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# PERIOD–LENGTH EQUALITY FOR THE NEAREST INTEGER AND NEAREST SQUARE CONTINUED FRACTION EXPANSIONS OF A QUADRATIC SURD

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Abstract. We prove equality of the period–lengths of the nearest integer continued fraction and the nearest square continued fraction, for arbitrary real quadratic irrationals.

## 1. Introduction

The oldest method known to give the general solution to Pell's equation is the *cyclic* method ([1, 14]), studied by Jayadeva, Bhaskara II, and others beginning in the 10th century or earlier ([15, p. 35]). The *nearest square* continued fraction (denoted by NSCF) is a variant of the cyclic method and so is one of the earliest continued fractions discovered. Despite its great age, it has not been studied to the same extent as many other continued fractions. The first systematic study of the nearest square continued fraction was done by A.A.K. Ayyangar ([2]). The nearest square continued fraction has nice properties similar to the more-studied *regular* continued fraction [12, p. 22]) and *nearest integer* continued fraction ([12, pp. 143, 160]) (denoted by RCF and NICF, respectively), such as easy criteria for finding the middle of the period of the expansion of $\sqrt{D}$ without computing the whole period, with the NICF and NSCF at times having some superiority over the RCF (see [8], [11] and [16]); symmetry properties for periods for certain classes of quadratic surd; and easy criteria for determining whether a quadratic surd has a purely periodic expansion. We were quite astounded to discover that the period length for the NSCF is the same as that for the NICF, despite the more

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complicated definition of the NSCF. The NSCF expansion of a quadratic surd is closely related to the *optimal continued fraction* (OCF) of W. Bosma ([3]) and is the basis for a recent computer algorithm [9] by the first author, for the finding the OCF of a quadratic surd.

The RCF, NICF and NSCF are expansions of an irrational number $\xi_0$ as a *semi–regular* continued fraction ([12, p. 137]): $\xi_0 = a_0 + \frac{\epsilon_1\rfloor}{\lceil a_1} + \cdots + \frac{\epsilon_n\rfloor}{\lceil a_n} + \cdots$, where $\epsilon_n = \pm 1$ for each $n$ and the $a_n$ are generated by the recurrence relations

$$\xi_n = a_n + \frac{\epsilon_{n+1}}{\xi_{n+1}}, \quad n \geq 0, \tag{1.1}$$

$$a_n = \begin{cases} \lfloor \xi_n \rfloor & \text{if } \epsilon_{n+1} = 1, \\ \lfloor \xi_n \rfloor + 1 & \text{if } \epsilon_{n+1} = -1, \end{cases} \tag{1.2}$$

where $\lfloor \xi_n \rfloor$ denotes the integer part of $\xi_n$. The $\xi_n$ are called the *complete quotients* and $\xi_n > 1$ if $n \geq 1$, by (1.1) and (1.2). The $\epsilon_{n+1}$ and $a_n$ are called *partial numerators* and *denominators*, respectively.

The RCF is defined by $a_n = \lfloor \xi_n \rfloor$, and $\epsilon_{n+1} = 1$. The NICF is defined by $a_n = [\xi_n]$, (the nearest integer to $\xi_n$) and $\epsilon_{n+1} = \text{sign}(\xi_n - a_n)$, so that $|\xi_n - a_n| < \frac{1}{2}$, $\xi_{n+1} > 2$ for $n \geq 0$ and hence $a_n \geq 2$ for $n \geq 1$.

The NSCF is defined only for real quadratic surds $\xi_0 = \frac{P_0 + \sqrt{D}}{Q_0}$ in standard form, i.e., $D$ is a non-square positive integer and $P_0, Q_0 \neq 0, (D - {P_0}^2)/Q_0$ are integers, having no common factor other than 1. Then for $n \geq 0$, with $\xi_n = \frac{P_n + \sqrt{D}}{Q_n}$ in standard form and $c_n = \lfloor \xi_n \rfloor$, we have *positive* and *negative* representations

$$\xi_n = \frac{P_n + \sqrt{D}}{Q_n} = c_n + \frac{Q'_{n+1}}{P'_{n+1} + \sqrt{D}} = c_n + 1 - \frac{Q''_{n+1}}{P''_{n+1} + \sqrt{D}}, \tag{1.3}$$

where $\frac{P'_{n+1} + \sqrt{D}}{Q'_{n+1}} > 1$ and $\frac{P''_{n+1} + \sqrt{D}}{Q''_{n+1}} > 1$ are also in standard form. Then the NSCF is defined by choosing

(a) $a_n = \begin{cases} c_n & \text{if } |Q'_{n+1}| < |Q''_{n+1}|, \text{ or } |Q'_{n+1}| = |Q''_{n+1}| \text{ and } Q_n < 0, \\ c_n + 1 & \text{if } |Q'_{n+1}| > |Q''_{n+1}|, \text{ or } |Q'_{n+1}| = |Q''_{n+1}| \text{ and } Q_n > 0, \end{cases}$

(b) $\epsilon_{n+1} = \text{sign}(\xi_n - a_n)$.

If $Q_n, Q'_{n+1}, Q''_{n+1}$ are all positive, (a) simplifies to

$$a_n = \begin{cases} c_n & \text{if } Q'_{n+1} < Q''_{n+1}, \\ c_n + 1 & \text{if } Q'_{n+1} \geq Q''_{n+1}. \end{cases}$$

Then (1.1) and (1.3) give

$$\xi_{n+1} = \begin{cases} (P'_{n+1} + \sqrt{D})/Q'_{n+1} & \text{if } \epsilon_{n+1} = 1, \\ (P''_{n+1} + \sqrt{D})/Q''_{n+1} & \text{if } \epsilon_{n+1} = -1. \end{cases}$$

The RCF, NICF and NSCF expansions of a quadratic surd become periodic, i.e., the complete quotients $\xi_n$ satisfy $\xi_i = \xi_{i+k}$ for $i \geq i_0$ for some $k \geq 1$. Then $\epsilon_{i+1} = \epsilon_{i+k+1}$ and $a_i = a_{i+k}$ for all $i \geq i_0$. The least such $k$ is called the *period–length* (see Satz 3.2 of [12, p. 66] for the RCF, where the proof of periodicity also holds for the NICF, by Satz 5.18(B) of [12, p. 161] and Theorem II of [2, p. 25] for the NSCF).

Let *L-RCF, L-NICF* and *L-NSCF* be the period–lengths of the RCF, NICF and NSCF expansions of $\xi_0$. Also let *N-NICF* and *N-NSCF* be the number of partial numerators $\epsilon_i = -1$ in the respective NICF and NSCF periods of $\xi_0$.

We prove *L-NICF* = *L-NSCF* by showing that if $\xi_0$ is not equivalent to $(1+\sqrt{5})/2$, i.e., its RCF period has at least one $a_i > 1$, then

$$\textit{L-NICF} + \textit{N-NICF} = \textit{L-RCF}, \tag{1.4}$$

$$\textit{L-NSCF} + \textit{N-NSCF} = \textit{L-RCF}, \tag{1.5}$$

$$\textit{N-NICF} = \textit{N-NSCF}, \tag{1.6}$$

while if $\xi_0$ is equivalent to $(1+\sqrt{5})/2$, then *L-NICF* = *L-NSCF* = 1.

We remark that (1.4) is an immediate consequence of the RCF to NICF singularization algorithm described in Section 2.11(i) of [7].

To prove (1.5) and (1.6), we reduce the problem to the case of a purely periodic regular continued fraction and study its transformation into the NSCF expansion, using Theorem 2.4. It is then a matter of studying the effect on strings of consecutive RCF partial quotients equal to 1. We have to make use of certain *approximation constants* $\Theta_n$. We prove *N-NSCF*=*N-NICF* by showing that if there are $k$ strings of consecutive 1's among the partial quotients of an RCF period and the length of the $i$-th string is $l_i$, then

$$\textit{N-NSCF} = \sum_{i=1}^{k} \left\lfloor \frac{l_i + 1}{2} \right\rfloor . \tag{1.7}$$

We note that (1.7) holds with *N-NSCF* replaced by *N-NICF*, as a consequence of the RCF to NICF singularization process in Section 2.11(i) of [7].

## 2. Selenius' Lemma and the RCF to NSCF transformation

C.-O. Selenius ([13, §43, p. 63]) gave an algorithm for converting the RCF expansion of $\sqrt{D}$ to its NSCF expansion. The algorithm generalizes to a wider class of quadratic irrationals and is given as Theorem 2.4.

We call a quadratic surd $\xi_0$ *quasi–reduced* if either

(i) $\xi_0$ is an RCF–reduced quadratic irrational (i.e., $\xi_0$ has a purely periodic RCF expansion, or equivalently ([12, §22]) $\xi_0 > 1$ and $-1 < \overline{\xi_0} < 0$), or

(ii) $0 < Q_0 < 2\sqrt{D}$ and $\xi_1$ is an RCF–reduced quadratic irrational.

LEMMA 2.1. *If $\xi_0$ is quasi–reduced and $\xi_0, \xi_1, \ldots$ denote the complete quotients of the RCF expansion of $\frac{P_0+\sqrt{D}}{Q_0}$, with positive and negative representations for $\nu \geq 0$:*

$$\xi_\nu = \frac{P_\nu + \sqrt{D}}{Q_\nu} = a_\nu + \frac{Q_{\nu+1}}{P_{\nu+1} + \sqrt{D}} = a_\nu + 1 - \frac{Q''_{\nu+1}}{P''_{\nu+1} + \sqrt{D}}, \tag{2.1}$$

*where $a_\nu = \lfloor \xi_\nu \rfloor$, then $Q_{\nu+1}, Q''_{\nu+1}, P_{\nu+1}, P''_{\nu+1}$ are positive for $\nu \geq 0$.*

PROOF. This follows from [2, Theorem I(iv), p. 22]. □

LEMMA 2.2. *With the notation of (2.1),*

(i) *If $\xi_0$ is a quadratic irrational and $a_\nu = 1$, where $\nu \geq 1$, then*
  (a) *$Q''_\nu = Q_{\nu+1}$ and conversely,*
  (b) *$P''_\nu = P_{\nu+1} + Q_{\nu+1}$.*

(ii) *If $\xi_0$ is quasi–reduced, then $Q''_\nu \leq Q_\nu$ implies $a_\nu = 1$.*

PROOF. See Satz 37 of [13, p. 62], where the results are given for $\xi_0 = \sqrt{D}$, but remain valid for the more general case here. □

REMARK 2.3. We note from Lemma 2.2 that if $Q''_\nu \leq Q_\nu$, then

$$\frac{P''_\nu + \sqrt{D}}{Q''_\nu} = \frac{P_{\nu+1} + Q_{\nu+1} + \sqrt{D}}{Q_{\nu+1}} = \xi_{\nu+1} + 1. \tag{2.2}$$

THEOREM 2.4. *Let $\xi_0$ be a quasi–reduced quadratic surd with RCF complete quotients $\xi_n = \frac{P_n+\sqrt{D}}{Q_n}$ and partial quotients $a_n$. Let $\epsilon_m$ and $f(m)$ be recursively defined for $m \geq 0$, as follows: Let $\epsilon_0 = 1$, $f(0) = 0$ and suppose $\epsilon_m$ and $f(m)$ are defined and $\xi_{f(m)}$ has positive and negative representations*

$$\xi_{f(m)} = a_{f(m)} + \frac{Q_{f(m)+1}}{P_{f(m)+1} + \sqrt{D}} = a_{f(m)} + 1 - \frac{Q''_{f(m)+1}}{P''_{f(m)+1} + \sqrt{D}}. \tag{2.3}$$

*Let*

$$\epsilon_{m+1} = \begin{cases} 1, & \text{if } Q_{f(m)+1} < Q''_{f(m)+1}, \\ -1, & \text{if } Q_{f(m)+1} \geq Q''_{f(m)+1} \end{cases} \tag{2.4}$$

*and*

$$f(m+1) = \begin{cases} f(m)+1, & \text{if } \epsilon_{m+1} = 1, \\ f(m)+2, & \text{if } \epsilon_{m+1} = -1. \end{cases} \tag{2.5}$$

*Also for $m \geq 0$, let*

$$\tilde{\xi}_m = \begin{cases} \xi_{f(m)}, & \text{if } \epsilon_m = 1, \\ \xi_{f(m)} + 1, & \text{if } \epsilon_m = -1. \end{cases} \tag{2.6}$$

*and*

$$\tilde{a}_m = \begin{cases} a_{f(m)}, & \text{if } \epsilon_m = 1,\ \epsilon_{m+1} = 1, \\ a_{f(m)} + 1, & \text{if } \epsilon_m \epsilon_{m+1} = -1, \\ a_{f(m)} + 2, & \text{if } \epsilon_m = -1,\ \epsilon_{m+1} = -1. \end{cases} \tag{2.7}$$

*Then $\tilde{\xi}_m, \epsilon_{m+1}$ and $\tilde{a}_m$ are the complete quotients, partial numerators and denominators of the NSCF expansion of $\xi_0$.*

Proof. We use induction on $m \geq 0$ to prove that $\tilde{\xi}_m$ is the $m$-th NSCF complete quotient. As $\epsilon_0 = 1$, (2.6) gives $\tilde{\xi}_0 = \xi_0$. Now assume that $\tilde{\xi}_m$ is the $m$-th complete NSCF quotient of $\xi_0$. Then (2.6) gives the positive and negative representations

$$\tilde{\xi}_m = c_m + \frac{Q_{f(m)+1}}{P_{f(m)+1} + \sqrt{D}} = c_m + 1 - \frac{Q''_{f(m)+1}}{P''_{f(m)+1} + \sqrt{D}}, \tag{2.8}$$

where

$$c_m = \begin{cases} a_{f(m)} & \text{if } \epsilon_m = 1, \\ a_{f(m)} + 1 & \text{if } \epsilon_m = -1. \end{cases} \tag{2.9}$$

If $\xi$ denotes the $(m+1)$-th NSCF complete quotient of $\xi_0$, from (2.8) we have

$$\xi = \begin{cases} \frac{P_{f(m)+1} + \sqrt{D}}{Q_{f(m)+1}}, & \text{if } Q_{f(m)+1} < Q''_{f(m)+1}, \\ \frac{P''_{f(m)+1} + \sqrt{D}}{Q''_{f(m)+1}}, & \text{if } Q_{f(m)+1} \geq Q''_{f(m)+1}. \end{cases} \tag{2.10}$$

But $\epsilon_{m+1} = 1 \implies Q_{f(m)+1} < Q''_{f(m)+1}$. Also $\epsilon_{m+1} = -1 \implies Q_{f(m)+1} \geq Q''_{f(m)+1}$ and hence $\frac{P''_{f(m)+1} + \sqrt{D}}{Q''_{f(m)+1}} = \xi_{f(m)+2} + 1$, by Lemma 2.2. Then (2.10) gives

$$\begin{aligned} \xi &= \begin{cases} \xi_{f(m)+1} = \xi_{f(m+1)}, & \text{if } \epsilon_{m+1} = 1, \\ \xi_{f(m)+2} + 1 = \xi_{f(m+1)} + 1, & \text{if } \epsilon_{m+1} = -1, \end{cases} \\ &= \tilde{\xi}_{m+1} \end{aligned}$$

and the induction goes through.

From (2.8), we see that the $m$-th NSCF partial denominator $a$ is given by

$$a = \begin{cases} c_m, & \text{if } Q_{f(m)+1} < Q''_{f(m)+1}, \text{ i.e., if } \epsilon_{m+1} = 1, \\ c_m + 1, & \text{if } Q_{f(m)+1} \geq Q''_{f(m)+1}, \text{ i.e., if } \epsilon_{m+1} = -1, \end{cases} \tag{2.11}$$

$$= \begin{cases} a_{f(m)}, & \text{if } \epsilon_m = 1 = \epsilon_{m+1}, \\ a_{f(m)} + 1, & \text{if } \epsilon_m = 1, \epsilon_{m+1} = -1, \\ a_{f(m)} + 1, & \text{if } \epsilon_m = -1, \epsilon_{m+1} = 1, \\ a_{f(m)} + 2, & \text{if } \epsilon_m = -1, \epsilon_{m+1} = -1, \end{cases}$$

$$= \tilde{a}_m.$$

Next, from (2.8) and (2.11),

$$\tilde{\xi}_m - \tilde{a}_m = \begin{cases} \frac{Q_{f(m)+1}}{P_{f(m)+1} + \sqrt{D}} > 0, & \text{if } \epsilon_{m+1} = 1, \\ -\frac{Q''_{f(m)+1}}{P''_{f(m)+1} + \sqrt{D}} < 0, & \text{if } \epsilon_{m+1} = -1. \end{cases}$$

Hence $\operatorname{sign}(\tilde{\xi}_m - \tilde{a}_m) = \epsilon_{m+1}$. ∎

Table 1 from [13, pp. 65–66] gives the RCF and NSCF expansions of $\sqrt{97}$ as far as the end of the first RCF period. The RCF expansion is

$$\sqrt{97} = [9, \overline{1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18}].$$

If $\tilde{\xi}_m$ occurs at line $n = f(m)$ of the RCF positive and negative representation and $Q_{n+1} < Q''_{n+1}$, then $\tilde{\xi}_{m+1} = \xi_{n+1}$; otherwise we proceed to line $n+2$ and $\tilde{\xi}_{m+1} = \xi_{n+2} + 1$. Here $(P, Q)$ denotes $\frac{P+\sqrt{97}}{Q}$, $\xi_j = (P_j + \sqrt{97})/Q_j$ and $\tilde{\xi}_k = (\tilde{P}_k + \sqrt{97})/\tilde{Q}_k$. Then

$f(1) = 2, f(2) = 4, f(3) = 6, f(4) = 7, f(5) = 9, f(6) = 11, f(7) = 13,$
$\epsilon_1 = -1, \epsilon_2 = -1, \epsilon_3 = -1, \epsilon_4 = 1, \epsilon_5 = -1, \epsilon_6 = -1, \epsilon_7 = -1$ and

$$\sqrt{97} = 10 - \frac{1|}{|\underset{*}{7}} - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|2} - \frac{1|}{|7} - \frac{1|}{|\underset{*}{20}} - \cdots$$

where the asterisks denote the period $\tilde{\xi}_1 = \tilde{\xi}_7$.

By contrast, we have the NICF expansion:

$$\sqrt{97} = 10 - \frac{1|}{|\underset{*}{7}} - \frac{1|}{|3} - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|6} - \frac{1|}{|\underset{*}{20}} - \cdots$$

LEMMA 2.5. *For each $\xi_n$, $n \geq 1$ in the RCF to NSCF transformation where $a_n > 1$, there exists an $m \geq 0$ such that $n = f(m)$.*

PROOF. NSCF: Let $a_n > 1$ and $f(m) \leq n < f(m+1)$. If $f(m) < n$, then $f(m+1) = f(m) + 2$ and $\epsilon_{m+1} = -1$; also $n = f(m) + 1$. Hence $Q_{f(m)+1} \geq Q''_{f(m)+1}$ and so by Lemma 2.2(ii), $a_n = a_{f(m)+1} = 1$, a contradiction. Hence $n = f(m)$. ∎

| RCF | NSCF |
|---|---|
| $\xi_0 = (0,1) = 9 + (9,16)^{-1} = 10 - (10,3)^{-1}$ | $\tilde{\xi}_0 = 9 + (9,16)^{-1} = \boxed{10} - (10,3)^{-1}$ (16>3) |
| $\xi_1 = (9,16) = 1 + (7,3)^{-1} = 2 - (23,27)^{-1}$ | |
| $\xi_2 = (7,3) = 5 + (8,11)^{-1} = 6 - (11,8)^{-1}$ | $\tilde{\xi}_1 = 6 + (8,11)^{-1} = \boxed{7} - (11,8)^{-1}$ (11>8) |
| $\xi_3 = (8,11) = 1 + (3,8) = 2 - (14,9)$ | |
| $\xi_4 = (3,8) = 1 + (5,9)^{-1} = 2 - (13,9)^{-1}$ | $\tilde{\xi}_2 = 2 + (5,9)^{-1} = \boxed{3} - (13,9)^{-1}$ (9=9) |
| $\xi_5 = (5,9) = 1 + (4,9)^{-1} = 2 - (13,8)^{-1}$ | |
| $\xi_6 = (4,9) = 1 + (5,8)^{-1} = 2 - (14,11)^{-1}$ | $\tilde{\xi}_3 = \boxed{2} + (5,8)^{-1} = 3 - (14,11)^{-1}$ (8<11) |
| $\xi_7 = (5,8) = 1 + (3,11)^{-1} = 2 - (11,3)^{-1}$ | $\tilde{\xi}_4 = 1 + (3,11)^{-1} = \boxed{2} - (11,3)^{-1}$ (11>3) |
| $\xi_8 = (3,11) = 1 + (8,3)^{-1} = 2 - (19,24)^{-1}$ | |
| $\xi_9 = (8,3) = 5 + (7,16)^{-1} = 6 - (10,1)^{-1}$ | $\tilde{\xi}_5 = 6 + (7,16)^{-1} = \boxed{7} - (10,1)^{-1}$ (16>1) |
| $\xi_{10} = (7,16) = 1 + (9,1)^{-1} = 2 - (25,33)^{-1}$ | |
| $\xi_{11} = (9,1) = 18 + (9,16)^{-1} = 19 - (10,3)^{-1}$ | $\tilde{\xi}_6 = 19 + (9,16)^{-1} = \boxed{20} - (10,3)^{-1}$ (16>3) |

TABLE 1. RCF to NSCF algorithm for $\sqrt{97}$.

## 3. THE $Q$-$\gamma$ LAW OF SELENIUS

Selenius defined his SK continued fraction expansion of a real number $\xi_0$ by comparing the approximation constants $\Theta_n$ and $\Theta_{n-1}$. In the case $\xi_0 = \sqrt{D}$, he demonstrated a closeness with the NSCF expansion in Satz 38, [13, p. 67], using the following result.

LEMMA 3.1. *Let $\Theta_n = B_n|B_n\xi_0 - A_n|$, where $A_n/B_n$ is the $n$-th RCF convergent to $\xi_0 = (P_0 + \sqrt{D})/Q_0$. Suppose $Q_n$ and $Q_{n+1}$ are positive for all large $n \geq 0$.*

(a) *If $n$ is sufficiently large (e.g., $B_nB_{n-1} > |Q_0|$) and $Q_{n+1} \neq Q_n$, then*

$$Q_{n+1} < Q_n \iff \Theta_n < \Theta_{n-1}. \tag{3.1}$$

*Moreover if $\xi_0 = \sqrt{D}$, then equation* (3.1) *holds for $n \geq 1$.*

(b) *If $Q_{n+1} = Q_n$, then for $n \geq 1$,*

$$(-1)^n(\Theta_n - \Theta_{n-1}) > 0.$$

Selenius stated his result in terms of $\gamma_n = 1/\Theta_{n-1}$.

PROOF. See Satz 29, [13, p. 52]. ∎

## 4. INEQUALITIES FOR THE $\Theta_n$

Fortunately there exist inequalities for the $\Theta_n$, which by virtue of Lemma 3.1, translate to inequalities between $Q_n$ and $Q_{n+1}$. The former inequalities are due to Selenius ([13, §24, p. 37]) and subsequently W. Bosma and C. Kraaikamp.

LEMMA 4.1. *Let $\xi_0$ be an irrational number with RCF expansion*

$$x = [a_0, a_1, \ldots, a_n, 1^m, a_{n+m+1}, \ldots],$$

*where* $1^m$ *denotes a sequence of consecutive partial quotients equal to* 1*, i.e.,* $a_n > 1$ *if* $n \geq 1$ *and* $a_{n+1} = \cdots = a_{n+m} = 1, a_{n+m+1} > 1$*. Then*

(i) *If* $m$ *is odd,*

$$\Theta_{n+e} > \Theta_{n+e+1} \text{ if } 0 \leq e \leq m-1, e \text{ even.} \tag{4.1}$$

(ii) *If* $m$ *is even,* $m = 2k$*,*

$$\Theta_{n+e} > \Theta_{n+e+1} \text{ if } 0 \leq e \leq k-2, e \text{ even,} \tag{4.2}$$

$$\Theta_{n+f} > \Theta_{n+f+1} \text{ if } k \leq f \leq 2k-1, f \text{ odd.} \tag{4.3}$$

(iii) *If* $m = 2k, k$ *even,*

$$\Theta_{n+k} < \Theta_{n+k+1}. \tag{4.4}$$

(iv) *If* $m = 2k, k$ *odd,* $k \geq 3$*,*

$$\Theta_{n+k+1} < \Theta_{n+k+2}. \tag{4.5}$$

PROOF. These follow from [4, Theorem 2.2, p. 485], except for the case $f = 2k-1$ of (4.3), which is easily proved using Lemma 2.1 of [4, p. 485].

□

## 5. EQUALITY OF CONSECUTIVE $Q_i$'S IN A UNISEQUENCE

Lemma 3.1 gives little information when $Q_{n+1} = Q_n$. The following result identifies $n$ and is used in the proof of Lemma 7.1.

LEMMA 5.1. *Suppose* $\xi_0$ *is RCF–reduced with period–length* $l$*. Then if* $a_n > 1, a_{n+1} = \cdots = a_{n+m} = 1, a_{n+m+1} > 1, n+m+1 \leq l$ *and* $Q_v = Q_{v+1}, n+1 \leq v < n+m$*, we have* $m = 2k$ *and* $v = n+k$*.*

PROOF. Suppose $Q_v = Q_{v+1}$. Then $D = P_{v+1}^2 + Q_vQ_{v+1} = P_{v+1}^2 + Q_{v+1}^2$ and $\xi_{v+1} = (q + \sqrt{p^2+q^2})/p$, where $p = Q_{v+1}$ and $q = P_{v+1}$. Now $\xi_{v+1}$ is RCF–reduced and the RCF expansion is purely periodic. There are two cases:

(i) $p > 2q$. Lemma 2 of [2, p. 106] dealt with this case. The period begins and ends with an odd number $k$ of unit partial quotients and hence $m = 2k$, with $k$ odd.
(ii) $p < 2q$. The proof also shows that the period begins and ends with an even number $k$ of unit partial quotients and hence $m = 2k$, with $k$ even.

It follows that $v = n+k$. □

## 6. Connections between RCF and NSCF period–lengths

Ayyangar ([2, p. 27]) gave a definition of NSCF–reduced quadratic surd that is less explicit than the one for regular continued fractions. A surd $\frac{\tilde{P}_\nu+\sqrt{D}}{\tilde{Q}_\nu}$ is said to be *special* if $\tilde{Q}_\nu^2 + \frac{1}{4}\tilde{Q}_{\nu+1}^2 \le D$ and $\tilde{Q}_{\nu+1}^2 + \frac{1}{4}\tilde{Q}_\nu^2 \le D$; it is *semi–reduced* if it is the successor of a special surd. A *reduced* surd is the successor of a semi-reduced one. Ayyangar proved that a reduced surd is special ([2, p. 28]) and that a quadratic surd has a purely periodic NSCF expansion if and only if it is reduced ([2, p. 101-102]). We remark that a more explicit variant of the definition of NSCF–reduced surd has been given by the authors in [10].

Lemma 6.1. *If $\xi_0 = \frac{P_0+\sqrt{D}}{Q_0}$ is an RCF–reduced quadratic surd with period–length $l$ and positive–negative representations*

$$\xi_j = a_j + \frac{Q_{j+1}}{P_{j+1}+\sqrt{D}} = a_j + 1 - \frac{Q''_{j+1}}{P''_{j+1}+\sqrt{D}}, 0 \le j \le l-1,$$

*the numbers $\frac{P_{j+1}+\sqrt{D}}{Q_{j+1}}, \frac{P''_{j'+1}+\sqrt{D}}{Q''_{j'+1}}, 0 \le j, j' \le l-1$, are distinct.*

Proof. For suppose $\frac{Q_{i+1}}{P_{i+1}+\sqrt{D}} = \frac{Q''_{j+1}}{P''_{j+1}+\sqrt{D}}$, $0 \le i, j \le l-1$. Then

$$\frac{Q_{j+1}}{P_{j+1}+\sqrt{D}} + \frac{Q''_{j+1}}{P''_{j+1}+\sqrt{D}} = 1, \frac{Q_{j+1}}{P_{j+1}+\sqrt{D}} + \frac{Q_{i+1}}{P_{i+1}+\sqrt{D}} = 1$$

and hence $1/\xi_{j+1} + 1/\xi_{i+1} = 1$. Taking conjugates gives $1/\overline{\xi}_{j+1} + 1/\overline{\xi}_{i+1} = 1$ and this contradicts that fact that $\overline{\xi}_{j+1}$ and $\overline{\xi}_{j+1}$ are negative, being reduced surds. ∎

Lemma 6.2. *Suppose $\xi$ is a NSCF–reduced quadratic surd. Then $\xi$ or $\xi - 1$ is an RCF–reduced quadratic surd.*

Proof. By [2, Corollary 1, p. 30], we have $\xi > \frac{1+\sqrt{5}}{2}$ and $-1 < \overline{\xi} < 1$. So if $-1 < \overline{\xi} < 0$, $\xi$ is RCF–reduced, as $\xi > 1$. If $0 < \overline{\xi}$, let $\xi = \frac{P+\sqrt{D}}{Q}$. Then $\overline{\xi} < 1 < \xi$ implies $0 < Q$. Also as $\xi$ is special, we have $Q < \sqrt{D}$. Then

$$2 < \frac{2\sqrt{D}}{Q} = \xi - \overline{\xi} < \xi.$$

Hence $1 < \xi - 1$ and $-1 < \overline{\xi - 1} < 0$, so $\xi - 1$ is RCF–reduced. ∎

Lemma 6.3. *Suppose $\xi_0$ is NSCF–reduced with NSCF and RCF period–lengths $k$ and $l$, respectively, where $\eta_0 = \xi_0$ or $\xi_0 - 1$ is RCF–reduced. Also assume $\xi_0$ is not equivalent to $\frac{1+\sqrt{5}}{2}$. Then under the RCF to NSCF transformation of $\eta_0$, we have $f(k) = l$.*

Proof. Let $\eta_0 = [\overline{b_0, \ldots, b_{l-1}}]$. Then $\tilde{\eta}_j = \tilde{\xi}_j$ for $j \geq 1$. As $\eta_0 \neq \frac{1+\sqrt{5}}{2}$, we have $b_K > 1$, for some least $K \geq 0$. Then by Lemma 2.5, there exist $m$ and $n$ such that under the RCF to NSCF transformation performed on $\eta_0$, $f(m) = K, f(n) = K + l$. Hence $\eta_{f(m)} = \eta_{f(n)}$ and $\tilde{\eta}_{m+1} = \tilde{\eta}_{n+1}$. Also by Lemma 6.1, $\tilde{\eta}_{m+1}, \ldots, \tilde{\eta}_n$ are distinct. Hence $n - m = k$, the period–length of the NSCF expansion of $\xi_0$. Also $\tilde{\eta}_k = \tilde{\xi}_k = \xi_0$, i.e., $\eta_{f(k)}$ or $\eta_{f(k)} + 1$ is equal to $\eta_0$ or $\eta_0 + 1$. Hence $\eta_{f(k)} = \eta_0$ and $f(k) = tl, t \geq 1$. However

$$f(k) \leq f(m + k) = f(n) = K + l \leq 2l - 1,$$

so $t = 1$. ∎

Example 6.4. (a) $\xi_0 = \frac{235+\sqrt{31683}}{158}$ is NSCF–reduced and $\xi_0 - 1 = \eta_0 = [\overline{1, 1, 1, 1, 1, 2, 3, 1, 1, 1}]$. Then $l = 10, k = 6, f(6) = 10$ and

$$\eta_0 = [1, 1, 1, 1, 1, 2, 3, 1, 1, 1, 1, \ldots].$$

(b) $\xi_0 = \frac{81+\sqrt{31683}}{159}$ is NSCF–reduced and $\xi_0 = \eta_0 = [\overline{1, 1, 1, 1, 2, 3, 1, 1, 1, 1}]$. Then $l = 10, k = 6, f(6) = 10$ and

$$\eta_0 = [1, 1, 1, 1, 2, 3, 1, 1, 1, 1, 1, \ldots].$$

Theorem 6.5. *Let L-RCF and L-NSCF be the period–lengths of the RCF and NSCF expansions of $\xi_0$. Also let N-NSCF be the number of partial numerators $\epsilon_i = -1$ in a NSCF period of $\xi_0$. Then if $\xi_0$ is not equivalent to $(1 + \sqrt{5})/2$,*

$$\textit{L-NSCF} + \textit{N-NSCF} = \textit{L-RCF}. \tag{6.1}$$

Proof. Let $\tilde{\xi}_i$ be the first NSCF–reduced complete quotient of $\xi_0$. Then by Lemma 6.3, $\tilde{\xi}_i = \eta_0$ or $\tilde{\xi}_i - 1 = \eta_0$, where $\eta_0 = [\overline{b_0, \ldots, b_{l-1}}]$ is an RCF–reduced surd with period $l$. Because $\eta_0$ is equivalent to $\xi_0$, by Satz 2.24 of [12], $b_0, \ldots, b_{l-1}$ is also a period of the RCF expansion of $\xi_0$, so $l = \textit{L-RCF}$. If $\xi_0$ is not equivalent to $\frac{1+\sqrt{5}}{2}$, then neither is $\tilde{\xi}_i$ and so by Lemma 6.3, with $\tilde{\xi}_i$ instead of $\xi_0$ and $k = \textit{L-NSCF}$, under the RCF to NSCF transformation of $\eta_0$, we have $f(k) = l$. Also $k = r + s$ and $l = 2r + s$, where $r = \textit{N-NSCF}$ and $s$ are the number of jumps of 2 and 1 respectively, which are made in reaching $\eta_l$. Hence $l = k + r$ and (6.1) holds. ∎

Example 6.6. $\xi_0 = \frac{16137-\sqrt{31683}}{25323} = [0, 1, 1, 1, 2, 2, \overline{1, 1, 1, 1, 1, 2, 3, 1, 1, 1}]$. Then $\tilde{\xi}_4 = \frac{235+\sqrt{31683}}{158}$ is the first reduced NSCF complete quotient and $\eta_0 = \tilde{\xi}_4 - 1 = [\overline{1, 1, 1, 1, 1, 2, 3, 1, 1, 1}]$. The RCF to NSCF transformation, when applied to this period, gives $[1, 1, 1, 1, 1, 2, 3, 1, 1, 1]$, producing the period of NSCF complete quotients $\tilde{\xi}_5, \ldots, \tilde{\xi}_{10} = \tilde{\xi}_4$. Here $r = 4, s = 2, k = 6, l = 10$.

## 7. Equality of N-NICF and N-NSCF

In this section we prove equality of period–lengths *L-NSCF* and *L-NICF*. If $\xi_0$ is equivalent to $(1+\sqrt{5})/2$, $\xi_0$ eventually has the same NSCF and NICF expansion $3-\frac{1|}{|3}-\cdots$. Hence *L-NSCF* $=1=$ *L-NICF* $=1$. Henceforth we can assume that $\xi_0$ is not equivalent to $(1+\sqrt{5})/2$.

In order to prove that *L-NICF* $=$ *L-NSCF*, it suffices by (6.1) and (1.4) to prove *N-NSCF* $=$ *N-NICF*. By virtue of the proof of Theorem 6.5, we can assume $\xi_0 = [\overline{a_0,\ldots,a_{l-1}}]$, where $\xi_0$ or $\xi_0+1$ is NSCF–reduced. It is convenient to determine *N-NSCF* by considering an RCF period $a_N,\ldots,a_{N+l-1}$, where $a_N>1$. We require additionally that $B_NB_{N+1}>Q_0$, for then if $n\geq N$ and $Q_{n+1}\neq Q_n$, by Lemma 3.1, we have the equivalence $Q_{n+2}<Q_{n+1} \iff \Theta_{n+1}<\Theta_n$.

We note also that in the RCF to NSCF transformation, a jump $\xi_i\to\xi_{i+1}$, where $a_i>1, a_{i+1}>1$, produces a partial numerator 1. Hence it suffices to count the number of partial numerators $-1$ arising from an $m$–unisequence:

$$a_n, 1, \ldots, 1, a_{n+m+1}, \tag{7.1}$$

where $N\leq n, n+m+1\leq N+l$ and $a_n>1, a_{n+m+1}>1$.

Lemma 7.1. (i) *The RCF to NSCF transformation acts on an $m$–unisequence* (7.1) *to produce one of the following patterns of partial numerators:*

(a) *If $m$ is odd, we get*

$$\epsilon_{j+1}=\cdots=\epsilon_{j+\frac{m+1}{2}}=-1.$$

(b) (i) *If $m=4t$, we get* $\overbrace{-1,\ldots,-1}^{t}, 1, \overbrace{-1,\ldots,-1}^{t}$.

(ii) *If $m=4t+2$ and $Q_{n+2t+1}<Q_{n+2t+2}$,*

$$\overbrace{-1,\ldots,-1}^{t}, 1, \overbrace{-1,\ldots,-1}^{t+1},$$

*while if $Q_{n+2t+1}\geq Q_{n+2t+2}$, we get*

$$\overbrace{-1,\ldots,-1}^{t+1}, 1, \overbrace{-1,\ldots,-1}^{t}.$$

(ii) *If $N_m$ is the number of $m$–unisequences occurring in a least period of the RCF expansion of $\xi_0$, then*

$$N\text{-}NSCF=\sum_{m\geq 1}\left\lfloor\frac{m+1}{2}\right\rfloor N_m. \tag{7.2}$$

Proof. (i) Consider the RCF to NSCF transformation and assume that $Q_v\neq Q_{v+1}$ for $n+1\leq v<n+m$.

(a) If $m$ is odd, we know from Lemma 4.1, inequalities (4.1), that

$$\Theta_n > \Theta_{n+1}, \Theta_{n+2} > \Theta_{n+3}, \ldots, \Theta_{n+m-1} > \Theta_{n+m}$$

and hence

$$Q_{n+1} > Q_{n+2}, Q_{n+3} > Q_{n+4}, \ldots, Q_{n+m} > Q_{n+m+1}.$$

So by Lemma 2.2,

$$Q_{n+1} > Q''_{n+1}, Q_{n+3} > Q''_{n+3}, \ldots, Q_{n+m} > Q''_{n+m}$$

and we get $\epsilon_{j+1} = \cdots = \epsilon_{j+\frac{m+1}{2}} = -1$.

(b) Now assume $m$ is even, $m = 2k$. Then we know from Lemma 4.1, inequalities (4.2) and (4.3), that

$$\Theta_{n+e} > \Theta_{n+e+1} \text{ if } 0 \leq e \leq k-2, e \text{ even},$$
$$\Theta_{n+f} > \Theta_{n+f+1} \text{ if } k \leq f \leq 2k-1, f \text{ odd}.$$

Hence

$$Q_{n+e+1} > Q_{n+e+2} \text{ if } 0 \leq e \leq k-2, e \text{ even}, \tag{7.3}$$
$$Q_{n+f+1} > Q_{n+f+2} \text{ if } k \leq f \leq 2k-1, f \text{ odd}. \tag{7.4}$$

Case (i): Now assume $k$ is even, $k = 2t$. Then (7.3), (7.4) give

$$Q_{n+e+1} > Q''_{n+e+1} \text{ if } 0 \leq e \leq 2t-2, e \text{ even}, \tag{7.5}$$
$$Q_{n+f+1} > Q''_{n+f+1} \text{ if } 2t+1 \leq f \leq 4t-1, f \text{ odd}. \tag{7.6}$$

Also Lemma 4.1, inequality (4.4) gives $\Theta_{n+2t} < \Theta_{n+2t+1}$, so

$$Q_{n+2t+1} < Q''_{n+2t+1}. \tag{7.7}$$

Then inequalities (7.5) and (7.6) give $\epsilon_{j+1} = \cdots = \epsilon_{j+t} = -1$ and $\epsilon_{j+t+2} = \cdots = \epsilon_{j+2t+1} = -1$, while (7.7) gives $\epsilon_{j+t+1} = 1$.

Case (ii): Assume $k$ is odd, $k = 2t+1$. Then (7.3) and (7.4) give

$$Q_{n+e+1} > Q''_{n+e+1} \text{ if } 0 \leq e \leq 2t-2, e \text{ even}, \tag{7.8}$$
$$Q_{n+f+1} > Q''_{n+f+1} \text{ if } 2t+1 \leq f \leq 4t+1, f \text{ odd}. \tag{7.9}$$

($\alpha$) Assume $\Theta_{n+2t} < \Theta_{n+2t+1}$. Then

$$Q_{n+2t+1} < Q''_{n+2t+1}. \tag{7.10}$$

Then inequalities (7.8) and (7.9) give $\epsilon_{j+1} = \cdots = \epsilon_{j+t} = -1$ and $\epsilon_{j+t+2} = \cdots = \epsilon_{j+2t+2} = -1$, while (7.10) gives $\epsilon_{j+t+1} = 1$.

($\beta$) Assume $\Theta_{n+2t} > \Theta_{n+2t+1}$. Then

$$Q_{n+2t+1} > Q''_{n+2t+1}. \tag{7.11}$$

Then (7.8) and (7.11) give $\epsilon_{j+1} = \cdots = \epsilon_{j+t+1} = -1$. Also from (4.5), if $k \geq 3$, i.e., $t \geq 1$,

$$\Theta_{n+2t+2} < \Theta_{n+2t+3}, \text{ so } Q_{n+2t+3} < Q''_{n+2t+3}. \tag{7.12}$$

Then (7.12) implies $\epsilon_{j+t+2} = 1$ and (7.9) implies $\epsilon_{j+t+3} = \cdots = \epsilon_{j+2t+2} = -1$. If $k = 1$, i.e., $t = 0$, as $\epsilon_{j+1} = -1$, we must have $\epsilon_{j+2} = 1$, as a jump of 1 takes us from $a_{n+m}$ to $a_{n+m+1}$.

Finally, we assume $Q_v = Q_{v+1}$, where $n+1 \leq v < n+m$. Then by Lemma 5.1, $m = 2k$ and $v = n+k$.

Case (1): If $k$ is even, $k = 2t$, we have $Q_{n+2t} = Q''_{n+2t}$ and there is no change to the corresponding earlier argument.

Case (2): If $k$ is odd, $k = 2t+1$, we have $Q_{n+2t+1} = Q''_{n+2t+1}$ and the corresponding earlier argument where $\Theta_{n+2t} > \Theta_{n+2t+1}$, goes over with (7.11) replaced by $Q_{n+2t+1} = Q''_{n+2t+1}$.

(ii) Part (i) tells us that the RCF to NSCF transformation produces

$$\lfloor (m+1)/2 \rfloor = \begin{cases} t+1 & \text{if } m = 2t+1, \\ 2t & \text{if } m = 4t, \\ 2t+1 & \text{if } m = 4t+2, \end{cases} \tag{7.13}$$

partial numerators $\epsilon_i = -1$ and this gives (7.2).

∎

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