GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 325 – 331

ON A VARIANT OF A DIOPHANTINE EQUATION OF CASSELS

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Abstract. Recently, Yuan and Li considered a variant $y^2 = px(Ax^2 - 2)$ of Cassels' equation $y^2 = 3x(x^2+2)$. They proved that the equation has at most five solutions in positive integers (x, y). In this note, we improve Yuan-Li's result by showing that for any prime p and any odd positive integer A, the Diophantine equation $y^2 = px(Ax^2 - 2)$ has at most three solutions in positive integers (x, y).

1. Introduction

In 1985, J. W. S. Cassels ([6]) was asked to find all the cases when the sum of three consecutive integral cubes is a square. The problem consists in solving the Diophantine equation $y^2 = 3x(x^2 + 2)$. He used some elementary results from the theory of algebraic number fields to find $x = 0, 1, 2, 24$, i.e., the solutions are $(x, y) = (0, 0), (1, 3), (2, 6)$, and $(24, 204)$. Using the classical work of Wilhelm Ljunggren, Luca and Walsh ([8]) obtained a generalization. They proved that the equation $y^2 = px(x^2 + 2)$ has at most three solutions in positive integers (x, y), where p is a prime. An analogous result was obtained by Bennett ([4]) when he studied the Diophantine equation $y^2 = nx(x+1)(x+2)$.

In [13], the second author and Li studied another extension of Cassels' theorem by replacing the factor $x^2 + 2$ by $x^2 - 2$. In fact, using some results on the Diophantine equations of the forms $ax^4 - by^2 = 2$, $2 \nmid ab$, $ax^4 - by^2 = 1$ ([1–3, 10, 12]) and $ax^4 - y^2 = 1$ ([7, 11]), they proved the following two results.

2010 *Mathematics Subject Classification.* 11D25, 11B39.

Key words and phrases. Diophantine equations.

Supported by the Guangdong Provincial Natural Science Foundation (No. 81510275 01000114) and NSF of China (No. 10971072).

Theorem 1.1. *For any prime p and any odd positive integer $A > 1$, the Diophantine equation*

$$y^2 = px(Ax^2 - 2) \tag{1.1}$$

has at most five solutions in positive integers (x, y).

Theorem 1.2. *For any prime p, the equation*

$$y^2 = px(x^2 - 2) \tag{1.2}$$

has no solutions when $p \equiv 3 \pmod 8$, at most one positive solution (x, y) when $p \equiv 5, 7 \pmod 8$, and at most three such solutions when $p \equiv 1 \pmod 8$.

The aim of this paper is to improve Theorem 1.1 by obtaining a sharper bound of the number of solutions to equation (1.1). So we will prove the following result.

Theorem 1.3. *For any prime p and any odd positive integer $A > 1$ one of the following holds.*

- (i) *If $(A, p) \equiv (3, 1), (1, 7), (5, 3) \pmod 8$, then Diophantine equation (1.1) has at most two positive integer solutions (x, y).*
- (ii) *If $(A, p) \equiv (3, 5), (1, 1), (7, 5), (5, 1) \pmod 8$, then Diophantine equation (1.1) has at most three positive integer solutions (x, y).*
- (iii) *If $(A, p) \equiv (3, 3), (3, 7), (1, 5), (1, 3) \pmod 8$, then Diophantine equation (1.1) has at most one positive integer solution (x, y).*
- (iv) *Diophantine equation (1.1) has no solutions in other cases.*

The organization of this paper is as follows. In Section 2, we will recall some results on the Diophantine equation $aX^4 - bY^2 = c$, where a, b are positive integers and $c = 1, 2$. In the last section, we use the results recalled in Section 2 to prove Theorem 1.3. We consider all possible cases and obtain sharp bounds for the number of solutions of equation (1.1) in each case.

2. Preliminary results

In this section, we will recall the following three results that will be used to prove Theorem 1.3. The first result can be found in [7, 11].

Lemma 2.1. *Let d denote a positive non-square integer and let ϵ_d denote the fundamental solution of the Pell equation $x^2 - dy^2 = -1$. Then the Diophantine equation*

$$x^2 + 1 = dy^4 \tag{2.1}$$

has at most one positive integer solution (x, y) except when $d = 2$.

The next result was proved by Akhtari ([1]). It deals with the Diophantine equation

$$aX^4 - bY^2 = 1, \tag{2.2}$$

where a, b are positive integers.

Lemma 2.2. *Let a and b be positive integers. Then Diophantine equation (2.2) has at most two solutions in positive integers X and Y.*

The last result that we recall here was obtained by the second author and Li ([12]). They completely solved the Diophantine equation

$$aX^4 - bY^2 = 2. \tag{2.3}$$

Lemma 2.3. *For any positive odd integers a, b, Diophantine equation (2.3) has at most one solution in positive integers, and a such solution must arise from the fundamental solution to the quadratic equation $aX^2 - bY^2 = 2$.*

3. Proof of Theorem 1.3

Let p be a prime, A an odd positive integer. Moreover, let x, y be positive integers satisfying

$$y^2 = px(Ax^2 - 2).$$

If $p = 2$, then we have $x = 2x_1, y = 2y_1$. So we get

$$y_1^2 = 2x_1(2Ax_1^2 - 1).$$

Since $\gcd(x_1, 2Ax_1^2 - 1) = 1$, it follows that

$$x_1 = 2a^2, \quad 2Ax_1^2 - 1 = b^2.$$

This is impossible by taking modulo 8.

Now, we assume that p is an odd prime. We consider two cases.

Case 1: x is odd. As p is square-free, replacing y/p by w, equation (1.1) becomes

$$pw^2 = x(Ax^2 - 2).$$

Then, there exist odd integers u, v such that

$$x = pu^2, \quad Ax^2 - 2 = v^2,$$

i.e.,

$$Ap^2u^4 - v^2 = 2, \tag{3.1}$$

or

$$x = u^2, \quad Ax^2 - 2 = pv^2,$$

i.e.,

$$Au^4 - pv^2 = 2. \tag{3.2}$$

By Lemma 2.3, equation (3.1) or equation (3.2) has at most one positive integer solution (u, v). Moreover, equation (3.1) has a positive integer solution (u, v) only when $A \equiv 3 \pmod 8$ and equation (3.2) has a positive integer solution (u, v) only if $A \equiv p + 2 \pmod 8$.

CASE 2: x IS EVEN. Put $x = 2z$. As before, since p is square-free, replacing $y/2p$ by w we have

$$pw^2 = z(2Az^2 - 1).$$

Then there are positive integers u, v such that

$$z = pu^2, \quad 2Az^2 - 1 = v^2,$$

that is

$$2Ap^2u^4 - v^2 = 1, \tag{3.3}$$

or

$$z = u^2, \quad 2Az^2 - 1 = pv^2,$$

so

$$2Au^4 - pv^2 = 1. \tag{3.4}$$

From Lemma 2.1, equation (3.3) has at most one positive integer solution (u, v). Also using Lemma 2.2, one can see that equation (3.4) has at most two positive integer solutions (u, v). Moreover, equation (3.3) has a positive integer solution (u, v) only if $A \equiv 1 \pmod 8$ and equation (3.4) has a positive integer solution (u, v) only when $2A \equiv p + 1 \pmod 8$.

Therefore we consider the following four subcases:

SUBCASE 1: $A \equiv 1 \pmod 8$. In this first subcase, equation (3.3) has at most one solution, equation (3.1) has no solutions, equation (3.2) has a solution only when $p \equiv 7 \pmod 8$, and equation (3.4) has a solution only if $p \equiv 1 \pmod 8$. It follows that if $A \equiv 1 \pmod 8$, then equation (1.1) has at most two solutions when $p \equiv 7 \pmod 8$ and equation (1.1) has at most three solutions when $p \equiv 1 \pmod 8$.

SUBCASE 2: $A \equiv 3 \pmod 8$. Here equation (3.1) has at most one solution, equation (3.3) has no solutions, equation (3.2) has a solution only when $p \equiv 1 \pmod 8$, and equation (3.4) has a solution only when $p \equiv 5 \pmod 8$. Therefore, if $A \equiv 3 \pmod 8$, then equation (1.1) has at most two solutions when $p \equiv 1 \pmod 8$ and equation (1.1) has at most three solutions when $p \equiv 5 \pmod 8$.

SUBCASE 3: $A \equiv 5 \pmod 8$. Then, both equations (3.1) and (3.3) have no solutions, equation (3.2) has a solution only if $p \equiv 3 \pmod 8$, equation (3.4) has a solution only when $p \equiv 1 \pmod 8$. Thus, if $A \equiv 5 \pmod 8$, then equation (1.1) has at most one solution when $p \equiv 3 \pmod 8$ and equation (1.1) has at most two solutions when $p \equiv 1 \pmod 8$.

SUBCASE 4: $A \equiv 7 \pmod 8$. In the last subcase, both equations (3.1) and (3.3) have no solutions, equation (3.2) has a solution only when $p \equiv 5 \pmod 8$, and equation (3.4) has a solution only when $p \equiv 5 \pmod 8$. It follows that if $A \equiv 7 \pmod 8$, then equation (1.1) has at most three solutions when $p \equiv 5 \pmod 8$.

Therefore, we conclude as follows: If $(A,p) \equiv (3,1), (1,7), (5,3) \pmod 8$, then equation (1.1) has at most two positive integer solutions (x,y). If $(A,p) \equiv (3,5), (1,1), (7,5), (5,1) \pmod 8$, then equation (1.1) has at most three positive integer solutions (x,y). If $(A,p) \equiv (3,3), (3,7), (1,5), (1,3) \pmod 8$, then equation (1.1) has at most one positive integer solution (x,y). Moreover equation (1.1) has no solutions in the other cases.

4. Final remarks

In Table 1 below, we list the result of our computations done by Magma ([5]). To do these computations, we first transform equation (1.1) into the form

$$y^2 = pAx^3 - 2px. \tag{4.1}$$

Multiplying both sides by p^2A^2, we get

$$V^2 = U^3 - 2p^2AU, \tag{4.2}$$

where $U = pAx, V = pAy$. Using Magma, we determine the rational points (U,V) on the elliptic curve (4.2), then we compute the corresponding values of x, y which should be positive integers. It took a few seconds to obtain the solutions of each equation. Here, we remind the reader that

type (i): $(A,p) \equiv (3,1), (1,7), (5,3) \pmod 8$;

type (ii): $(A,p) \equiv (3,5), (1,1), (7,5), (5,1) \pmod 8$;

type (iii): $(A,p) \equiv (3,3), (3,7), (1,5), (1,3) \pmod 8$.

Notice that if $A = p+2$, then equation (1.1) has the solution $(x,y) = (1,p)$. Also if $2A-1 = p$, then equation (1.1) has the solution $(x,y) = (2,2p)$. Therefore, we make the following conjecture.

Conjecture 4.1. *Diophantine equation* (1.1) *has at most one positive integer solution.*

To give some comments on the above conjecture, we first recall the following conjecture of Walsh ([9]) on the diophantine equation

$$AX^4 - BY^2 = 1.$$

Conjecture 4.2. *Let $t > 1$ denote a positive integer. Then the only positive integer solution to*

$$(t+1)X^4 - tY^2 = 1$$

is $(X,Y) = (1,1)$, unless $t = m^2 + m$ for some positive integer m, in which case there is also the solution $(X,Y) = (2m+1, 4m^2+4m+3)$.

Since $m^2 + m + 1$ is odd and $2A$ is even, Conjecture 4.2 implies that equation (3.4) has at most one positive integer solution. To prove Conjecture 4.1, first we must prove that for given $A > 0$, p, where p is an odd prime, equation (3.4) has at most one positive integer solution, which is a special

Type	(A,p)	(U,V)	Solutions (x,y)
(i)	$(3,41)$	$(0,0),(2,142),(5043,358053)$	$(41,2911)$
	$(25,7)$	$(0,0),(49,49),(50,50),$	
		$(56,196),(1575,62475)$	$(9,357)$
	$(21,19)$	$(0,0),(38,722),(175,1645),$	
		$(343,5929),(399,7581)$	$(1,19)$
	$(25,23)$	$(0,0),(-46,1058),(575,13225)$	$(1,23)$
	$(33,31)$	$(0,0),(62,1922),(1023,31713),$	
		$(16399,2099785)$	$(1,31)$
	$(43,41)$	$(0,0),(82,3362),(387,1419),(1763,72283)$	$(1,41)$
	$(5,11)$	$(0,0)$	None
	$(3,73)$	$(0,0),(243,2565)$	None
	$(9,23)$	$(0,0),(184,2116)$	None
	$(5,347)$	$(0,0),(43375,9030675)$	None
(ii)	$(7,5)$	$(0,0),(10,50),(35,175)$	$(1,5)$
	$(15,13)$	$(0,0),(26,338),(195,2535)$	$(1,13)$
	$(31,29)$	$(0,0),(58,1682),(899,26071)$	$(1,29)$
	$(39,37)$	$(0,0),(312,1716),(74,2738)$	
		$(675,15345),(1443,53391),(5547,412413)$	$(1,37)$
	$(7,13)$	$(0,0),(13,169),(182,2366)$	$(2,26)$
	$(9,17)$	$(0,0),(72,36),(17,289),(306,5202)$	$(2,34)$
	$(15,29)$	$(0,0),(120,1140),(29,841),(5,355)$	
		$(294,4242),(870,25230),(5046,358266)$	$(2,38)$
	$(11,5)$	$(0,0),(99,957)$	None
	$(9,73)$	$(0,0),(72,2556)$	None
	$(15,5)$	$(0,0),(54,342)$	None
	$(13,17)$	$(0,0),(25,415)$	None
(iii)	$(11,11)$	$(0,0),(50,90)$	None
	$(3,7)$	$(0,0)$	None
	$(17,29)$	$(0,0)$	None
	$(9,3)$	$(0,0),(9,27),(8,28),(18,54),(3042,167778)$	None

TABLE 1. Examples

case of Conjecture 4.2. Next, from the proof in Section 3, we must show that equations (3.1) and (3.2), (3.1) and (3.4), (3.2) and (3.4), (3.3) and (3.4) cannot have positive integer solutions simultaneously, which seems to be very difficult.

ACKNOWLEDGEMENTS.

The authors are deeply grateful to the referees for their many valuable suggestions to improve the manuscript. The work on this paper was completed during a very enjoyable visit of first author at *l'Institut de Mathématiques de Bordeaux*. He thanks the people of this institution for their hospitality.

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Received: 29.4.2010.
Revised: 25.8.2010. & 20.9.2010. & 5.11.2010.