapunov function V_3 given by

$$V_3 = e^T [e_d^T P_1 e_d] e + \frac{2}{|k_p|} [\phi^T \phi] \quad (23)$$

where $e^T = [e_1, e_2, \dots]$ and, for example, when

$$e^T = [e_1 \quad e_2] e_d = e_d^T = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} e_s^T = [e_1^2, e_2^2];$$

$$e_s^T = e^T e_d = e^T e_d^T$$

Thus, with $P_3 = P_3^T = e_d^T P_1 e_d > 0$

$$V_3 = e^T P_3 e + \frac{2}{|k_p|} [\phi^T \phi]. \quad (24)$$

Differentiating Equation (13) with respect to time, using Lemma 5.1 in [2] and choosing the adaptive laws for the controller parameters in vector θ of Equation (5), as

$$\dot{\phi} = \dot{\theta} = -\text{sgn}(k_p)e_1^3 w. \quad (25)$$

Gives with $Q_3 = e_d^T Q_1 e_d$ and $e_s^T = e^T e_d$ (26)

$$\dot{V}_3 = -e^T Q_3 e = -e_s^T Q_1 e_s \leq 0. \quad (27)$$

Since $e \in L^4 \subset L^2$ [6] and its derivative is bounded from Equations (18, 19), the error signal $e_1 \rightarrow 0$ as time $t \rightarrow \infty$. Thus, stability of the system for this case is also assured as shown in [2, 7, 8, 9]. Detailed development of this section is given in [8].

4 SIMULATION RESULTS

For the simulation study, the SIMNON (Simulation of Nonlinear Systems) package has been used, which was originally developed at the Lund Institute of Technology in Sweden. For this study, an example is presented for a second order system. The responses of the system error are obtained, using adaptive laws based on QLF and the new NQLF.

Example:

This example is similar to that given in [2]. The plant and the reference model parameters are: $a_{p1} = -2, a_{p0} = 2; a_{m1} = 2, a_{m0} = 1; b_{m0} = b_{p0} = 1; \lambda_0 = l = 1$ leading to $\theta_o^* = -4$ and $\theta_2^* = 5$. Thus, initially the plant was unstable. Response of the system error signal e_1 in the adaptive system is shown in Figure 3, for using: 1) the adaptive laws based on QLF; and 2) the adaptive laws based on NQLF.

The initial values chosen were: $e_1(0) = 2$ and $\theta_o(0) = 1.0$ and $\theta_2(0) = 1.0$. The reference input $r = 5$.

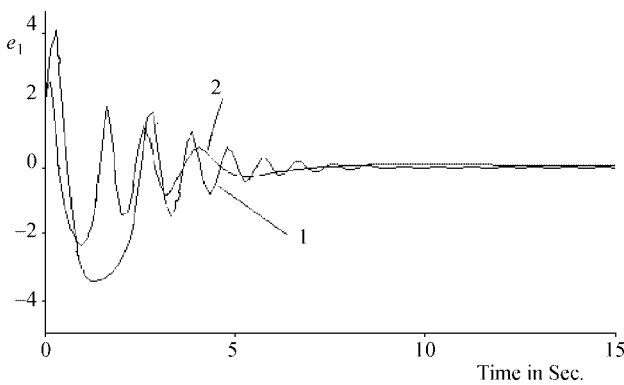


Fig. 3 Example system error responses: 1) using e_1 signal; 2) using e_1^3 signal

It can be seen from Figure 3 that the new adaptive laws improve the response of the system.

5 CONCLUSION

This paper considers the development of adaptive laws for an n^{th} order plant (with $n^* = 1$) in the MRAC system, for the disturbance-free case. It develops these laws using first Quadratic Lyapunov Function (QLF) and next using a recently developed Non-Quadratic Lyapunov Function (NQLF). While Quadratic Lyapunov Function (QLF) had led to the use of the system output error signal in the adaptive laws, the new Non-Quadratic Lyapunov Function (NQLF) leads to the use of the cube of the same error signal in the adaptive laws. Mathematical arguments offer improved convergence of the system error signal to zero; simulation results support these arguments. Thus, Non-Quadratic Lyapunov Function and the new adaptive laws seem to offer improvement to MRAC systems in general. There appears to be some scope for further work

on developing MRAC schemes for plants with relative degree $n^* \geq 2$ and for plants in the presence of disturbances.

ACKNOWLEDGEMENT

The first author gratefully thank Prof. K. S. Narendra of Yale University and Prof. A. M. Annaswamy of M.I.T., U.S.A. for their encouragement.

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Novi zakoni adaptivnog upravljanja referentnim modelom zasnovani na nekvadratnoj Ljapunovljevoj funkciji.

Za projektiranje sustava adaptivnog upravljanja referentnim modelom (MRAC) razvijeni su adaptivni zakoni koji se zasnivaju na Ljapunovljevoj teoriji stabilnosti. U zadnjih se nekoliko desetljeća uobičajeno koristi kvadratna Ljapunovljeva funkcija (QLF). Suprotno tome, u ovome se radu zakoni upravljanja referentnim modelom izvode na osnovi nove nekvadratne Ljapunovljeve funkcije (NQLF). Za izvedene je nove zakone upravljanja cjelokupni sustav stabilan, kao i kod prijašnjih zakona adaptivnog upravljanja kvadratnom Ljapunovljevom funkcijom. Povrh toga predloženim novim zakonima upravljanja NQLF funkcijom poboljšava se konvergencija izlaznog signala pogreške prema nuli. Na kraju su u radu predstavljeni rezultati simulacija koji podupiru navedene tvrdnje.

Ključne riječi: adaptivno upravljanje, adaptivni zakoni, adaptivno upravljanje referentnim modelom, Ljapunovljeva funkcija, adaptivno upravljanje referentnim modelom po Ljapunovu

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Received: 2004–11–30