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DIOPHANTINE m-TUPLES FOR QUADRATIC POLYNOMIALS

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Abstract. In this paper, we prove that there does not exist a set with more than 98 nonzero polynomials in $\mathbb{Z}[X]$, such that the product of any two of them plus a quadratic polynomial n is a square of a polynomial from $\mathbb{Z}[X]$ (we exclude the possibility that all elements of such set are constant multiples of a linear polynomial $p \in \mathbb{Z}[X]$ such that $p^2|n$). Specially, we prove that if such a set contains only polynomials of odd degree, then it has at most 18 elements.

1. Introduction

Diophantus of Alexandria ([2]) first studied the problem of finding sets with the property that the product of any two of its distinct elements increased by one is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m-tuple. Diophantus found the first Diophantine quadruple of rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$, while the first Diophantine quadruple of integers $\{1, 3, 8, 120\}$ was found by Fermat. Many generalizations of this problem were considered since then, for example by adding a fixed integer n instead of 1, looking at kth powers instead of squares, or considering the problem over other domains than $\mathbb{Z}$ or $\mathbb{Q}$.

Definition 1.1. *Let n be a nonzero integer. A set of m different positive integers $\{a_1, a_2, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple if the product $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$.*

Diophantus ([2]) found the first such quadruple $\{1, 33, 68, 105\}$ with the property $D(256)$. The first $D(1)$-quadruple is the above mentioned Fermat's

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set. The folklore conjecture is that there does not exist a $D(1)$-quintuple. Baker and Davenport ([1]) proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella ([6]) proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. But, for example, the set $\{1, 33, 105, 320, 18240\}$ has the property $D(256)$ ([3]), and the set $\{99, 315, 9920, 32768, 44460, 19534284\}$ has the property $D(2985984)$ ([12]). The natural question is to find upper bounds for the numbers M_n defined by

$$M_n = \sup\{|S| : S \text{ has the property } D(n)\}$$

where $|S|$ denotes the number of elements in the set S. Dujella ([4,5]) proved that $M_n \leq 31$ for $|n| \leq 400$, and $M_n < 15.476 \log |n|$ for $|n| > 400$.

The first polynomial variant of the above problem was studied by Jones ([13,14]) and it was for the case $n = 1$.

Definition 1.2. *Let $n \in \mathbb{Z}[X]$ and let $\{a_1, a_2, \ldots, a_m\}$ be a set of m nonzero polynomials with integer coefficients. We assume that there does not exist a polynomial $p \in \mathbb{Z}[X]$ such that $\frac{a_1}{p}, \ldots, \frac{a_m}{p}$ and $\frac{n}{p^2}$ are integers. The set $\{a_1, a_2, \ldots, a_m\}$ is called a polynomial $D(n)$-m-tuple if for all $1 \leq i < j \leq m$ the following holds: $a_i \cdot a_j + n = b_{ij}^2$ where $b_{ij} \in \mathbb{Z}[X]$.*

We mention that for $n \in \mathbb{Z}$ the assumption concerning the polynomial p means that not all elements of $\{a_1, a_2, \ldots, a_m\}$ are allowed to be constant.

In analogy to the above results we are interested in the size of

$$P_n = \sup\{|S| : S \text{ is a polynomial } D(n)\text{-tuple}\}.$$

Dujella and Fuchs ([7]) proved that $P_{-1} = 3$ and their result from [8] implies that $P_1 = 4$. Moreover, from [11, Theorem 1] it follows that $P_n \leq 7$ for all $n \in \mathbb{Z}\backslash\{0\}$. It is an improvement of the previous bound $P_n \leq 22$, which follows from [4, Theorem 1].

Dujella and Fuchs, jointly with Tichy ([9]) and later with Walsh ([10]), considered the case $n = \mu_1 X + \mu_0$ with integers $\mu_1 \neq 0$ and μ_0. They defined

$$L = \sup\{|S| : S \text{ is a polynomial } D(\mu_1 X + \mu_0)\text{-tuple for some } \mu_1 \neq 0, \mu_0 \in \mathbb{Z}\},$$

and they denoted by L_k the number of polynomials of degree k in a polynomial $D(\mu_1 X + \mu_0)$-tuple S. The results from [10] are sharp bounds $L_0 \leq 1$, $L_1 \leq 4$, $L_k \leq 3$ for all $k \geq 2$, and finally

$$L \leq 12.$$

In this paper, we handle the case where n is a quadratic polynomial in $\mathbb{Z}[X]$, which is more complicated than the case with linear n, mostly because quadratic polynomials need not be irreducible. Let us define

$$Q = \sup\{|S| : S \text{ is a polynomial } D(\mu_2 X^2 + \mu_1 X + \mu_0)\text{-tuple for some } \mu_2 \neq 0, \mu_1, \mu_0 \in \mathbb{Z}\}.$$

Let us also denote by Q_k the number of polynomials of degree k in a polynomial $D(\mu_2 X^2 + \mu_1 X + \mu_0)$-tuple S. The main goal of this paper is to prove the following theorem:

THEOREM 1.3. *There are at most* 98 *elements in a polynomial* $D(n)$*-tuple for a quadratic polynomial* n*, i.e.,*

$$Q \leq 98.$$

In the proof of Theorem 1.3, we also prove the following statement.

COROLLARY 1.4. *If a polynomial* $D(n)$*-*m*-tuple for a quadratic* n *contains only polynomials of odd degree, then* $m \leq 18$.

In order to prove Theorem 1.3, we follow the strategy used in [9] and [10] for linear n. First, we estimate the numbers Q_k of polynomials of degree k.

PROPOSITION 1.5.
1) $Q_0 \leq 2$.
2) $Q_1 \leq 4$.

Proposition 1.5 completely solves the problem for constant and linear polynomials because, for example, the set $\{3, 5\}$ is a polynomial $D(9X^2 + 24X + 1)$-pair, and the set

$$\{2X, 10X + 20, 4X + 14, 2X + 8\} \tag{1.1}$$

is a polynomial $D(-4X^2 - 16X + 9)$-quadruple. By further analysis, we get:

PROPOSITION 1.6.
1) $Q_2 \leq 81$.
2) $Q_3 \leq 5$.
3) $Q_4 \leq 6$.
4) $Q_k \leq 3$ *for* $k \geq 5$.

Let us mention that it is not obvious that the number Q_2 is bounded, so the result from Proposition 1.6 1) is nontrivial. Quadratic polynomials have the major contribution to the bound from Theorem 1.3. The bound from Proposition 1.6 4) is sharp. For example, the set

$$\{X^{2l-1} + X, X^{2l-1} + 2X^l + 2X, 4X^{2l-1} + 4X^l + 5X\}$$

is a polynomial $D(-X^2)$-triple for any integer $l \geq 2$, and the set

$$\{X^{2l} + X^l, X^{2l} + X^l + 4X, 4X^{2l} + 4X^l + 8X\}$$

is a polynomial $D(4X^2)$-triple for any integer $l \geq 1$.

In Section 2, we consider the cases of equal degrees separately and give proofs of Propositions 1.5 and 1.6. In Section 3, we adapt the gap principle for the degrees of the elements of S, proved in [9] for linear n, to quadratic n. Using the bounds from Section 2 and by combining the gap principle with an upper bound for the degree of the largest element in a polynomial $D(n)$-quadruple, obtained in [10], in Section 4 we give the proof of Theorem 1.3.

2. Sets with polynomials of equal degree

The first step which leads us to the proof of Theorem 1.3 is to estimate the numbers Q_k for $k \geq 0$.

2.1. *Constant and linear polynomials.* Here we give the proofs of the sharp bounds from Proposition 1.5.

Proof of Proposition 1.5 1). Suppose that, for given $\pi \in \mathbb{Z} \setminus \{0\}$, there exist two different nonzero integers ν_1 and ν_2 such that

$$\pi\nu_i + \mu_2 X^2 + \mu_1 X + \mu_0 = r_i^2 \tag{2.1}$$

where $r_i \in \mathbb{Z}[X]$ for $i = 1, 2$. From this, it follows that $r_i = \varrho_i X + \kappa_i$ where $\varrho_i \neq 0, \kappa_i \in \mathbb{Z}$ for $i = 1, 2$. Comparing the coefficients in (2.1), we get $\varrho_i^2 = \mu_2$, $2\varrho_i\kappa_i = \mu_1$, $\pi\nu_i = \kappa_i^2 - \mu_0$ for $i = 1, 2$, so $\varrho_1 = \pm\varrho_2$, $\kappa_1 = \pm\kappa_2$. From that, we obtain $\nu_1 = \nu_2$, a contradiction. □

Proof of Proposition 1.5 2). Let $\{\alpha X + \beta, \gamma X + \delta, \varepsilon X + \varphi\}$ be a polynomial $D(\mu_2 X^2 + \mu_1 X + \mu_0)$-triple. First, we show that we may assume that one of the polynomials in the triple is a multiple of X. Observe that $\{\alpha^2 X + \alpha\beta, \alpha\gamma X + \alpha\delta, \alpha\varepsilon X + \alpha\varphi\}$ is a polynomial $D(\alpha^2\mu_2 X^2 + \alpha^2\mu_1 X + \alpha^2\mu_0)$-triple. By substituting $\alpha X = Y$, we obtain a polynomial $D(\mu_2 Y^2 + \alpha\mu_1 Y + \alpha^2\mu_0)$-triple $\{\alpha Y + \alpha\beta, \gamma Y + \alpha\delta, \varepsilon Y + \alpha\varphi\}$. Finally, by substituting $Y + \beta = Z$, we get a polynomial $D(\mu_2 Z^2 + (\alpha\mu_1 - 2\mu_2\beta)Z + \gamma')$-triple

$$\{\alpha Z, \gamma Z + \delta', \varepsilon Z + \varphi'\}$$

where $\delta' = \alpha\delta - \gamma\beta$, $\varphi' = \alpha\varphi - \varepsilon\beta$, $\gamma' = \alpha^2\mu_0 - \alpha\beta\mu_1 + \beta^2\mu_2$. This implies that

$$\alpha\gamma + \mu_2 = A^2, \ \alpha\varepsilon + \mu_2 = B^2, \ \gamma\varepsilon + \mu_2 = C^2$$

with integers $A, B, C \geq 0$. By specializing $Z = 0$, we see that $\gamma' = D^2$ with $D \in \mathbb{Z}$. Now, by comparing the coefficients in

$$\alpha Z(\gamma Z + \delta') + \mu_2 Z^2 + (\alpha\mu_1 - 2\mu_2\beta)Z + D^2 = (AZ \pm D)^2,$$

it follows that $\delta' = \dfrac{\pm 2AD - \alpha\mu_1 + 2\mu_2\beta}{\alpha}$. Analogously,

$$\varphi' = \frac{\pm 2BD - \alpha\mu_1 + 2\mu_2\beta}{\alpha}.$$

If we denote $\mu_1' := 2\mu_2\beta - \alpha\mu_1$, we obtain the set

$$\Big\{\alpha Z, \gamma Z + \frac{\pm 2AD + \mu_1'}{\alpha}, \varepsilon Z + \frac{\pm 2BD + \mu_1'}{\alpha}\Big\}, \tag{2.2}$$

which is a polynomial $D(\mu_2 Z^2 - \mu_1' Z + D^2)$-triple. It means that

$$\Big(\gamma Z + \frac{\pm 2AD + \mu_1'}{\alpha}\Big)\Big(\varepsilon Z + \frac{\pm 2BD + \mu_1'}{\alpha}\Big) + \mu_2 Z^2 - \mu_1' Z + D^2 \tag{2.3}$$

is a square of a linear polynomial or a square of an integer. Observe that $\gamma Z + \frac{\pm 2AD+\mu_1'}{\alpha} = \frac{A^2Z - \mu_2 Z \pm 2AD + \mu_1'}{\alpha}$ and $\varepsilon Z + \frac{\pm 2BD+\mu_1'}{\alpha} = \frac{B^2Z-\mu_2Z\pm 2BD+\mu_1'}{\alpha}$. Assume first that (2.3) is a square of an integer P. Then, by comparing the coefficients in (2.3), we obtain a system of three equations with unknowns α, B and P. For each combination of the signs $\pm$ in (2.3) we get only two possibilities $B_{1,2}$ for B, so the set (2.2) can be extended at most to a polynomial $D(n)$-quadruple

$$\Big\{\alpha Z, \frac{A^2Z - \mu_2 Z \pm 2AD + \mu_1'}{\alpha}, \frac{B_1^2 Z - \mu_2 Z \pm 2B_1 D + \mu_1'}{\alpha}, \frac{B_2^2 Z - \mu_2 Z \pm 2B_2 D + \mu_1'}{\alpha}\Big\}.$$

Assume now that (2.3) is a square of a linear polynomial. Then the discriminant of this quadratic polynomial is equal to 0. If both signs $\pm$ in (2.3) are equal, we obtain a discriminant which can be factored into three factors

$$\begin{aligned}&(A-B+\alpha)(A-B-\alpha)(4A^2B^2D^2 + 4\alpha^2 D^2\mu_2 + 8ABD^2\mu_2\\ &+ 4D^2\mu_2^2 \pm 4A^2BD\mu_1' \pm 4AB^2D\mu_1' \pm 4AD\mu_2\mu_1' \pm 4BD\mu_2\mu_1' - \alpha^2\mu_1'^2\\ &+ A^2\mu_1'^2 + 2AB\mu_1'^2 + B^2\mu_1'^2) = 0.\end{aligned} \tag{2.4}$$

By solving this equation in B, we obtain four possibilities

$$\begin{aligned}&B_1 = A - \alpha, \qquad B_2 = A + \alpha,\\ &B_{3,4} = \frac{-\mu_1' A - 2\mu_2 D \pm \sqrt{-4D^2\alpha^2\mu_2 + \alpha^2\mu_1'^2}}{2AD + \mu_1'}.\end{aligned} \tag{2.5}$$

Analogously, if the signs in (2.3) are different: From

$$\begin{aligned}&(A+B-\alpha)(A+B+\alpha)(4A^2B^2D^2 + 4\alpha^2D^2\mu_2 - 8ABD^2\mu_2\\ &+4D^2\mu_2^2 \pm 4A^2BD\mu_1' \mp 4AB^2D\mu_1' \mp 4AD\mu_2\mu_1' \pm 4BD\mu_2\mu_1' - \alpha^2\mu_1'^2\\ &+A^2\mu_1'^2 - 2AB\mu_1'^2 + B^2\mu_1'^2) = 0,\end{aligned}$$

we get

$$\begin{aligned}&B_1 = -A + \alpha, \qquad B_2 = -A - \alpha,\\ &B_{3,4} = \frac{\mu_1' A + 2\mu_2 D \pm \sqrt{-4D^2\alpha^2\mu_2 + \alpha^2\mu_1'^2}}{2AD + \mu_1'}.\end{aligned} \tag{2.6}$$

We now conclude that the set (2.2) can be extended at most to a polynomial $D(n)$-sextuple. Observe first that the term $-4D^2\mu_2 + \mu_1'^2$ in (2.5) and (2.6) is a discriminant of a polynomial n. If that term is not a square of an integer, we already have at most a $D(n)$-quadruple. Suppose now that $\gamma \le \varepsilon$. If $\alpha > 0$, we obtain $A^2 \leq B^2$, so $A \leq B$. For this case, in (2.5) we have $B_1 < A$, a contradiction. Also, in (2.6) we have a contradiction $B_2 < 0$. If $\alpha < 0$, from $\gamma \leq \varepsilon$ we get $A^2 \geq B^2$, so $A \geq B$. In (2.5) we have $B_1 > A$, a contradiction. Also, in (2.6) we obtain a contradiction $B_1 < 0$. Analogously we conclude for $\gamma \geq \varepsilon$. Therefore, neither in (2.5) nor (2.6) we can have the possibilities B_1 and B_2 for the same $D(n)$-tuple.

Let us take $B = B_1$ from (2.5) or (2.6) and consider the polynomial $D(\mu_2 Z^2 - \mu_1' Z + D^2)$-triple (2.2). The analogous situation is for $B = B_2$. Hence, we have the set

$$\left\{\alpha Z, \gamma Z + \frac{\pm 2AD + \mu_1'}{\alpha}, \varepsilon Z + \frac{\pm 2(A-\alpha)D + \mu_1'}{\alpha}\right\} \tag{2.7}$$

where both signs $\pm$ are the same. It is sufficient to look only at the case with positive signs $\pm$ in (2.7) because the signs depend on the sign of the integer D.[1]

Let us now extend the set (2.7) with the element $\zeta Z + \frac{\pm 2B_3 D + \mu_1'}{\alpha}$, obtained by the above construction, where $B_3 = \frac{-\mu_1' A - 2\mu_2 D + \sqrt{-4D^2\alpha^2\mu_2 + \alpha^2\mu_1'^2}}{2AD + \mu_1'}$. Observe that for B_3 the sign $\pm$ is the same as the other signs $\pm$ in (2.7) and depends only on the sign of D, so we may assume that this sign is $+$. Inserting $A - \alpha$ and B_3 into (2.4), instead of A and B, we obtain five solutions for the unknown α.

1) $\alpha = 0$, a contradiction.
2) $\alpha = 2\frac{DA^2 + \mu_1' A + \mu_2 D}{\sqrt{-4D^2\mu_2 + \mu_1'^2}}$, for which $B_3 = A$. From $\alpha\gamma + \mu_2 = A^2$ and $\alpha\zeta + \mu_2 = B_3^2$ we get $\gamma = \zeta$, so we have two equal elements in a quadruple, again a contradiction.
3) $\alpha = 2\frac{DA^2 + \mu_1' A + \mu_2 D}{4DA + 2\mu_1' + \sqrt{-4D^2\mu_2 + \mu_1'^2}}$, for which we have $B_3 = A - 2\alpha$.[2]
4) $\alpha = \frac{2AD + \mu_1' - \frac{1}{2}\sqrt{-4D^2\mu_2 + \mu_1'^2}}{D}$, for which $B_3 = -\frac{1}{2}\frac{\mu_1' - 2\sqrt{-4D^2\mu_2 + \mu_1'^2}}{D}$. This is also a possible case.[3]
5) $\alpha = -\frac{1}{2}\frac{\sqrt{-4D^2\mu_2 + \mu_1'^2}}{D}$ and then $B_3 = -\frac{1}{2}\frac{\mu_1'}{D}$. From $\alpha\zeta + \mu_2 = B_3^2$, we obtain $\zeta = \alpha$. Hence, we have a quadruple with two equal elements αZ and ζZ, a contradiction.

We conclude that the set

$$\left\{\alpha Z, \gamma Z + \frac{\pm 2AD + \mu_1'}{\alpha}, \varepsilon Z + \frac{\pm 2(A-\alpha)D + \mu_1'}{\alpha}, \zeta Z + \frac{\pm 2B_3 D + \mu_1'}{\alpha}\right\}, \tag{2.8}$$

with equal signs $\pm$, can be a polynomial $D(\mu_2 Z^2 - \mu_1' Z + D^2)$-quadruple.

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[1] For example, let $n := Z^2 + 6Z + 9$. For $D = 3$ and $D = -3$, we get polynomial $D(n)$-triples $\{3Z, 8Z + 8, Z + 2\}$ and $\{3Z, 8Z - 12, Z - 6\}$, respectively.

[2] For example, for $A = 4$, we obtain the set (1.1).

[3] For $A = 1$, we obtain the polynomial $D(-X^2 - 8X + 9)$-quadruple $\{X, 2X + 2, X + 8, 5X + 20\}$.

We are left to check the possibility

$$B = B_4 = \frac{-\mu_1' A - 2\mu_2 D - \sqrt{-4D^2\alpha^2\mu_2 + \alpha^2\mu_1'^2}}{2AD + \mu_1'}.$$

By the above construction, we obtain the element $\eta Z + \frac{\pm 2B_4 D + \mu_1'}{\alpha}$, which we want to adjoin to the set (2.8). This element has the same sign $\pm$ as the others in (2.8), which depends only on the sign of D, so it is enough to look at the case with the positive signs. Inserting B_3 and B_4 into (2.4), instead of A and B, we obtain five solutions for the unknown α.

1) $\alpha = 0$, a contradiction.
2) $\alpha = \frac{\sqrt{-4D^2\mu_2 + \mu_1'^2}}{2D}$, so $B_4 = -\frac{\mu_1'}{2D}$. From $\alpha\eta + \mu_2 = B_4^2$, it follows that $\eta = \alpha$. Hence, we have two equal elements αZ and ηZ in a quadruple, a contradiction.
3) $\alpha = -\frac{\sqrt{-4D^2\mu_2 + \mu_1'^2}}{2D}$, for which we get a contradiction as in the previous case.
4) $\alpha = 2\frac{\sqrt{-4D^2\mu_2 + \mu_1'^2}(\mu_1' A + A^2 D + \mu_2 D)}{4D^2\mu_2 - \mu_1'^2}$, from which $B_4 = A$. We get $\eta = \gamma$, a contradiction.
5) $\alpha = -2\frac{\sqrt{-4D^2\mu_2 + \mu_1'^2}(\mu_1' A + A^2 D + \mu_2 D)}{4D^2\mu_2 - \mu_1'^2}$, which is a contradiction as in the case 4).

Therefore, we cannot adjoin the fifth element to the set (2.8) and $Q_1 \leq 4$. ◻

2.2. *Polynomials of degree* $k \geq 2$. Let $\mathbb{Z}^+[X]$ denote the set of all polynomials with integer coefficients with positive leading coefficient. For $a, b \in \mathbb{Z}[X]$, $a < b$ means that $b - a \in \mathbb{Z}^+[X]$.

Let $\{a, b, c\}$ be a polynomial $D(n)$-triple, containing only polynomials of degree k for some $k \geq 2$ and with quadratic $n \in \mathbb{Z}[X]$. Let

$$ab + n = r^2, \quad ac + n = s^2, \quad bc + n = t^2 \tag{2.9}$$

where $r, s, t \in \mathbb{Z}^+[X]$. Assume that $a < b < c$ and denote by α, β, γ the leading coefficients of the polynomials a, b, c, respectively. Observe that α, β, γ must have the same sign, so there is no loss of generality in assuming that $a, b, c \in \mathbb{Z}^+[X]$. We may also assume that $\gcd(\alpha, \beta, \gamma) = 1$ since otherwise we substitute $Y = \gcd(\alpha, \beta, \gamma)X$. This implies that α, β and γ are perfect squares, say

$$\alpha = A^2, \quad \beta = B^2, \quad \gamma = C^2$$

where $A, B, C \in \mathbb{N}$.

The following lemma, which is [9, Lemma 1], will play the key role in our proofs. It is a very useful construction with the elements of a polynomial $D(n)$-triple where n is a polynomial with integer coefficients.

Lemma 2.1. *Let* $\{a,b,c\}$ *be a polynomial* $D(n)$*-triple for which (2.9) holds. Then there exist polynomials* $e,u,v,w\in\mathbb{Z}[X]$ *such that*

$$ae+n^2=u^2,\ be+n^2=v^2,\ ce+n^2=w^2.$$

More precisely,

$$e=n(a+b+c)+2abc-2rst. \tag{2.10}$$

Furthermore, it holds $c=a+b+\frac{e}{n}+\frac{2}{n^2}(abe+ruv)$ *where* $u=at-rs$, $v=bs-rt$.

The above construction is a direct modification from the integer case [4, Lemma 3]. The analogous statement for polynomial $D(1)$-triples was proved by Jones ([14]) and it was also used in [7] for the case $n=-1$. We define

$$\overline{e}=n(a+b+c)+2abc+2rst. \tag{2.11}$$

By easy computation, we obtain the relation

$$e\cdot\overline{e}=n^2(c-a-b-2r)(c-a-b+2r), \tag{2.12}$$

which we will use for determining all possible e-s. From (2.11), $\deg(\overline{e})=3k$ and then, from (2.12), we obtain that

$$\deg(e)\leq 4-k. \tag{2.13}$$

Also, from (2.10), using (2.9) and the expressions for u,v,w from Lemma 2.1, we get

$$e=n(a+b-c)+2rw, \tag{2.14}$$

$$e=n(a-b+c)+2sv, \tag{2.15}$$

$$e=n(-a+b+c)+2tu. \tag{2.16}$$

In order to bound the number of elements of degree k in a polynomial $D(n)$-tuple, we are interested to find the number of possible c-s, for fixed a and b, such that (2.9) holds. The first step is finding all possible e-s from Lemma 2.1. In the following lemma, we adapt for quadratic n the important result from [10].

Lemma 2.2. *Let* $\{a,b\}$, $a<b$, *be a polynomial* $D(n)$*-pair with* $ab+n=r^2$. *Let*

$$ae+n^2=u^2,\ be+n^2=v^2$$

where $u,v\in\mathbb{Z}^+[X]$ *and* $e\in\mathbb{Z}[X]$. *Then for each such* e *there exists at most one polynomial* $c>b$ *such that* $\{a,b,c\}$ *is a polynomial* $D(n)$*-triple.*

Proof. Suppose that $\{a,b,c\}$ is a polynomial $D(n)$-triple. Since u and v are fixed up to the sign, from Lemma 2.1 it follows that, for e defined by (2.10) and for fixed a and b, two possible c-s come from

$$c_{\pm}=a+b+\frac{e}{n}+\frac{2}{n^2}(abe\pm ruv). \tag{2.17}$$

From this, we obtain

$$c_+ \cdot c_- = b^2 + a(a-2b) + \frac{e^2}{n^2} - \frac{2ae}{n} - \frac{2be}{n} - 4n.$$

From $a < b < 2b$ and (2.13), it follows that $c_- < b$. Hence, the only possible c is c_+. ∎

2.2.1. *Quadratic polynomials.* Let $\deg(a) = \deg(b) = \deg(c) = 2$. The proof of Proposition 1.6 1) is based on the construction from Lemma 2.1 and the results from the next few lemmas.

LEMMA 2.3. *Let $\{a, b, c\}$ be a polynomial $D(n)$-triple. Then at most one of the polynomials a, b, c is divisible by n.*

PROOF. Let a and b be divisible by n. Suppose first that n is irreducible over $\mathbb{Q}$. Then, from (2.9), it follows that $n|r$. Hence, $n^2|n$, a contradiction.

Assume now that $n = n_1 n_2$ where n_1,n_2 are linear polynomials over $\mathbb{Q}$. Let $n_1 \nmid n_2$. From (2.9), it follows that $n_1^2|r^2$ and $n_2^2|r^2$, so we obtain the contradiction $n^2|n$ again. Assume finally that $n = \lambda n_1^2$ where $\lambda \in \mathbb{Q}\backslash\{0\}$. Now $a = \delta_1 n_1^2$, $b = \delta_2 n_1^2$ where $\delta_1, \delta_2 \in \mathbb{Q}\backslash\{0\}$. Since the leading coefficients of the polynomials a and b are squares of positive integers, we have $\delta_1 = D_1^2$ and $\delta_2 = D_2^2$, $D_1, D_2 \in \mathbb{Q}\backslash\{0\}$, so $D_1^2 D_2^2 n_1^4 + \lambda n_1^2 = r^2$. Hence, $n_1^2|r^2$ and we obtain

$$(D_1 D_2 n_1 + r_1)(D_1 D_2 n_1 - r_1) = -\lambda$$

where r_1 is a linear polynomial over $\mathbb{Q}$ and $r = n_1 r_1$. Both factors on the left side of the previous equation must be constant. If we denote by μ_1 and ϱ_1 the leading coefficients of the polynomials n_1 and r_1, respectively, then we obtain $\mu_1 = \varrho_1 = 0$, a contradiction. The proof is analogous if a and c or b and c are divisible by n. ∎

Let us now find all possible e-s for fixed a and b. By (2.13), we have

$$\deg(e) \leq 2.$$

Moreover, from (2.12), we will find possible common factors of n and e. Obviously, $n^2 \nmid e$ and $n_1 n \nmid e$, if $n = n_1 n_2$ and n_1, n_2 are linear polynomials over $\mathbb{Q}$.

LEMMA 2.4. *Let $e \in \mathbb{Z}[X]$ be defined by (2.10)*[4] *and let $n|e$. Then $n = \lambda n_1^2$ where $\lambda \in \mathbb{Q}\backslash\{0\}$ and n_1 is a linear polynomial over $\mathbb{Q}$. For fixed a and b, there is at most one such e.*

[4]Here and also in the following lemmas, we are looking at extensions of $\{a, b\}$ to a polynomial $D(n)$-triple $\{a, b, c\}$ with $c > b$ and then at the corresponding $e \in \mathbb{Z}[X]$ defined by (2.10).

Proof. Let $e = \tau n$, $\tau \in \mathbb{Q}\backslash\{0\}$. Suppose that n is irreducible over $\mathbb{Q}$. By Lemma 2.1, there exists $u \in \mathbb{Z}[X]$ such that

$$a\tau n + n^2 = u^2.$$

From that, we have $n|u$ and then $n|a$. Analogously, we obtain that $n|b$, which is a contradiction with Lemma 2.3.

Assume now that $n = n_1 n_2$ where n_1, n_2 are linear polynomials over $\mathbb{Q}$. Let $n_1 \nmid n_2$. By Lemma 2.1, there exists $u \in \mathbb{Z}[X]$ such that

$$a\tau n_1 n_2 + n_1^2 n_2^2 = u^2. \tag{2.18}$$

Hence, $n_1^2|u^2$ and $n_2^2|u^2$, so $n|a$. Analogously, we obtain $n|b$, a contradiction. So we have that $n = \lambda n_1^2$, $\lambda \in \mathbb{Q}\backslash\{0\}$. Now $e = \tau n = \tau\lambda n_1^2 = \nu n_1^2$, $\nu \in \mathbb{Q}\backslash\{0\}$ and thus (2.18) takes the form $a\nu n_1^2 + \lambda^2 n_1^4 = u^2$. We conclude that $n_1^2|u^2$, so

$$a\nu + \lambda^2 n_1^2 = u_1^2 \tag{2.19}$$

where $u = n_1 u_1$ and $u_1 \in \mathbb{Q}[X]$, $\deg(u_1) \leq 1$. Assume that, for fixed a and b, two distinct e-s exist.[5] We call them e and f. Let $f = \nu' n_1^2$ with $\nu' \in \mathbb{Q}\backslash\{0\}$, $\nu' \neq \nu$. From (2.19), we see that a is a product of two linear polynomials. Hence,

$$a = A^2(X - \phi_1)(X - \phi_2)$$

with $A \in \mathbb{N}$. Denote $n_1' := \lambda n_1$ and assume that

$$\begin{aligned} u_1 - n_1' &= \varepsilon_1(X - \phi_1), \\ u_1 + n_1' &= \varepsilon_2(X - \phi_2) \end{aligned}$$

where $\varepsilon_1\varepsilon_2 = A^2\nu$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{Q}\backslash\{0\}$. It implies

$$2n_1' = X(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\phi_1 - \varepsilon_2\phi_2.$$

Analogously, we have $a\nu' + (n_1')^2 = u_2^2$ where $af + n^2 = (u')^2$, $u' = n_1 u_2$ and $u_2 \in \mathbb{Q}[X]$, $\deg(u_2) \leq 1$. We conclude that

$$\begin{aligned} u_2 - n_1' &= \varphi_1(X - \phi_1), \\ u_2 + n_1' &= \varphi_2(X - \phi_2), \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} u_2 - n_1' &= \varphi_1(X - \phi_2), \\ u_2 + n_1' &= \varphi_2(X - \phi_1) \end{aligned} \tag{2.21}$$

where $\varphi_1\varphi_2 = A^2\nu'$ and $\varphi_1, \varphi_2 \in \mathbb{Q}\backslash\{0\}$. Let us first consider the case (2.20). We get

$$2n_1' = X(\varphi_2 - \varphi_1) + \varphi_1\phi_1 - \varphi_2\phi_2.$$

Hence, $\varepsilon_2 - \varepsilon_1 = \varphi_2 - \varphi_1$ and $\varepsilon_1\phi_1 - \varepsilon_2\phi_2 = \varphi_1\phi_1 - \varphi_2\phi_2$. Consequently, we have $\phi_1(\varepsilon_1 - \varphi_1) = \phi_2(\varepsilon_2 - \varphi_2) = \phi_2(\varepsilon_1 - \varphi_1)$, from which it follows that

[5] We follow the approach from [9, Proposition 3].

$\phi_1 = \phi_2$ or $\varepsilon_1 = \varphi_1$. If $\varepsilon_1 = \varphi_1$, then $\varepsilon_2 = \varphi_2$ so $\nu = \nu'$, a contradiction. If $\phi_1 = \phi_2$, then from (2.20) it follows that $(X - \phi_1)|n_1'$, so we get

$$(X - \phi_1)^2 | n. \tag{2.22}$$

Therefore, $a|n$. Assume now that (2.21) holds. Then,

$$2n_1' = X(\varphi_2 - \varphi_1) + \varphi_1\phi_2 - \varphi_2\phi_1.$$

Hence, $\varepsilon_2 - \varepsilon_1 = \varphi_2 - \varphi_1$ and $\varepsilon_1\phi_1 - \varepsilon_2\phi_2 = \varphi_1\phi_2 - \varphi_2\phi_1$. This yields $\phi_1(\varepsilon_1 + \varphi_2) = \phi_2(\varepsilon_2 + \varphi_1) = \phi_2(\varepsilon_1 + \varphi_2)$, so $\phi_1 = \phi_2$ or $\varepsilon_1 = -\varphi_2$. For $\varepsilon_1 = -\varphi_2$, it follows that $\varepsilon_2 = -\varphi_1$ and we obtain $\nu = \nu'$, a contradiction. If $\phi_1 = \phi_2$, then analogously as for the previous case, we obtain (2.22) and we conclude that $a|n$. Completely analogously, for b we get a contradiction except when $b|n$. Now we have $a|n$ and $b|n$, a contradiction with Lemma 2.3. Therefore, for fixed a and b, there is at most one e with the above form and it exists only for $n = \lambda n_1^2$ with $\lambda \in \mathbb{Q}\backslash\{0\}$ and n_1 a linear polynomial over $\mathbb{Q}$. ∎

LEMMA 2.5. *Let $e \in \mathbb{Z}[X]$ be defined by (2.10) and let n and e have a common linear factor but $n \nmid e$. Then $n = n_1n_2$ where n_1, n_2 are linear polynomials over $\mathbb{Q}$ such that $n_1 \nmid n_2$. For fixed a and b, there exist at most two such e-s.*

PROOF. Suppose that $e = \tau n_1$, $\tau \in \mathbb{Q}\backslash\{0\}$. By Lemma 2.1, there is $u \in \mathbb{Z}[X]$ such that

$$a\tau n_1 + n_1^2n_2^2 = u^2.$$

We have $n_1^2|u^2$, so $n_1|a$. Analogously, we obtain that $n_1|b$ and $n_1|c$. From (2.9), it follows that $n_1^2|r^2$, so $n = \lambda n_1^2$, $\lambda \in \mathbb{Q}\backslash\{0\}$. Also, from (2.9), we get that $n_1|s$ and $n_1|t$. Since $e \neq 0$, by (2.10), $n_1^3|e$ which is a contradiction.

Assume now that $n = n_1n_2$ and $e = \tau n_1e_1$ where n_1, n_2, e_1 are linear polynomials over $\mathbb{Q}$ and $\tau \in \mathbb{Q}\backslash\{0\}$. By Lemma 2.1, there exists $u \in \mathbb{Z}[X]$ such that

$$a\tau n_1e_1 + n_1^2n_2^2 = u^2. \tag{2.23}$$

Hence, $n_1^2|u^2$. If $n_1 \nmid e_1$, then $n_1|a$ and analogously, by Lemma 2.1, we obtain that $n_1|b$ and $n_1|c$. As for the previous case, we get the contradiction $n_1^3|e$. Hence, $n_1|e_1$, so $e = \nu n_1^2$, $\nu \in \mathbb{Q}\backslash\{0\}$. Observe that if $n_1|n_2$, then $n|e$. Therefore, $n_1 \nmid n_2$. Assume now that, for fixed a and b, there are two such e-s. From (2.23), we see that $u = n_1u_1$ where $u_1 \in \mathbb{Q}[X]$, $\deg(u_1) \leq 1$, so

$$a\nu + n_2^2 = u_1^2.$$

This equation has the same form as (2.19), so the proof follows analogously to the proof of Lemma 2.4. The only difference is that here the proof stops whenever we obtain (2.22). We conclude that there exists at most one e which has the same linear factor n_1 as n has. Analogously, there is at most one e which has with n a common linear factor n_2. Hence, for fixed a and b, there exist at most two e-s of the above form. In this case, $n = n_1 n_2$ with n_1, n_2 linear polynomials over $\mathbb{Q}$ such that $n_1 \nmid n_2$. □

We are left with the possibility that e and n do not have a common nonconstant factor. For $e = 0$ and for fixed a and b, by Lemma 2.2, $c = a + b + 2r$ is the only possible c. An example for this is the polynomial $D(X^2 + 2X + 1)$-triple

$$\{X^2 + 1, X^2 + 2X + 3, 4X^2 + 4X + 8\}.$$

For $e \neq 0$ we have the following lemma.

LEMMA 2.6. *Let $e \in \mathbb{Z} \setminus \{0\}$ be defined by (2.10). Then, for fixed a and b, there is at most one such e.*

PROOF. By Lemma 2.1, there is $w \in \mathbb{Z}[X]$ such that $ce + n^2 = w^2$. Therefore, $\deg(w) = 2$ and $\mu = \pm\omega$ where μ and ω are the leading coefficients of n and w, respectively. Also,

$$c = \frac{(w - n)(w + n)}{e} \tag{2.24}$$

where one factor in the numerator is constant and the other has degree 2. From (2.14), we see that $\deg(a+b-c) = 2$. Also, by comparing the coefficients in (2.14), we get

$$0 = \mu(A^2 + B^2 - C^2) + 2AB\omega.$$

Now it follows $C^2 = (A \pm B)^2$. Since $A - B < A \leq C$, we get $\omega = \mu$. By (2.24), we have $w - n = \xi e$ where $\xi \in \mathbb{Q}\backslash\{0\}$. Then, from (2.14), it follows that

$$e(1 - 2r\xi) = n(a + b - c + 2r).$$

Therefore,

$$1 - 2r\xi = \sigma n, \tag{2.25}$$

$$\sigma e = a + b - c + 2r \tag{2.26}$$

where $\sigma \in \mathbb{Q}\backslash\{0\}$.

Suppose that there exists another nonzero integer $f \neq e$ for which Lemma 2.1 holds. For a polynomial $D(n)$-triple $\{a, b, c'\}$, $a < b < c'$, by Lemma 2.1 there is $w' \in \mathbb{Z}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e,

$$1 - 2r\xi' = \sigma' n \tag{2.27}$$

where $a + b - c' + 2r = \sigma' f$, $w' - n = \xi' f$ and $\sigma', \xi' \in \mathbb{Q}\backslash\{0\}$. From (2.25) and (2.27), we get

$$-2r(\xi - \xi') = n(\sigma - \sigma').$$

If $n|r$, we obtain a contradiction with (2.25). Therefore, $\xi = \xi'$ and $\sigma = \sigma'$.

By (2.24), we get $c = \xi(\xi e + 2n)$ and, inserting this into (2.26), we obtain

$$e = \frac{1}{\xi^2 + \sigma}(a + b + 2r - 2n\xi). \tag{2.28}$$

Analogously, it follows

$$f = \frac{1}{\xi'^2 + \sigma'}(a + b + 2r - 2n\xi').$$

Comparing that with (2.28), we conclude $f = e$. Hence, for fixed a and b, there is at most one $e \in \mathbb{Z}\backslash\{0\}$. □

Lemma 2.7. *Let $e \in \mathbb{Z}[X]$ be a linear polynomial defined by (2.10), which does not divide n. Then, for fixed a and b, there is at most one such e.*

Proof. The proof is analogous to the proof of Lemma 2.6. The only difference is that here, in (2.24), we have $w - n = \theta e$, $\theta \in \mathbb{Q}\backslash\{0\}$ or $w + n = qe$, $q \in \mathbb{Q}[X]$, $\deg(q) = 1$. From that, two possibilities for e arise. □

Let us now consider the last possibility for e.

Lemma 2.8. *Let $\{a, b, c\}$, $a < b < c$, be a polynomial $D(n)$-triple. Let $e \in \mathbb{Z}[X]$ be a quadratic polynomial defined by (2.10), which does not have a common nonconstant factor with n. Then one of the following three possibilities holds:*

1) *There is at most one polynomial $c' \neq c$ such that $\{a, b, c'\}$, $a < b < c'$, is a polynomial $D(n)$-triple and $f \in \mathbb{Z}[X]$, obtained by applying (2.10) on that triple, is a quadratic polynomial which does not have a common nonconstant factor with n.*
2) *There is at most one polynomial $b' \neq b$ such that $\{a, b', c\}$, $a < b' < c$, is a polynomial $D(n)$-triple and $f \in \mathbb{Z}[X]$, obtained by applying (2.10) on that triple, is a quadratic polynomial which does not have a common nonconstant factor with n.*
3) *There is at most one polynomial $a' \neq a$ such that $\{a', b, c\}$, $a' < b < c$, is a polynomial $D(n)$-triple and $f \in \mathbb{Z}[X]$, obtained by applying (2.10) on that triple, is a quadratic polynomial which does not have a common nonconstant factor with n.*

Proof. Assume that for the $D(n)$-triple $\{a, b, c\}$ there is a quadratic polynomial e, which does not have a common nonconstant factor with n and for which Lemma 2.1 holds. By this lemma, there exists $w \in \mathbb{Z}[X]$ such that (2.24) holds where $\deg(w) \leq 2$ and both factors in the numerator have their degrees equal to 2. Also, e divides one of those factors or $e = e_1 e_2$ where e_1, e_2 are linear polynomials over $\mathbb{Q}$ and $e_1|(w - n)$, $e_2|(w + n)$. It is clear that at most one of these two cases holds.

If $w \pm n = \psi e$ where $\psi \in \mathbb{Q}\backslash\{0\}$, then from (2.14) we obtain

$$1 - 2r\psi = \phi n, \tag{2.29}$$

$$\phi e = a + b - c \mp 2r \tag{2.30}$$

where $\phi \in \mathbb{Q}\backslash\{0\}$. If we have $e = e_1e_2$ and $w-n = m_1e_1$, $w+n = m_2e_2$ where e_1, e_2, m_1, m_2 are linear polynomials over $\mathbb{Q}$, $m_1m_2 = c$, then from (2.14) we get

$$\begin{aligned} nd_1 &= 2rm_1 - e_2, \\ nd_2 &= 2rm_2 - e_1 \end{aligned} \tag{2.31}$$

and

$$e_1d_1 = c - a - b - 2r, \tag{2.32}$$

$$e_2d_2 = c - a - b + 2r \tag{2.33}$$

where d_1, d_2 are linear polynomials over $\mathbb{Q}$.

We first treat the case $e|(w \pm n)$. Assume also that for a polynomial $D(n)$-triple $\{a, b, c'\}$, $a < b < c'$, there is a quadratic polynomial f with the same properties as e. By Lemma 2.1, there is $w' \in \mathbb{Z}[X]$ such that $c'f + n^2 = (w')^2$.

Let $f = f_1f_2$ where f_1, f_2 are linear polynomials over $\mathbb{Q}$ and assume

$$\begin{aligned} w' - n &= h_1f_1, \\ w' + n &= h_2f_2 \end{aligned} \tag{2.34}$$

where h_1, h_2 are linear polynomials over $\mathbb{Q}$, $h_1h_2 = c'$. Analogously as for e, we obtain

$$\begin{aligned} l_1n &= 2rh_1 - f_2, \\ l_2n &= 2rh_2 - f_1 \end{aligned} \tag{2.35}$$

with l_1, l_2 linear polynomials over $\mathbb{Q}$. Using (2.29), from (2.35), we get

$$\begin{aligned} -n(\psi l_1 + \phi h_1) &= f_2\psi - h_1, \\ -n(\psi l_2 + \phi h_2) &= f_1\psi - h_2, \end{aligned}$$

which is a contradiction unless $f_2\psi = h_1$ and $f_1\psi = h_2$. Now we have that $c'|f$. Also, from (2.34), we get $c'|2n$, a contradiction.

Assume now that $w' \pm n = \psi' f$ where $\psi' \in \mathbb{Q}\backslash\{0\}$. Analogously as for e, we have

$$1 - 2r\psi' = \phi' n, \tag{2.36}$$

$$\phi' f = a + b - c' \mp 2r \tag{2.37}$$

with $\phi' \in \mathbb{Q}\backslash\{0\}$. From (2.29) and (2.36), we get that $\phi = \phi'$ and $\psi = \psi'$. By (2.24), $c = \psi(\psi e \mp 2n)$. Inserting that into (2.30), we obtain

$$e = \frac{1}{\psi^2 + \phi}(a + b \mp 2r \pm 2n\psi). \tag{2.38}$$

Analogously, using (2.37), we obtain

$$f = \frac{1}{\psi'^2 + \phi'}(a + b \mp 2r \pm 2n\psi'). \tag{2.39}$$

From (2.38) and (2.39), we conclude that for fixed a and b there is at most one $f \neq e$. For such f, by (2.37), we have $c' = -\phi' f + a + b \mp 2r$.

Now we come to the second case, i.e., that $e = e_1 e_2$ where e_1, e_2 are linear polynomials over $\mathbb{Q}$ and $e_1|(w-n)$, $e_2|(w+n)$. By adding the equations (2.17) (with the sign +) and (2.32), we obtain $\frac{e}{n} + \frac{2}{n^2}(abe + ruv) = 2r + d_1 e_1$. From that, using (2.9) and (2.31), it follows that

$$uv - n^2 = e_1(m_1 n - re_2). \tag{2.40}$$

For $u, v \in \mathbb{Z}[X]$, from Lemma 2.1, it follows that $u \pm n = k_1 e_1$ and $v \pm n = z_1 e_1$ where $k_1, z_1 \in \mathbb{Q}[X]$, $\deg(k_1) = \deg(z_1) = 1$. Using that, from (2.40), we get

$$e_1|(k_1 z_1 e_1^2 \pm k_1 e_1 n \pm z_1 e_1 n \pm n^2 - n^2),$$

so both signs in the equations $u \pm n = k_1 e_1$ and $v \pm n = z_1 e_1$ must be the same. Analogously, from (2.17) (with the sign +), (2.33) and the equations $u \mp n = k_2 e_2$, $v \mp n = z_2 e_2$ where $k_2, z_2 \in \mathbb{Q}[X]$, $\deg(k_2) = \deg(z_2) = 1$, we obtain

$$e_2|(k_2 z_2 e_2^2 \pm k_2 e_2 n \pm z_2 e_2 n \pm n^2 + n^2).$$

Therefore, signs in the equations $u \mp n = k_2 e_2$ and $v \mp n = z_2 e_2$ must be different. So $u \pm n = \kappa e_1 e_2$ where $\kappa \in \mathbb{Q}\backslash\{0\}$ and $v \pm n = z_1 e_1$, $v \mp n = z_2 e_2$, or $v \pm n = \mu e_1 e_2$ where $\mu \in \mathbb{Q}\backslash\{0\}$ and $u \pm n = k_1 e_1$, $u \mp n = k_2 e_2$. If we have both possibilities at the same time, then e and n have a common linear factor, so we obtain a contradiction.

Suppose first that

$$e|(u \pm n).$$

Using Lemma 2.1 and (2.16), analogously as in the case where $e|(w \pm n)$, we obtain

$$\begin{aligned} 1 - 2t\kappa &= \vartheta n, \\ a &= -\vartheta e + b + c \mp 2t \end{aligned} \tag{2.41}$$

where $\vartheta \in \mathbb{Q}\backslash\{0\}$. Also, for fixed b and c, there is at most one $f \neq e$ with the properties from the assumption of the lemma. For the triple $\{a', b, c\}$, $a' < b < c$, by Lemma 2.1, there is $u' \in \mathbb{Z}[X]$ such that $a'f + n^2 = (u')^2$. If $u' \pm n = \kappa' f$, $\kappa' \in \mathbb{Q}\backslash\{0\}$, we have

$$\begin{aligned} 1 - 2t\kappa' &= \vartheta' n, \\ a' &= -\vartheta' f + b + c \mp 2t \end{aligned} \tag{2.42}$$

where $\vartheta' \in \mathbb{Q}\backslash\{0\}$. Then, it follows that

$$e = \frac{1}{\kappa^2 + \vartheta}(b + c \mp 2t \pm 2n\kappa)$$

and

$$f = \frac{1}{\kappa'^2 + \vartheta'}(b + c \mp 2t \pm 2n\kappa').$$

Analogously as for (2.38) and (2.39), we have $\kappa' = \kappa$, $\vartheta' = \vartheta$, so there exists at most one $f \neq e$. For e and f, we have at most one a and at most one a', given by (2.41) and (2.42), respectively.

Finally in the case when $e|(v \pm n)$, analogously, from Lemma 2.1 and (2.15), it follows that for fixed a and c there exist at most two different e-s with the properties from the assumption of the lemma. For them, we obtain b and b' such that $\{a, b, c\}$ and $\{a, b', c\}$, where $a < b' < c$, are a polynomial $D(n)$-triples. ∎

Examples for the case 1) are the polynomial $D(16X^2 + 9)$-triples $\{X^2, 16X^2 + 8, 100X^2 + 44\}$ and $\{X^2, 16X^2 + 8, 36X^2 + 20\}$ for which $e = 273X^2 + 126$ and $f = 33X^2 + 18$, respectively.

Now we are ready to estimate the number Q_2.

Proof of Proposition 1.6 1). Let $a, b \in \mathbb{Z}^+[X]$, $a < b$, be quadratic polynomials. Let $ab + n = r^2$ with $r \in \mathbb{Z}^+[X]$. We want to find the number of possible $D(n)$-triples $\{a, b, c\}$ where $c \in \mathbb{Z}^+[X]$, $c > b$, is also a quadratic polynomial. First we look for the possible e-s coming from Lemma 2.1 applied to such a triple.

By Lemma 2.4 and Lemma 2.5, we have at most two[6] quadratic polynomials e_i, $i = 1, 2$, with common linear or quadratic factor with n. From Lemma 2.2, we obtain

$$c_i = a + b + \frac{e_i}{n} + \frac{2}{n^2}(abe_i + ru_iv_i)$$

for $i = 1, 2$ where $u_i, v_i \in \mathbb{Z}^+[X]$ such that $ae_i + n^2 = u_i^2$, $be_i + n^2 = v_i^2$. Moreover, $a < b < c_i$ for $i = 1, 2$. For $e = 0$, we get

$$c_3 = a + b + 2r$$

and $a < b < c_3$. By Lemma 2.6, we have at most one nonzero integer e_4 for which

$$c_4 = a + b + \frac{e_4}{n} + \frac{2}{n^2}(abe_4 + ru_4v_4)$$

where $u_4, v_4 \in \mathbb{Z}^+[X]$ such that $ae_4 + n^2 = u_4^2$, $be_4 + n^2 = v_4^2$. Also, $a < b < c_4$. By Lemma 2.7, there is at most one linear polynomial e_5 which does not divide n. For e_5, we have

$$c_5 = a + b + \frac{e_5}{n} + \frac{2}{n^2}(abe_5 + ru_5v_5)$$

where $u_5, v_5 \in \mathbb{Z}^+[X]$ such that $ae_5 + n^2 = u_5^2$, $be_5 + n^2 = v_5^2$. It holds $a < b < c_5$.

Finally, by Lemma 2.8, there does not exist a set $\{a, b, c_1', c_2', c_3', c_4', c_5'\}$, $b < c_1' < c_2' < c_3' < c_4' < c_5'$, of quadratic polynomials from $\mathbb{Z}^+[X]$ such that

[6] There does not exist a quadratic polynomial n for which Lemma 2.4 and Lemma 2.5 both hold.

every three of its elements have for e a quadratic polynomial from $\mathbb{Z}[X]$ which does not have a common nonconstant factor with n. Namely, in that case for the set $\{c_1', c_2', c_5'\}$ Lemma 2.8 does not hold.

Let us consider a set $\{a, b, c_1', c_2', c_3', c_4'\}$ with the property that every three of its elements correspond to an e that is a quadratic polynomial from $\mathbb{Z}[X]$ which does not have a common nonconstant factor with n. We have seen that a set with this property cannot be larger. Every two elements from this set have at most five extensions to a polynomial $D(n)$-triple which does not have the above property. If we add all this elements, then the set has the size at most

$$\binom{6}{2} \cdot 5 + 6 = 81.$$

Clearly, in a set with more than 81 elements we would be able to find a subset consisting of 7 elements which has the property that every three elements contained in the set have a quadratic e which has no common nonconstant factor with n. Since this is impossible, it follows that $Q_2 \leq 81$. ∎

2.2.2. *Cubic polynomials.* The proof of Proposition 1.6 2) is based on the construction from Lemma 2.1 and the following lemmas which deal with a polynomial $D(n)$-triple $\{a, b, c\}$ where $\deg(a) = \deg(b) = \deg(c) = 3$. An example of such a set is the following $D(-7X^2 + 8X)$-triple

$$\{X^3 + 2X, X^3 + 4X^2 + 4X - 4, 4X^3 + 8X^2 + 8X - 4\}. \tag{2.43}$$

First, we are looking for the possible e-s for fixed a and b. By (2.13), we have that

$$\deg(e) \leq 1.$$

From (2.12), we determine possible relations between e and n. We have $n \nmid e$ and we will prove that n and e do not have a common linear factor. For $e = 0$, by Lemma 2.2, $c = a + b + 2r$. An example for such a triple is (2.43).

Lemma 2.9. *For fixed a and b, there is at most one $e \in \mathbb{Z}\backslash\{0\}$ defined by (2.10).*

Proof. By Lemma 2.1, we have $w \in \mathbb{Z}[X]$ for which (2.24) holds. Observe that $\deg(w) = 2$ and the leading coefficients of n and w are equal up to sign. Also, one of the factors in the numerator of (2.24) has degree 1 and the other one has degree 2. From (2.14), we obtain that $\omega = \mu$ where μ, ω are the leading coefficients of n, w, respectively. Therefore, in (2.24), $w - n = ge$ with g a linear polynomial over $\mathbb{Q}$ which divides c. From (2.14), it follows that

$$1 - 2rg = hn, \tag{2.44}$$

$$he = a + b - c + 2r \tag{2.45}$$

where $h \in \mathbb{Q}[X]$, $\deg(h) = 2$.

Assume that $f \neq e$ is another nonzero integer for which Lemma 2.1 holds. Then, for the polynomial $D(n)$-triple $\{a, b, c'\}$, $a < b < c'$, there is $w' \in \mathbb{Z}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e, it holds

$$1 - 2rg' = h'n \tag{2.46}$$

where $a + b - c' + 2r = h'f$, $w' - n = g'f$, $g', h' \in \mathbb{Q}[X]$ and $\deg(g') = 1$, $\deg(h') = 2$. From (2.44) and (2.46), we obtain that $g = g'$ and $h = h'$. By (2.24) and (2.45), we get

$$e = \frac{1}{g^2 + h}(a + b + 2r - 2ng). \tag{2.47}$$

Analogously, we obtain $f = \frac{1}{g'^2+h'}(a + b + 2r - 2ng')$, so $f = e$. $\square$

Lemma 2.10. *Let $e \in \mathbb{Z}[X]$ be a linear polynomial defined by (2.10). Then $e \nmid n$. For fixed a and b, there exist at most two such e-s.*

Proof. Assume on the contrary, that $n = n_1 n_2$ where n_1, n_2 are linear polynomials over $\mathbb{Q}$ and $e = \tau n_1$, $\tau \in \mathbb{Q}\backslash\{0\}$. By Lemma 2.1, there exists $u \in \mathbb{Z}[X]$ such that

$$a\tau n_1 + n_1^2 n_2^2 = u^2.$$

We have $n_1^2 | u^2$, so $n_1 | a$. Analogously, we obtain that $n_1 | b$. Now, from (2.9), we conclude that $n_1^2 | n$ and $n_1 | s$. Then, from (2.10), we get $n_1^2 | e$, a contradiction. Therefore, $e \nmid n$.

By Lemma 2.1, there is $w \in \mathbb{Z}[X]$ for which (2.24) holds. Observe that $\deg(w) \leq 2$, that both factors in the numerator of (2.24) have the degree 2 and that $w \pm n = pe$ where $p \in \mathbb{Q}[X]$, $\deg(p) = 1$. From (2.14), it follows that

$$1 - 2rp = qn, \tag{2.48}$$

$$qe = a + b - c \mp 2r \tag{2.49}$$

where $q \in \mathbb{Q}[X]$, $\deg(q) = 2$.

Let $f \neq e$ be another linear polynomial which does not divide n and for which Lemma 2.1 holds. Then, for the polynomial $D(n)$-triple $\{a, b, c'\}$, $a < b < c'$, there is $w' \in \mathbb{Z}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e, we have

$$1 - 2rp' = q'n \tag{2.50}$$

where $a + b - c' \mp 2r = q'f$, $w' \pm n = p'f$ and $p', q' \in \mathbb{Q}[X]$, $\deg(p') = 1$, $\deg(q') = 2$. From (2.48) and (2.50), we have $p = p'$ and $q = q'$. Using (2.24) and (2.49), we obtain

$$e = \frac{1}{p^2 + q}(a + b \mp 2r \pm 2np). \tag{2.51}$$

Analogously, $f = \frac{1}{p'^2+q'}(a + b \mp 2r \pm 2np')$, so there is at most one $f \neq e$. $\square$

Now we are able to determine the upper bound for Q_3.

Proof of Proposition 1.6 2). Let $\{a, b, c\}$, $a < b < c$, be a polynomial $D(n)$-triple which contains only cubic polynomials. Let us fix a and b. By Lemma 2.9, there is at most one nonzero integer e for which Lemma 2.1 holds. From Lemma 2.2, for such e, it follows that there is at most one possibility for c. For $e = 0$ we obtain $c = a + b + 2r$. By Lemma 2.10, we have at most two linear e-s which do not divide n. For each of that e-s, by Lemma 2.2, we obtain at most one possible c. In Lemma 2.10, we also excluded the last option which comes from (2.12), those that e and n have a common linear factor. Therefore, the pair $\{a, b\}$ can be extended with at most 4 cubic polynomials. From (2.44) and (2.48), we obtain that $g = p$ and $h = q$, so there exist at most two between three possible e-s, given by (2.47) and (2.51). Hence, $Q_3 \leq 5$. □

2.2.3. *Polynomials of degree* 4. Now we determine the upper bound for the number of polynomials of degree 4 in a polynomial $D(n)$-tuple. Let $\{a, b, c\}$ be a polynomial $D(n)$-triple, $\deg(a) = \deg(b) = \deg(c) = 4$. By (2.13), we have that

$$\deg(e) \leq 0.$$

For $e = 0$ and for fixed a and b, from Lemma 2.2 we obtain, for example, the polynomial $D(4X^2)$-triple

$$\{X^4 + X^2, X^4 + X^2 + 4X, 4X^4 + 4X^2 + 8X\}.$$

Lemma 2.11. *Let $e \in \mathbb{Z}\backslash\{0\}$ be defined by (2.10). Then, for fixed a and b, there exist at most three such e-s.*

Proof. By Lemma 2.1, there is $u \in \mathbb{Z}[X]$ such that

$$a = \frac{(u - n)(u + n)}{e} \tag{2.52}$$

where $\deg(u) \leq 2$. If we denote $y := \frac{u-n}{e}$, then $u + n = ye + 2n$, $y \in \mathbb{Q}[X]$ and $\deg(y) = 2$. From (2.16), we obtain that

$$n|(1 - 2ty). \tag{2.53}$$

Suppose that $f \neq e$ is another nonzero integer for which Lemma 2.1 holds. Hence, for the polynomial $D(n)$-triple $\{a, b, c'\}$, $a < b < c'$, there is $u' \in \mathbb{Z}[X]$ such that

$$a = \frac{(u' - n)(u' + n)}{f}. \tag{2.54}$$

If we denote $y' := \frac{u'-n}{f}$, then $u' + n = y'f + 2n$, $y' \in \mathbb{Q}[X]$, $\deg(y') = 2$. By (2.16)

$$n|(1 - 2t'y') \tag{2.55}$$

where $bc' + n = (t')^2$. From (2.52) and (2.54), it follows that

$$y^2 e - (y')^2 f + 2n(y - y') = 0 \tag{2.56}$$

and then

$$(y-y')(e(y+y')+2n)=(y')^2(f-e).$$

Therefore, $\deg(y-y')=2$ and $\deg(e(y+y')+2n)=2$. If

$$y'|(y-y') \text{ and } y'|(e(y+y')+2n),$$

then $y'|y$ and $y'|n$, so we obtain a contradiction with (2.55). Therefore, $y'=y'_1\cdot y'_2$ where y'_1, y'_2 are linear polynomials over $\mathbb{Q}$ and

$$(y'_1)^2|(y-y') \text{ and } (y'_2)^2|(e(y+y')+2n). \tag{2.57}$$

From that, y and y' have a common linear factor y'_1. If $y|y'$ and by (2.56), we get $y|n$, which is in contradiction with (2.53). Also, from (2.57), it follows that $y'_2|(ye+2n)$. Therefore, y' and $ye+2n$ have a common linear factor, but if $y'|(ye+2n)$, then we have $y'_1|n$, a contradiction with (2.55).

Analogously, we transform (2.56) into

$$(y-y')(f(y+y')+2n)=y^2(f-e)$$

and we conclude that y and $y'f+2n$ have a common linear factor, but are not equal up to a constant.

From (2.52) and (2.54), $a=y(ye+2n)=y'(y'f+2n)$. Also, it must hold

$$a=A^2(X-\phi_1)(X-\phi_2)(X-\phi_3)(X-\phi_4)$$

where $A\in\mathbb{N}$ and $\phi_i\in\mathbb{Q}$ for $i=1,2,3,4$. Let $(X-\phi_1)|y$ and $(X-\phi_1)|y'$. If $(X-\phi_1)|(ye+2n)$, it leads to a contradiction with (2.53). Analogously, $(X-\phi_1)\nmid(y'f+2n)$ because it contradicts (2.55). Hence, we also have $(X-\phi_1)^2\nmid y$ and $(X-\phi_1)^2\nmid y'$. Suppose next that $y=\pi_1(X-\phi_1)(X-\phi_2)$, $\pi_1\in\mathbb{Q}\backslash\{0\}$. Then, we have $(X-\phi_2)|(y'f+2n)$. Also, $(X-\phi_2)\nmid(ye+2n)$ because otherwise it would be a contradiction with (2.53). Let $y'=\pi_2(X-\phi_1)(X-\phi_3)$, $\pi_2\in\mathbb{Q}\backslash\{0\}$. Then, we have $(X-\phi_3)|(ye+2n)$. Also, $(X-\phi_3)\nmid(y'f+2n)$ because otherwise it would contradict (2.55). Finally, we have $ye+2n=\pi_3(X-\phi_3)(X-\phi_4)$ and $y'f+2n=\pi_4(X-\phi_2)(X-\phi_4)$, $\pi_3,\pi_4\in\mathbb{Q}\backslash\{0\}$.

Assume now that for $g\in\mathbb{Z}\backslash\{0\}$ Lemma 2.1 also holds and that $g\neq e$, $g\neq f$. Therefore, $ag+n^2=(u'')^2$ where $u''\in\mathbb{Z}[X]$. As for e and f, we obtain that

$$a=y''(y''g+2n)$$

with $y'':=\frac{u''-n}{g}$, $y''\in\mathbb{Q}[X]$ and $\deg(y'')=2$. Then, y'' and y have a common linear factor, but $y''\nmid y$. The same holds for y'' and y'. Observe now that

$$(X-\phi_1)|y'' \quad\text{or}\quad (X-\phi_2)(X-\phi_3)|y''. \tag{2.58}$$

If $(X-\phi_1)|y''$, then $(X-\phi_2)\nmid y''$ and $(X-\phi_3)\nmid y''$ because we would have that $y''|y$ or $y''|y'$, a contradiction in both cases. We conclude that $y''=\pi_5(X-\phi_1)(X-\phi_4)$, $\pi_5\in\mathbb{Q}\backslash\{0\}$. Hence, $y''g+2n=\pi_6(X-\phi_2)(X-\phi_3)$, $\pi_6\in\mathbb{Q}\backslash\{0\}$. If $y''=\pi_7(X-\phi_2)(X-\phi_3)$ where $\pi_7\in\mathbb{Q}\backslash\{0\}$, then $y''g+2n=\pi_8(X-\phi_1)(X-\phi_4)$ for $\pi_8\in\mathbb{Q}\backslash\{0\}$. If we have both possibilities (2.58), then

$\pi_5(X-\phi_1)(X-\phi_4)|\pi_8(X-\phi_1)(X-\phi_4)$, a contradiction. Therefore, we have at most one g. ∎

Proof of Proposition 1.6 3). Let $\{a,b,c\}$, $a<b<c$, be a polynomial $D(n)$-triple which contains only polynomials of degree 4. Let us fix a and b. By Lemma 2.11, there are at most three nonzero integers e for which Lemma 2.1 holds. Then, from Lemma 2.2, it follows that for each such e, we have at most one possibility for c. For $e=0$, $c=a+b+2r$. Since there are no other possibilities for e which come from (2.12), we have $Q_4\le 6$. ∎

2.2.4. *Polynomials of degree k, $k\ge 5$.* We determine a sharp bound for the number of polynomials of degree k, for $k\ge 5$, in a polynomial $D(n)$-tuple.

Proof of Proposition 1.6 4). Let $\{a,b,c\}$, $a<b<c$, be a polynomial $D(n)$-triple for which (2.9) holds. Let $\deg(a)=\deg(b)=\deg(c)=k\ge 5$. By (2.13),

$$\deg(e)\le -1,$$

which is a contradiction except for $e=0$. Therefore, for fixed a and b, there is only one possible c, which is $c=a+b+2r$. ∎

3. Gap principle

We will prove a gap principle for the degrees of the elements in a polynomial $D(n)$-quadruple. This result will be used in the proof of Theorem 1.3, together with the bounds from Section 2 and with the upper bound for the degree of the element in a polynomial $D(n)$-quadruple ([10, Lemma 1]), given in the following lemma.

Lemma 3.1. *Let $\{a,b,c,d\}$, $a<b<c<d$, be a polynomial $D(n)$-quadruple with $n\in\mathbb{Z}[X]$. Then*

$$\deg(d)\le 7\deg(a)+11\deg(b)+15\deg(c)+14\deg(n)-4.$$

The proof of this lemma is based on the theory of function fields, precisely it is obtained by using Mason's inequality ([15]).

Now we will adjust the result from [9, Lemma 3], for linear n, to achieve the needed gap principle.

Lemma 3.2. *Let $\{a,b,c,d\}$, where $a<b<c<d$ and $\deg(a)\ge 5$, be a polynomial $D(n)$-quadruple for quadratic $n\in\mathbb{Z}[X]$. Then*

$$\deg(d)\ge \deg(b)+\deg(c)-4.$$

Proof. By Lemma 2.1, for a polynomial $D(n)$-triple $\{a,c,d\}$, there exist $e,u,w\in\mathbb{Z}[X]$ such that $ae+n^2=u^2$, $ce+n^2=w^2$. If $e<0$, then $ae+n^2<0$, which is a contradiction. Therefore, $e=0$ or $e\in\mathbb{Z}^+[X]$.

Assume that $n > 0$. Using the relations $a^2 < ad + s^2 = a(c + d) + n$, $ad + n = x^2$ and $cd + n = z^2$, we obtain

$$a^2 z^2 < (ac + n)(ad + n) = s^2 x^2.$$

It follows that $u < 0$. Analogously, $c^2 < cd + s^2 = c(a + d) + n$, so we have

$$c^2 x^2 < (ac + n)(cd + n) = s^2 z^2.$$

It follows that $w < 0$. In analogue way, if $n < 0$, then $u, w > 0$.

For $e = 0$, by Lemma 2.1,

$$d = a + c + 2s. \tag{3.1}$$

For $e > 0$, by Lemma 2.1, using the relations $n^2 < a < c$ and $uw > 0$, we obtain

$$\begin{aligned} n^2 d &= n^2(a + c) + en + 2(ace + suw) \\ &> 2n^4 + n + 2ac > 2ac. \end{aligned} \tag{3.2}$$

Analogously, applying Lemma 2.1 to the polynomial $D(n)$-triple $\{b, c, d\}$, we obtain that either $d = b + c + 2t$ or $n^2 d > 2bc$. If $d = b + c + 2t$, then $s^2 = ac + n < bc + n = t^2$, so we have $s < t$. Hence, $a + c + 2s < b + c + 2t$, which contradicts (3.1). Also,

$$t^2 = bc + n \leq (c - 1)c + n = c^2 - c + n < c^2 - n^2 + n < c^2.$$

From that, we have $t < c$, so

$$n^2 d = n^2(b + c + 2t) < n^2 \cdot 4c < 2ac,$$

which is a contradiction with (3.2). Therefore,

$$n^2 d > 2bc.$$

□

Observe that Proposition 1.6 4) is a consequence of Lemma 3.2. Namely, if $\{a, b, c, d\}$ is a polynomial $D(n)$-quadruple where $\deg(a) = \deg(b) = \deg(c) = \deg(d) = k \geq 5$, by Lemma 3.2, we obtain the contradiction $4 \geq k \geq 5$.

4. Proof of Theorem 1.3.

We will combine the results from Section 2 and 3, using the approach from [10]. By [4, Theorem 1] and Propositions 1.5 and 1.6, we have $Q \leq 113$. Hence, we will improve that bound.

Proof of Theorem 1.3. Let $S = \{a_1, a_2, \ldots, a_m\}$, $a_1 < a_2 < \cdots < a_m$, be a polynomial $D(n)$-m-tuple where $n \in \mathbb{Z}[X]$ is a quadratic polynomial. Observe that if S contains a polynomial of degree ≥ 2, then it contains only polynomials of even or only polynomials of odd degree. By Proposition 1.5, in S we have at most 2 nonzero constants and at most 4 linear polynomials. By Proposition 1.6, the number of quadratic polynomials in S is at most 81, in S there are at most 5 cubic polynomials, at most 6 polynomials of degree four and at most 3 polynomials of degree k for every $k \geq 5$.

Assume that in S there is a polynomial of degree ≥ 2. Let us first consider the case where the degrees of all polynomials in S are odd. First, we have

$$\deg(a_1) \geq 1,\ \deg(a_2) \geq 1,\ \deg(a_3) \geq 1,\ \deg(a_4) \geq 1,$$

$$\deg(a_5) \geq 3,\ \deg(a_6) \geq 3,\ \deg(a_7) \geq 3,\ \deg(a_8) \geq 3,\ \deg(a_9) \geq 3$$

and

$$\deg(a_{10}) \geq 5,\ \deg(a_{11}) \geq 5,\ \deg(a_{12}) \geq 5.$$

Applying Lemma 3.2 to the polynomial $D(n)$-quadruple $\{a_{10}, a_{11}, a_{12}, a_{13}\}$ gives $\deg(a_{13}) \geq 6$ and, since this degree is odd, we conclude

$$\deg(a_{13}) \geq 7.$$

If we continue in analogue way, we obtain

$$\begin{array}{llll} \deg(a_{14}) \geq 9, & \deg(a_{15}) \geq 13, & \deg(a_{16}) \geq 19, & \deg(a_{17}) \geq 29, \\ \deg(a_{18}) \geq 45, & \deg(a_{19}) \geq 71, & \deg(a_{20}) \geq 113, & \deg(a_{21}) \geq 181, \ldots \end{array}$$

We will separate the cases depending on the number of linear polynomials in S. Assume first that $\deg(a_1) = \deg(a_2) = \deg(a_3) = \deg(a_4) = 1$. Applying Lemma 3.1 to a polynomial $D(n)$-quadruple $\{a_1, a_2, a_3, a_m\}$, we get $\deg(a_m) \leq 57$. Hence, in this case

$$m \leq 18.$$

Analogously, if $\deg(a_1) = \deg(a_2) = \deg(a_3) = 1$ and $\deg(a_4) \geq 3 \ldots$, we obtain $\deg(a_m) \leq 57$. From that, it follows $m \leq 17$.

Assume next that $\deg(a_1) = \deg(a_2) = 1$, $\deg(a_3) = A$ where $A \geq 3$ is an odd positive integer. As before, we obtain

$$\deg(a_m) \leq 7 + 11 + 15A + 28 - 4 = 15A + 42.$$

If $A = 3$, then

$$\begin{array}{lll} \deg(a_4) \geq A, & \deg(a_5) \geq A, & \deg(a_6) \geq A, \\ \deg(a_7) \geq A, & \deg(a_8) = B, & \deg(a_9) \geq B, \\ \deg(a_{10}) \geq B, & \deg(a_{11}) \geq 2B - 3, & \deg(a_{12}) \geq 3B - 7, \\ \deg(a_{13}) \geq 5B - 13, & \deg(a_{14}) \geq 8B - 23, & \deg(a_{15}) \geq 13B - 39, \\ \deg(a_{16}) \geq 21B - 65, & \deg(a_{17}) \geq 34B - 107, & \deg(a_{18}) \geq 55B - 175, \ldots \end{array}$$

where $B > A$ and B is an odd positive integer. Again, we have $m \leq 17$. If $A \geq 5$, then

$$\begin{array}{llll} \deg(a_4) \geq A, & \deg(a_5) \geq A, & \deg(a_6) \geq 2A - 3, & \ldots, \\ \deg(a_{12}) \geq 34A - 107, & \deg(a_{13}) \geq 55A - 175, & \deg(a_{14}) \geq 89A - 285, \ldots & \end{array}$$

Here, we get $m \leq 13$.

Suppose that $\deg(a_1) = 1, \deg(a_2) = A, \deg(a_3) = B$ where $3 \leq A \leq B$ and A, B are odd positive integers. From that, we obtain

$$\deg(a_m) \leq 7 + 11A + 15B + 28 - 4 \leq 26B + 31.$$

If $A = B = 3$, then

$$\begin{array}{lll} \deg(a_4) \geq B, & \deg(a_5) \geq B, & \deg(a_6) \geq B, \\ \deg(a_7) = C, & \deg(a_8) \geq C, & \deg(a_9) \geq C, \\ \deg(a_{10}) \geq 2C-3, & \dots, & \deg(a_{16}) \geq 34C-107, \\ \deg(a_{17}) \geq 55C-175, & \deg(a_{18}) \geq 89C-285, & \dots \end{array}$$

where $C \geq 5$ is an odd positive integer, so $m \leq 17$. If $A = 3$ and $B \geq 5$, then

$$\begin{array}{llll} \deg(a_4) \geq B, & \deg(a_5) \geq B, & \deg(a_6) \geq 2B-3, & \dots, \\ \deg(a_{13}) \geq 55B-175, & \deg(a_{14}) \geq 89B-285, & \deg(a_{15}) \geq 144B-463, & \dots \end{array}$$

Now, we have $m \leq 14$. Similarly, if $5 \leq A \leq B$, we obtain that $m \leq 13$.

Finally, suppose that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$ where $3 \leq A \leq B \leq C$ and A, B, C are odd positive integers. We get

$$\deg(a_m) \leq 7A + 11B + 15C + 28 - 4 \leq 33C + 24.$$

If $A = B = C = 3$ and

$$\begin{array}{lll} \deg(a_4) \geq C, & \deg(a_5) \geq C, & \deg(a_6) = D, \\ \deg(a_7) \geq D, & \deg(a_8) \geq D, & \deg(a_9) \geq 2D-3, \\ \deg(a_{10}) \geq 3D-7, & \dots, & \deg(a_{16}) \geq 55D-175, \\ \deg(a_{17}) \geq 89D-285, & \deg(a_{18}) \geq 144D-463, & \dots \end{array}$$

where $D \geq 5$ is an odd positive integer, then $m \leq 16$. If $A = B = 3$ and $C \geq 5$, we have

$$\begin{array}{llll} \deg(a_4) \geq C, & \deg(a_5) \geq C, & \deg(a_6) \geq 2C-3, & \dots, \\ \deg(a_{13}) \geq 55C-175, & \deg(a_{14}) \geq 89C-285, & \deg(a_{15}) \geq 144C-463, & \dots \end{array}$$

Hence, $m \leq 14$. Analogously, if $A = 3$ and $5 \leq B \leq C$, then $m \leq 13$. For $5 \leq A \leq B \leq C$, we have

$$\begin{array}{llll} \deg(a_4) \geq C, & \deg(a_5) \geq 2C-3, & \deg(a_6) \geq 3C-7, & \dots, \\ \deg(a_{12}) \geq 55C-175, & \deg(a_{13}) \geq 89C-285, & \deg(a_{14}) \geq 144C-463, & \dots, \end{array}$$

so $m \leq 13$.

We conclude that, if S contains only polynomials of odd degree, then $m \leq 18$.

Let all polynomials in S have even degree. Now we have

$$\deg(a_1) \geq 0, \ \deg(a_2) \geq 0,$$

$$\deg(a_3) \geq 2, \ \deg(a_4) \geq 2, \dots, \ \deg(a_{83}) \geq 2,$$

$$\deg(a_{84}) \geq 4, \ \deg(a_{85}) \geq 4, \dots, \ \deg(a_{89}) \geq 4$$

and

$$\deg(a_{90}) \geq 6, \ \deg(a_{91}) \geq 6, \ \deg(a_{92}) \geq 6.$$

Applying Lemma 3.2 to the polynomial $D(n)$-quadruple $\{a_{90}, a_{91}, a_{92}, a_{93}\}$ we get

$$\deg(a_{93}) \geq 8.$$

In analogue way, it follows

$$\begin{array}{llll} \deg(a_{94}) \ge 10, & \deg(a_{95}) \ge 14, & \deg(a_{96}) \ge 20, & \deg(a_{97}) \ge 30, \\ \deg(a_{98}) \ge 46, & \deg(a_{99}) \ge 72, & \deg(a_{100}) \ge 114, & \deg(a_{101}) \ge 182, \ldots \end{array}$$

Assume first that $\deg(a_1) = \deg(a_2) = 0$, $\deg(a_3) = A$ where $A \ge 2$ is an even positive integer. If we apply Lemma 3.1 to a polynomial $D(n)$-quadruple $\{a_1, a_2, a_3, a_m\}$, it follows that

$$\deg(a_m) \le 0 + 0 + 15A + 28 - 4 = 15A + 24.$$

Let $A = 2$. Then

$$\begin{array}{lll} \deg(a_4) \ge A, & \deg(a_5) \ge A, \ldots, & \deg(a_{83}) \ge A, \\ \deg(a_{84}) = B, & \deg(a_{85}) \ge B, \ldots, & \deg(a_{89}) \ge B, \\ \deg(a_{90}) = C, & \deg(a_{91}) \ge C, & \deg(a_{92}) \ge C, \\ \deg(a_{93}) \ge 2C - 4, & \deg(a_{94}) \ge 3C - 8, & \deg(a_{95}) \ge 5C - 16, \\ \deg(a_{96}) \ge 8C - 28, & \deg(a_{97}) \ge 13C - 48, & \deg(a_{98}) \ge 21C - 80, \ldots \end{array}$$

where $A < B < C$, and B, C are even positive integers, so

$$m \le 98.$$

Let $A = 4$. Since the set S contains at most 6 polynomials of degree 4, at most 3 polynomials of degree 6 and, by [4, Theorem 1], at most 21 polynomials of degree ≥ 8, in this case $m \le 32$. Analogously, $m \le 26$ if $A = 6$ and $m \le 23$ if $A \ge 8$.

Similarly, assume that $\deg(a_1) = 0, \deg(a_2) = A, \deg(a_3) = B$ where $2 \le A \le B$ and A, B are even positive integers. It follows that

$$\deg(a_m) \le 0 + 11A + 15B + 28 - 4 \le 26B + 24.$$

If $A = B = 2$, then

$$\begin{array}{lll} \deg(a_4) \ge B, & \deg(a_5) \ge B, \ldots, & \deg(a_{82}) \ge B, \\ \deg(a_{83}) = C, & \deg(a_{84}) \ge C, & \deg(a_{85}) \ge C, \ldots, \\ \deg(a_{88}) \ge C, & \deg(a_{89}) = D, & \deg(a_{90}) \ge D, \\ \deg(a_{91}) \ge D, & \deg(a_{92}) \ge 2D - 4, & \ldots, \\ \deg(a_{97}) \ge 21D - 80, & \deg(a_{98}) \ge 34D - 132, & \deg(a_{99}) \ge 55D - 216, \ldots \end{array}$$

where $B < C < D$ and C, D are even positive integers. Again, we have

$$m \le 98.$$

Let $A = 2$ and $B = 4$. Since in S there are at most 6 polynomials of degree 4, at most 3 polynomials of degree 6 and, by [4, Theorem 1], at most 21 polynomials of degree ≥ 8, we conclude that $m \le 32$. Analogously, $m \le 31$ if $A = B = 4$, $m \le 26$ if $A = 2$ or $A = 4$ and $B = 6$, $m \le 25$ if $A = B = 6$, and $m \le 23$ if $B \ge 8$.

Suppose finally that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$ where $2 \le A \le B \le C$ and A, B, C are even positive integers. We have

$$\deg(a_m) \le 7A + 11B + 15C + 28 - 4 \le 33C + 24.$$

If $A = B = C = 2$ and

$$\begin{array}{lll}
\deg(a_4) \geq C, \ldots, & \deg(a_{81}) \geq C, & \deg(a_{82}) = D, \\
\deg(a_{83}) \geq D, \ldots, & \deg(a_{87}) \geq D, & \deg(a_{88}) = E, \\
\deg(a_{89}) \geq E, & \ldots, & \deg(a_{96}) \geq 21E - 80, \\
\deg(a_{97}) \geq 34E - 132, & \deg(a_{98}) \geq 55E - 216, & \ldots
\end{array}$$

where $4 \leq D < E$ and D, E are even positive integers, then $m \leq 97$. Also, as before, $m \leq 32$ if $C = 4$, $m \leq 26$ if $C = 6$, and $m \leq 23$ if $C \geq 8$.

We conclude that the set S has at most 98 polynomials of even degree. Therefore, $Q \leq 98$. □

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