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IDENTITIES WITH DERIVATIONS IN RINGS

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Abstract. In this paper we investigate identities with derivations in rings. We prove, for example the following result. Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $6nm^2(2m+n-3)!$-torsion free semiprime ring. Suppose there exists a derivation $D : R \to R$ satisfying the relation $[[D(x^m), x^n], D(x^m)] = 0$ for all $x \in R$. In this case D maps R into its center.

Throughout, R will represent an associative ring with center $Z(R)$. Let $n > 1$ be an integer. A ring R is n-torsion free in case $nx = 0$, $x \in R$, implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. Recall that a ring R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = \{0\}$ implies that $a = 0$. We denote by $char(R)$ the characteristic of a prime ring R and by Q the maximal right ring of quotients of a semiprime ring R. For the explanation of maximal right ring of quotients we refer the reader to [1]. An additive mapping D is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A derivation $D : R \to R$ is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. A mapping $f : R \to R$, where R is an arbitrary ring, is called centralizing on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$. In a special case when $[f(x), x] = 0$ is fulfilled for all $x \in R$ a mapping f is called commuting on R. A classical result of Posner (Posner's second theorem, see [7]) asserts that the existence of a nonzero derivation $D : R \to R$, where R is a prime ring, which is centralizing on R, forces R to be commutative. Posner's second theorem in general cannot be proved for semiprime rings as shows the following example. Take prime rings R_1, R_2, where R_1 is commutative, and set $R = R_1 \oplus R_2$. Let $D_1 : R_1 \to R_1$

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be a nonzero derivation. A mapping $D : R \to R$, defined by $D((r_1, r_2)) = (D_1(r_1), 0)$, is then a nonzero commuting derivation. It is well known and easy to prove that if D is a commuting derivation on a semiprime ring R, then D maps R into $Z(R)$. From the result of Deng and Bell ([4]) it follows that if R is a $n!$-torsion free semiprime ring, where $n > 1$ is some fixed integer, and $D : R \to R$ a derivation satisfying the relation $[D(x), x^n] = 0$ for all $x \in R$, then D maps R into $Z(R)$ (see also [6]). Vukman ([9]) has proved the following result. Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $2mn(m+n-1)!$-torsion free semiprime ring. Suppose that there exist derivations $D, G : R \to R$ such that $D(x^m)x^n + x^n G(x^m) = 0$ is fulfilled for all $x \in R$. In this case D and G map R into $Z(R)$. Moreover, $D + G = 0$. Recently this result has been fairly generalized by A. Fošner, M. Fošner, and Vukman ([5]).

If D is a derivation on a noncommutative prime ring R into itself, such that the mapping $x \mapsto D(x)^2$ is commuting on R, then one cannot prove in general that $D = 0$ as shows the following example. Let R be the ring of all 2×2 matrices and let $D(x) = [a, x], a \notin Z(R)$, be an inner derivation. Then a simple calculation shows that the mapping $x \mapsto D(x)^2$ is commuting on R, but $D \neq 0$ since $a \notin Z(R)$. We are talking about the relation $[D(x)^2, x] = 0, x \in R$, which can be written in the form

$$[D(x), x]D(x) + D(x)[D(x), x] = 0, \quad x \in R.$$

The question arises what can be proved in the case we have a Lie version of the above relation. More precisely, in this paper we investigate the following identity

$$[[D(x), x], D(x)] = 0, \quad x \in R.$$

Now we are in the position to write our first result.

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be a derivation satisfying the relation*

$$[[D(x), x], D(x)] = 0$$

for all $x \in R$. In this case D maps R into $Z(R)$.

The proof of the result above is, as we shall see, rather long but it is elementary in the sense that it requires no specific knowledge concerning semiprime rings in order to follow the proof. For the proof of Theorem 1 we shall need lemmas below.

Lemma 2. *[2, Lemma 4] Let R be a 2-torsion free semiprime ring and let $a, b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.*

Lemma 3. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be a derivation, such that the mapping $x \mapsto D^2(x)$ is commuting on R. In this case D is commuting on R.*

Proof. The result of the lemma is a special case of Theorem 4 in [8]. $\square$

Proof of Theorem 1. Let us introduce a mapping $B(.,.) : R \times R \to R$ by the relation

$$B(x,y) = [D(x), y] + [D(y), x], \quad x, y \in R.$$

Obviously, the mapping $B(.,.)$ is symmetric (i.e., $B(x,y) = B(y,x)$ for all pairs $x, y \in R$) and additive in both arguments. Moreover, a simple calculation shows that for all $x, y, z \in R$ the following relation holds

$$B(xy, z) = B(x,z)y + xB(y,z) + D(x)[y,z] + [x,z]D(y). \tag{1}$$

We shall write $f(x)$ for $B(x,x)$. Then

$$f(x) = 2[D(x), x], \quad x \in R. \tag{2}$$

It is easy to see that

$$f(x+y) = f(x) + f(y) + 2B(x,y) \tag{3}$$

is fulfilled for pairs $x, y \in R$. Throughout the proof we shall use the relations (1), (2) and (3) without specific reference. The assumption of the theorem can now be written as follows

$$[f(x), D(x)] = 0, \quad x \in R. \tag{4}$$

Let us prove that

$$[[f(x), x], D(x)] = 0, \quad x \in R. \tag{5}$$

We have

$$\begin{aligned} 0 &= 2[[f(x), D(x)], x] \\ &= 2[f(x), x]D(x) + f(x)^2 - f(x)^2 - 2D(x)[f(x), x] \\ &= 2[[f(x), x], D(x)] \end{aligned}$$

which proves the above relation. The linearization of the relation (4) gives

$$[f(y), D(x)] + 2[B(x,y), D(x)] + [f(x), D(y)] + 2[B(x,y), D(y)] = 0, \quad x, y \in R.$$

Replacing $-x$ for x and comparing so obtained relation with the above relation we obtain

$$[f(x), D(y)] + 2[B(x,y), D(x)] = 0, \quad x, y \in R. \tag{6}$$

The substitution yz for y in the above relation gives

$$\begin{aligned}
0 &= [f(x), D(yz)] + 2[B(yz, x), D(x)]\\
&=[f(x), D(y)z + yD(z)] + 2[B(y, x)z\\
&\quad + yB(z, x) + D(y)[z, x] + [y, x]D(z), D(x)]\\
&= [f(x), D(y)]z + D(y)[f(x), z] + [f(x), y]D(z) + y[f(x), D(z)]\\
&\quad + 2[B(y, x), D(x)]z + 2B(y, x)[z, D(x)]\\
&\quad + 2[y, D(x)]B(z, x) + 2y[B(z, x), D(x)]\\
&\quad + 2[D(y), D(x)][z, x] + 2D(y)[[z, x], D(x)]\\
&\quad + 2[[y, x], D(x)]D(z) + 2[y, x][D(z), D(x)].
\end{aligned}$$

According to the relation (6) the above relation reduces to

$$\begin{aligned}
0 &= D(y)[f(x), z] + [f(x), y]D(z) + 2B(y, x)[z, D(x)]\\
&\quad + 2[y, D(x)]B(z, x) + 2[D(y), D(x)][z, x] + 2D(y)[[z, x], D(x)]\\
&\quad + 2[[y, x], D(x)]D(z) + 2[y, x][D(z), D(x)], \quad x, y, z \in R
\end{aligned} \tag{7}$$

and in particular for $y = x$

$$\begin{aligned}
0 &= D(x)[f(x), y] + [f(x), x]D(y) + 2f(x)[y, D(x)] - f(x)B(y, x)\\
&\quad + 2D(x)[[y, x], D(x)], \quad x, y \in R.
\end{aligned} \tag{8}$$

The substitution yx for y in (8) gives

$$\begin{aligned}
0 &= D(x)[f(x), yx] + [f(x), x]D(yx)\\
&\quad + 2f(x)[yx, D(x)] - f(x)B(yx, x) + 2D(x)[[y, x]x, D(x)]\\
&= D(x)[f(x), y]x + D(x)y[f(x), x] + [f(x), x]D(y)x\\
&\quad + [f(x), x]yD(x) + 2f(x)[y, D(x)]x + 2f(x)y[x, D(x)]\\
&\quad - f(x)B(y, x)x - f(x)yf(x) - f(x)[y, x]D(x)\\
&\quad + 2D(x)[[y, x], D(x)]x + 2D(x)[y, x][x, D(x)], \quad x, y \in R
\end{aligned}$$

which reduces according to the relation (8) to

$$\begin{aligned}
0 &= D(x)y[f(x), x] + [f(x), x]yD(x) - 2f(x)yf(x)\\
&\quad - f(x)[y, x]D(x) - D(x)[y, x]f(x), \quad x, y \in R.
\end{aligned} \tag{9}$$

The substitution $D(x)y$ for y gives

$$\begin{aligned}
0 &= D(x)^2y[f(x), x] + [f(x), x]D(x)yD(x) - 2f(x)D(x)yf(x)\\
&\quad - f(x)[D(x)y, x]D(x) - D(x)[D(x)y, x]f(x)\\
&= D(x)^2y[f(x), x] + [f(x), x]D(x)yD(x) - 2f(x)D(x)yf(x)\\
&\quad - f(x)[D(x), x]yD(x) - f(x)D(x)[y, x]D(x)\\
&\quad - D(x)[D(x), x]yf(x) - D(x)^2[y, x]f(x).
\end{aligned}$$

We gave therefore

$$
\begin{aligned}
0 = {} & D(x)^2 y[f(x),x] + [f(x),x]D(x)yD(x) \\
& - 2f(x)D(x)yf(x) - f(x)[D(x),x]yD(x) \\
& - f(x)D(x)[y,x]D(x) - D(x)[D(x),x]yf(x) \\
& - D(x)^2[y,x]f(x), \quad x,y \in R.
\end{aligned} \tag{10}
$$

Left multiplication of the relation (9) by $D(x)$ gives

$$
\begin{aligned}
0 = {} & D(x)^2 y[f(x),x] + D(x)[f(x),x]yD(x) \\
& - 2D(x)f(x)yf(x) - D(x)f(x)[y,x]D(x) \\
& - D(x)^2[y,x]f(x), \quad x,y \in R.
\end{aligned} \tag{11}
$$

Subtracting the relation (11) from the relation (10) we obtain

$$
\begin{aligned}
0 = {} & [[f(x),x],D(x)]yD(x) - 2[f(x),D(x)]yf(x) \\
& - f(x)[D(x),x]yD(x) - [f(x),D(x)][y,x]D(x) \\
& - D(x)[D(x),x]yf(x), \quad x,y \in R,
\end{aligned}
$$

which reduces because of (4) and (5) to

$$f(x)^2 yD(x) + D(x)f(x)yf(x) = 0, \quad x,y \in R. \tag{12}$$

Right multiplication of the relation (12) by $f(x)$ gives

$$f(x)^2 yD(x)f(x) + D(x)f(x)yf(x)^2 = 0, \quad x,y \in R,$$

whence it follows according to Lemma 2 that

$$f(x)^2 yD(x)f(x) = 0 \tag{13}$$

holds for all pairs $x, y \in R$. The substitution $yD(x)$ for y in the relation (12) gives

$$f(x)^2 yD(x)^2 + D(x)f(x)yD(x)f(x) = 0, \quad x,y \in R.$$

Let us replace in the above relation y by $yD(x)f(x)zD(x)f(x)y$. Then we have

$$
\begin{aligned}
& f(x)^2 yD(x)f(x)zD(x)f(x)yD(x)^2 \\
& + D(x)f(x)yD(x)f(x)zD(x)f(x)yD(x)f(x) = 0, \quad x,y,z \in R,
\end{aligned}
$$

which reduces according of the relation (13) to

$$D(x)f(x)yD(x)f(x)zD(x)f(x)yD(x)f(x) = 0, \quad x,y,z \in R,$$

whence it follows

$$D(x)f(x) = 0, \quad x \in R, \tag{14}$$

according to the semiprimeness of R. Of course we also have

$$f(x)D(x) = 0, \quad x \in R. \tag{15}$$

This yields

$$f(x)D(y) + 2B(x,y)D(x) = 0, \quad x,y \in R. \tag{16}$$

Let us write in the above relation yz instead of y. Then we have

$$\begin{aligned} 0 &= f(x)D(yz) + 2B(x,yz)D(x) = f(x)D(y)z + f(x)yD(z) \\ &\quad + 2B(x,y)zD(x) + 2yB(x,z)D(x) + 2[y,x]D(z)D(x) \\ &\quad + 2D(y)[z,x]D(x), \quad x,y,z \in R. \end{aligned}$$

According to the relation (16) one can replace in the above equation $2yB(x,z)D(x)$ by $-yf(x)D(z)$ and $f(x)D(y)z$ by $-2B(x,y)D(x)z$ which gives

$$\begin{aligned} 0 &= [f(x),y]D(z) + 2B(x,y)[z,D(x)] + 2[y,x]D(z)D(x) \\ &\quad + 2D(y)[z,x]D(x), \quad x,y,z \in R. \end{aligned} \tag{17}$$

In particular, for $z = D(x)$ the relation (17) reduces according to (15) to

$$[f(x),y]D^2(x) + 2[y,x]D^2(x)D(x) = 0, \quad x,y \in R.$$

The substitution xy for y gives

$$[f(x),x]yD^2(x) = 0, \quad x,y \in R. \tag{18}$$

For $z = x$ the relation (17) reduces because of (15) to

$$f(x)yD(x) - B(x,y)f(x) + 2[y,x]D(x)^2 = 0, \quad x,y \in R. \tag{19}$$

Putting in the above relation yx for y we obtain

$$\begin{aligned} 0 &= f(x)yxD(x) - B(x,yx)f(x) + 2[yx,x]D(x)^2 \\ &= f(x)yxD(x) - B(x,y)xf(x) - yf(x)^2 - [y,x]D(x)f(x) \\ &\quad + 2[y,x]xD(x)^2, \quad x,y \in R, \end{aligned}$$

which reduces according to the relation (15) to

$$f(x)yxD(x) - B(x,y)xf(x) - yf(x)^2 + 2[y,x]xD(x)^2 = 0, \quad x,y \in R. \tag{20}$$

On the other hand right multiplication of the relation (19) by x gives

$$f(x)yD(x)x - B(x,y)f(x)x + 2[y,x]D(x)^2x = 0, \quad x,y \in R. \tag{21}$$

Subtracting (20) from (21) we obtain

$$\begin{aligned} &f(x)y[D(x),x] - B(x,y)[f(x),x] + yf(x)^2 \\ &+2[y,x]([D(x),x]D(x) + D(x)[D(x),x]) = 0, \quad x,y \in R, \end{aligned}$$

which reduces, according to (14) and (15), to

$$f(x)y[D(x),x] - B(x,y)[f(x),x] + yf(x)^2 = 0, \quad x,y \in R.$$

Right multiplication of the above relation by $zD^2(x)$ gives, according to (18),

$$f(x)yf(x)zD^2(x) + 2yf(x)^2zD^2(x) = 0, \quad x,y,z \in R. \tag{22}$$

Left multiplication of the relation (22) by $D(x)$ gives, because of (14),

$$2D(x)yf(x)^2zD^2(x)=0, \quad x,y,z\in R. \tag{23}$$

Putting in the relation above first xy for y then multiplying the same relation from the left side by x and subtracting the relations so obtained one from another we obtain

$$f(x)yf(x)^2zD^2(x)=0, \quad x,y,z\in R.$$

The substitution $f(x)zD^2(x)y$ for y in the above relation gives

$$(f(x)^2zD^2(x))y(f(x)^2zD^2(x))=0, \quad x,y,z\in R,$$

which gives

$$f(x)^2zD^2(x)=0, \quad x,z\in R,$$

which reduces the relation (22) to

$$f(x)yf(x)zD^2(x)=0, \quad x,y,z\in R.$$

The substitution $zD^2(x)y$ for y in the above relation gives

$$(f(x)zD^2(x))y(f(x)zD^2(x))=0, \quad x,y,z\in R,$$

whence it follows

$$f(x)yD^2(x)=0, \quad x,y\in R. \tag{24}$$

This yields

$$([D(x),z]+[D(z),x])yD^2(x)+[D(x),x]yD^2(z)=0, \quad x,y,z\in R.$$

Putting $z=D(x)$ in the above identity we obtain

$$[D^2(x),x]yD^2(x)+[D(x),x]yD^3(x)=0, \quad x,y\in R.$$

The substitution $zD^2(x)y$ for y in the above relation gives according to the relation (24)

$$[D^2(x),x]zD^2(x)yD^2(x)=0, \quad x,y,z\in R.$$

The substitution $y[D^2(x),x]z$ for y in the above relation gives

$$([D^2(x),x]zD^2(x))y([D^2(x),x])zD^2(x))=0, \quad x,y,z\in R,$$

which gives

$$[D^2(x),x]yD^2(x)=0, \quad x,y\in R,$$

because of the semiprimeness of R. Putting in the relation above first yx for y then multiplying the relation above from the right side by x and then subtracting the relations so obtained one from another we obtain

$$[D^2(x),x]y[D^2(x),x]=0, \quad x,y\in R.$$

which gives

$$[D^2(x),x]=0, \quad x\in R,$$

because of the semiprimeness of R. According to Lemma 3 one can conclude that $[D(x), x] = 0, \quad x \in R$. It is well known and easy to prove that a commuting derivation on semiprime ring maps the ring into its center (see, for example, the proof of Theorem 2.1 in [9]). The proof of the theorem is completed. ∎

In the proof of the theorem below as well as in the proof of Theorem 7 we shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see Theorem 3 in [6]). For the explanation of differential identities we refer the reader to [3]. Let $m \geq 1, n \geq 1$ be some fixed integers, R a $2mn(m+n-1)!$-torsion free semiprime ring, and let $D : R \to R$ be a derivation satisfying the relation $[D(x^m), x^n] = 0$ for all $x \in R$. A special case of Theorem 2.1 in [9] states that in this case D maps R into $Z(R)$. Obviously, Theorem 4 generalizes Theorem 1 as well as the result we have just mentioned.

Theorem 4. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $6nm^2(2m+n-3)!$-torsion free semiprime ring. Suppose there exists a derivation $D : R \to R$ satisfying the relation*

$$[[D(x^m), x^n], D(x^m)] = 0 \tag{25}$$

for all $x \in R$. In this case D maps R into $Z(R)$.

Proof. Applying complete linearization of the relation (25) we obtain

$$0 = \sum_{\pi \in S_{2m+n}} [[D(x_{\pi(1)} \dots x_{\pi(m)}), \dots x_{\pi(m+n)}], D(x_{\pi(m+n+1)} \dots x_{\pi(2m+n)})]$$

for all $x_1, x_2, \dots, x_{2m+n} \in R$. According to Theorem 3 in [6] the above relation holds for all $x_1, x_2, \dots, x_{2m+n} \in Q$ as well. Substituting

$$x_1 = x_2 = x_3 = x, \qquad x_4 = \dots = x_{2m+n} = 1,$$

where 1 denotes the identity element, and applying the fact that $D(1) = 0$ we obtain $[[D(x), x], D(x)] = 0$ for all $x \in R$, whence one can conclude that D maps R into $Z(R)$ according to Theorem 1. ∎

Corollary 5. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $6nm^2(2m+n-3)!$-torsion free noncommutative semiprime ring. Suppose there exists an inner derivation $D : R \to R$ satisfying the relation*

$$[[D(x^m), x^n], D(x^m)] = 0$$

for all $x \in R$. In this case $D = 0$.

Proof. An immediate consequence of Theorem 4 and Corollary 5 in [8]. ∎

Corollary 6. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a prime ring with*

$$6nm^2(2m+n-3)! < char(R).$$

Suppose there exists a nonzero derivation $D : R \to R$ satisfying the relation

$$[[D(x^m), x^n], D(x^m)] = 0$$

for all $x \in R$. In this case R is commutative.

Proof. An immediate consequence of Theorem 4 and Posner's second theorem. ∎

We are now ready for our next result.

Theorem 7. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $6nm^2(2m+n-3)!$-torsion free semiprime ring. Suppose there exists a derivation $D : R \to R$ satisfying the relation*

$$(D(x^m)x^n)^2 + (x^nD(x^m))^2 = D(x^m)^2x^{2n} + x^{2n}D(x^m)^2, \tag{26}$$

for all $x \in R$. In this case D maps R into $Z(R)$.

Proof. Applying complete linearization of the relation (26) we obtain

$$\begin{aligned}
&\sum_{\pi \in S_{2m+2n}} D(x_{\pi(1)} \dots x_{\pi(m)})x_{\pi(m+1)} \dots x_{\pi(m+n)} \cdot \\
&\qquad D(x_{\pi(m+n+1)} \dots x_{\pi(2m+n)})x_{\pi(2m+n+1)} \dots x_{\pi(2m+2n)} \\
&\quad + \sum_{\pi \in S_{2m+2n}} x_{\pi(1)} \dots x_{\pi(n)}D(x_{\pi(n+1)} \dots x_{\pi(n+m)}) \cdot \\
&\qquad x_{\pi(n+m+1)} \dots x_{\pi(2n+m)}D(x_{\pi(2n+m+1)} \dots x_{\pi(2m+2n)}) \\
&= \sum_{\pi \in S_{2m+2n}} D(x_{\pi(1)} \dots x_{\pi(m)})D(x_{\pi(m+1)} \dots x_{\pi(2m)}) \cdot \\
&\qquad x_{\pi(2m+1)} \dots x_{\pi(2m+n)}x_{\pi(2m+n+1)} \dots x_{\pi(2m+2n)} \\
&\quad + \sum_{\pi \in S_{2m+2n}} x_{\pi(1)} \dots x_{\pi(n)}x_{\pi(2m+n+1)} \dots x_{\pi(2n)} \cdot \\
&\qquad D(x_{\pi(2n+1)} \dots x_{\pi(2n+m)})D(x_{\pi(2n+m+1)} \dots x_{\pi(2m+2n)})
\end{aligned}$$

for all $x_1, x_2, \dots, x_{2m+2n} \in R$. According to Theorem 3 in [6] the above relation holds for all $x_1, x_2, \dots, x_{2n+2m} \in Q$ as well. Substituting

$$x_1 = x_2 = x_3 = x, \qquad x_4 = \ldots = x_{2m+2n} = 1,$$

where 1 denotes the identity element, and applying the fact that $D(1) = 0$ we obtain

$$2D(x)xD(x) = xD(x)^2 + D(x)^2x$$

for all $x \in Q$, which can be written in the form

$$[[D(x), x], D(x)] = 0$$

for all $x \in R$. According to Theorem 1 one can conclude that D maps R into $Z(R)$. ∎

Corollary 8. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a $6nm^2(2m+n-3)!$-torsion free noncommutative semiprime ring. Suppose there exists an inner derivation $D : R \to R$ satisfying the relation*

$$(D(x^m)x^n)^2 + (x^n D(x^m))^2 = D(x^m)^2 x^{2n} + x^{2n} D(x^m)^2,$$

for all $x \in R$. In this case $D = 0$.

Proof. An immediate consequence of Theorem 7 and Corollary 5 in [8]. ∎

Corollary 9. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a prime ring with*

$$6nm^2(2m+n-3)! < char(R).$$

Suppose there exists a nonzero derivation $D : R \to R$ satisfying the relation

$$(D(x^m)x^n)^2 + (x^n D(x^m))^2 = D(x^m)^2 x^{2n} + x^{2n} D(x^m)^2,$$

for all $x \in R$. In this case R is commutative.

Proof. An immediate consequence of Theorem 7 and Posner's second theorem. ∎

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