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# FINITE GROUPS WITH A NONLINEAR IRREDUCIBLE CHARACTER HAVING FEW ZEROS

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Abstract. We classify the finite groups with a nonlinear irreducible character $\chi$ such that the number of zeros $\mathrm{n}_\chi$ of $\chi$ satisfies a certain relation.

Let

- $G$ be a finite group,
- $\mathrm{Irr}(G)$ be the set of irreducible characters of $G$,
- $\mathrm{Lin}(G)$ be the set of linear characters of $G$,
- $\mathrm{Irr}_1(G)$ be the set of nonlinear irreducible characters of $G$,
- $\chi \in \mathrm{Irr}(G)$,
- $\mathrm{U}_\chi = \{g \in G \mid |\chi(g)| = 1\}$, the set of *unitary elements* of $\chi$ (it is easy to check that the set $\mathrm{U}_\chi$ is invariant under $G$-conjugation),
- $\mathrm{T}_\chi = \{g \in G \mid \chi(g) = 0\}$, the set of *zeros* of $\chi$ (it is easy to check that the set $\mathrm{T}_\chi$ is invariant under $G$-conjugation),
- $\mathrm{N}_\chi = \langle \mathrm{T}_\chi \rangle$, the subgroup generated by zeros of $\chi$ (by the previous paragraph, $\mathrm{N}_\chi$ is a normal subgroup of $G$),
- $\mathrm{n}_\chi = |\mathrm{T}_\chi|$, the number of zeros of $\chi$,
- $\tau_\chi(H) = |\mathrm{T}_\chi \cap H|$, the number of zeros of $\chi$ on a given subgroup $H$ of $G$,
- $\mathrm{Z}(\chi) = \{x \in G \mid |\chi(x)| = \chi(1)\}$ ($\mathrm{Z}(\chi)$ is normal in $G$ and $\mathrm{Z}(\chi)/\ker(\chi) = \mathrm{Z}(G/\ker(\chi))$.

A classical theorem by Burnside asserts that if $\chi \in \mathrm{Irr}_1(G)$, then the set $\mathrm{T}_\chi \neq \emptyset$.

One of the main results of this note is the following

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Professor E.M. Zhmud (1918–2007) wrote this paper about 10 years ago. Y. Berkovich prepared this paper using Zhmud's letter.

Theorem A. *If $\chi \in \mathrm{Irr}_1(G)$ and $g \in \mathrm{T}_\chi$ is of order $m$, then*

$$\mathrm{n}_\chi \geq m + \varphi(m),$$

*where $\varphi$ is Euler's totient function.*

In Theorem A, as $g$, one can take an element of maximal order in the set $\mathrm{T}_\chi$.

Definition 1. *A group $G$ is said to be an $\mathcal{A}$-*group *provided it satisfies*

$$(\mathcal{A}) \qquad \mathrm{n}_\chi = m + \varphi(m)$$

*for some $\chi \in \mathrm{Irr}(G)$ and $g \in \mathrm{T}_\chi$ of order $m$. A character $\chi$ satisfying the condition $(\mathcal{A})$, is said to be an $\mathcal{A}$-*character.

Example: $\mathrm{S}_3$, the symmetric group of degree 3 is an $\mathcal{A}$-group with respect to $\chi \in \mathrm{Irr}_1(G)$ (of degree 2) and $g \in \mathrm{S}_3$ of order $m = 2$ (all elements of the set $\mathrm{T}_\chi$ have the same order 2).

The $\mathcal{A}$-groups are classified in the following

Theorem B. *If $G$ is an $\mathcal{A}$-group with respect to $\chi \in \mathrm{Irr}_1(G)$ and $g \in \mathrm{T}_\chi$ of order $m$, then $\chi(1) = 2$ and one of the following holds:*

(a) *$G$ is a nonabelian 2-group of order $2^{\lambda+1}$ with a cyclic subgroup of index 2.*
(b) *$G = \langle a, b \mid a^3 = b^{2^\lambda} = 1,\ a^b = a^{-1}\rangle$, $\lambda \geq 1$.*
(c) *$G \cong \mathrm{SL}(2, 3)$.*

In part (a) of Theorem B, we have two cases. (i) If $G$ is of maximal class, then $\chi$ has kernel of index 8 in $G$, $\mathrm{T}_\chi = G - \Phi(G)$, $m = 2^\lambda$ and $\mathrm{n}_\chi = 2^\lambda + 2^{\lambda-1}$. (ii) Now let $G = \langle a, b \mid a^\lambda = b^2 = 1,\ a^b = a^{1+2^{\lambda-1}}\rangle$. Here $\chi$ is faithful, $\mathrm{T}_\chi = G - \Phi(G)$, $m = 2^\lambda$, $\mathrm{n}_\chi = 2^\lambda + 2^{\lambda-1}$.

In part (b) of Theorem B, $\chi = \tau^G$, where $\tau \in \mathrm{Lin}(\mathrm{C}_G(G'))$ is faithful, $\mathrm{T}_\chi = G - \mathrm{C}_G(G')$, $m = 2^\lambda$ and $\mathrm{n}_\chi = |G - \mathrm{C}_G(G')| = 2^\lambda + 2^{\lambda-1}$.

In part (c) of Theorem B, $\chi$ is faithful ($G$ has exactly three faithful irreducible characters and each of them can be taken as $\chi$), $\mathrm{T}_\chi = G' - \mathrm{Z}(G')$, $m = 4$, $\mathrm{n}_\chi = 4 + 2$.

Theorem B follows from a long series of lemmas.

The following theorem is cited many times in what follows.

Theorem 2 (A. I. Veitsblit; see [BZ, Theorem 21.1]). *If $\chi \in \mathrm{Irr}_1(G)$ and $H \leq G$, then*

$$(*) \qquad \langle \chi_H, \chi_H \rangle \leq 1 + \frac{|\mathrm{T}_\chi - H|}{|H|}$$

*with equality if and only if $G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$.*

Note that $|\mathrm{T}_\chi - H| = |\mathrm{T}_\chi| - |\mathrm{T}_\chi \cap H| = \mathrm{n}_\chi - \tau_\chi(H)$.

In what follows $\chi$ is a fixed nonlinear irreducible character of the group $G$.

Lemma 3. *If $H < G$, $\chi \in \mathrm{Irr}_1(G)$ and $\chi_H \notin \mathrm{Irr}(H)$, then $\mathrm{n}_\chi \geq |H| + \tau_\chi(H)$ with equality if and only if*

$$(1) \qquad (i)\ \langle \chi_H, \chi_H \rangle = 2, \qquad (ii)\ G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi.$$

Proof. See [BZ, Lemma 21.13]. ∎

Lemma 4. *Under the hypotheses of Lemma 3, if $g \in G - \mathrm{Z}(\chi)$, then*

$$\mathrm{n}_\chi \geq |\mathrm{C}_G(g)| + \tau_\chi(\mathrm{C}_G(g)).$$

Proof. See [BZ, Corollary 21.14]. ∎

Lemma 5. *Let $\chi \in \mathrm{Irr}_1(G)$, $\chi(1) = 2$. Then*

(a) $\mathrm{T}_\chi = \{g \in G - \mathrm{Z}(\chi) \mid g^2 \in \mathrm{Z}(\chi)\}$.
(b) $\mathrm{U}_\chi = \{g \in G - \mathrm{Z}(\chi) \mid g^3 \in \mathrm{Z}(\chi)\}$.

Proof. Let $\Gamma$ be an irreducible matrix representation affording the character $\chi$ and let $(\alpha, \beta)$ be the spectrum of the matrix $\Gamma(g)$. Then $(\alpha^n, \beta^n)$ is the spectrum of the matrix $\Gamma(g^n) = \Gamma(g)^n$ for every positive integer $n$.

(i) Suppose that $g \in G - \mathrm{Z}(\chi)$ and $g^2 \in \mathrm{Z}(\chi)$. Since $\Gamma(g^2)$ is a scalar matrix, we have $\alpha^2 = \beta^2$ so $\alpha = \pm\beta$. Since $g \notin \mathrm{Z}(\chi)$ (i.e., the matrix $\Gamma(g)$ is non-scalar), we get $\alpha = -\beta$ so that $\chi(g) = \mathrm{tr}(\Gamma(g)) = \alpha + \beta = 0$, i.e., $g \in \mathrm{T}_\chi$.

Conversely, if $g \in \mathrm{T}_\chi$, then $\alpha + \beta = \chi(g) = 0$, i.e., $\alpha = -\beta$. In this case, $\alpha^2 = \beta^2$ so that $\Gamma(g^2)$ is a scalar matrix, i.e., $g^2 \in \mathrm{Z}(\chi)$, and (a) is proven.

(ii) Now suppose that $g \in G - \mathrm{Z}(\chi)$ and $g^3 \in \mathrm{Z}(\chi)$. Then $\alpha^3 = \beta^3$ since $\Gamma(g^3)$ is a scalar matrix. Since $g \notin \mathrm{Z}(\chi)$, we get $\alpha \neq \beta$. Therefore, $\alpha = \epsilon\beta$, where $\epsilon$ is a primitive 3-th root of unity. Since $1 + \epsilon + \epsilon^2 = 0$, we get

$$\chi(g) = \mathrm{tr}(\Gamma(g)) = \alpha + \beta = (1 + \epsilon)\beta = -\epsilon^2\beta.$$

Since $\beta$ is a root of unity (indeed, $\beta^{|G|} = 1$), it follows that $|\chi(g)| = 1$, i.e., $g \in \mathrm{U}_\chi$.

Conversely, suppose that $g \in \mathrm{U}_\chi$. Then $g \in G - \mathrm{Z}(\chi)$ since $\chi(1) > 1$. Since $|\alpha+\beta| = |\chi(g)| = 1$ then, setting $\alpha\beta^{-1} = \epsilon$, we get $1 = |\beta||1+\epsilon| = |1+\epsilon|$ (recall that $\alpha$ and $\beta$ are roots of 1). Let $\epsilon = e^{i\phi}$, where $\phi = \arg(\epsilon)$. Then it follows from

$$1 = |1 + \epsilon|^2 = (1 + \epsilon)(1 + \bar{\epsilon}) = 2 + \epsilon + \bar{\epsilon}$$

that $\cos\phi = -\frac{1}{2}$ so that $\epsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, a primitive third root of 1. Since $\alpha = \epsilon\beta$, then $\alpha^3 = \beta^3$; in this case, $\Gamma(g^3)$ is a scalar matrix so that $g^3 \in \mathrm{Z}(\chi)$, and the proof of (b) is complete. ∎

Definition 6. *Let $\chi$ be an irreducible character of a group $G$. A proper subgroup $H$ of $G$ is said to be* weakly $\chi$-maximal *in $G$ provided $\chi_H \notin \mathrm{Irr}(H)$ and $\mathrm{n}_\chi = |H| + \tau_\chi(H)$ (or, what is the same, $\chi_H \notin \mathrm{Irr}(H)$ and $|H| = |\mathrm{T}_\chi - H|$).*

Lemma 3 shows that the weak $\chi$-maximality of $H < G$ is equivalent (under the assumption $\chi_H \notin \mathrm{Irr}(H)$) to condition (1). It is easy to see that $\chi$-maximal subgroups of $G$ are weakly $\chi$-maximal. Recall [BZ, §21.2] that $H < G$ is $\chi$-maximal if $\chi_H$ is reducible and $|H| = \mathrm{n}_\chi$ (in this case, $\tau_\chi(H) = 0$). It follows from condition (ii) of Lemma 3 that, if $H < G$ is weakly $\chi$-maximal, then $\mathrm{Z}(\chi) \leq H$.

Lemma 7. *If an abelian $H < G$ is weakly $\chi$-maximal with respect to $\chi \in \mathrm{Irr}_1(G)$, then $\chi(1) = 2$ and $g^2 \in \mathrm{Z}(\chi)$ for all $g \in \mathrm{T}_\chi$.*

Proof. By Lemma 3, $\langle \chi_H, \chi_H \rangle = 2$ so $\chi_H = \psi + \psi'$, where $\psi, \psi' \in \mathrm{Lin}(H)$ and $\psi \neq \psi'$ (here we use Frobenius' reciprocity law). Therefore, $\chi(1) = 2$. The second assertion follows from Lemma 5(a). □

Lemma 8. *If $\mathrm{N}_\chi = G$ and $H$ is a weakly $\chi$-maximal subgroup of a group $G$, then $H$ is maximal in $G$.*

Proof. Assume that $H < K < G$. Then $G - K \subset G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$ (Lemma 3). By (∗), we have

$$\langle \chi_K, \chi_K \rangle = 1 + \frac{|\mathrm{T}_\chi - K|}{|K|}.$$

The same holds for $H$ instead of $K$, and this is a contradiction since $0 < |\mathrm{T}_\chi - K| \leq |\mathrm{T}_\chi - H|$ and $|K| > |H|$. Thus, $K$ does not exist so $H$ is maximal in $G$. □

Lemma 9. *Suppose that $g \in \mathrm{T}_\chi$ and $H = \mathrm{C}_G(g)$ weakly $\chi$-maximal in $G$. Then*

(a) $g^2 \in \mathrm{Z}(\chi)$.
(b) *$\chi_H = \psi_1 + \psi_2$, where $\psi_1, \psi_2 \in \mathrm{Irr}(H)$ are distinct of equal degree.*

Proof. We have $\langle \chi_H, \chi_H \rangle = 2$ (Lemma 3) so $\chi_H = \psi_1 + \psi_2$, where $\psi_1, \psi_2 \in \mathrm{Irr}(H)$ are distinct. Let $\Gamma$ be an irreducible representation of $G$ affording the character $\chi$ and $\Gamma_i$ an irreducible representation of $H$ affording the character $\psi_i$, $i = 1, 2$. One can assume that $\Gamma(x) = \mathrm{diag}(\Gamma_1(x), \Gamma_2(x))$ for every $x \in H$. If, in particular, $x = g$, then $\Gamma_i(g) = \alpha_i \mathrm{I}_{n_i}$, where $n_i = \deg(\Gamma_i) = \psi_i(1)$ $(i = 1, 2)$ and $\mathrm{I}_{n_i}$ is the identity $n_i \times n_i$ matrix (recall that $g \in \mathrm{Z}(H)$). Therefore, $\Gamma(g) = \mathrm{diag}(\alpha_1 \mathrm{I}_{n_1}, \alpha_2 \mathrm{I}_{n_2})$. Here $\alpha_1$ and $\alpha_2$ are roots of 1.

It follows from $0 = \chi(g) = n_1\alpha_1 + n_2\alpha_2$ that $n_1 = |n_1\alpha_1| = |-n_2\alpha_2| = n_2$ since $|\alpha_1| = |\alpha_2| = 1$. Thus, $\psi_1(1) = \psi_2(1)$ and $\alpha_1 = -\alpha_2$, so we get $\Gamma(g) = \alpha_1 \mathrm{diag}(\mathrm{I}_{n_1}, -\mathrm{I}_{n_1})$ so $\Gamma(g^2) = \alpha_1^2 \mathrm{I}_n$, where $n = 2n_1 = \deg(\Gamma) = \chi(1)$. Therefore, $g^2 \in \mathrm{Z}(\chi)$, and the proof is complete. □

Lemma 10. *If $g \in \mathrm{T}_\chi$ and $H = \mathrm{C}_G(g)$, then $\mathrm{n}_\chi \geq |H| + \tau_\chi(H)$.*

Proof. Since $g \notin \mathrm{Z}(\chi)$, the character $\chi_H$ is reducible, and the result follows from Lemma 3. ∎

Lemma 11. *If $H \lhd G$ is weakly $\chi$-maximal, then $|G : H| = 2$.*

Proof. As in the proof of Lemma 9, $\langle \chi_H, \chi_H \rangle = 2$ (Lemma 3) so $\chi_H = \psi + \psi'$, where $\psi, \psi' \in \mathrm{Irr}(H)$ are distinct. Since $H \lhd G$, we have $|G : \mathrm{I}_G(\psi)| = 2$ (this follows by Clifford theory; here $\mathrm{I}_G(\psi)$ is the inertia group of $\psi$ in $G$). Since $\chi$ is induced from $H$, we get $\mathrm{T}_\chi \supseteq G - H$; then $\mathrm{N}_\chi = G$. Therefore, $H$ is maximal in $G$ (Lemma 8). Thus, $H = \mathrm{I}_G(\psi)$, and so $|G : H| = |G : \mathrm{I}_G(\psi)| = 2$, and we are done. ∎

Proof of Theorem A. Let $\chi \in \mathrm{Irr}_1(G)$, $g \in \mathrm{T}_\chi$ and $o(g) = m$. Setting $H = \mathrm{C}_G(g)$, we get, by Lemma 10, $\mathrm{n}_\chi \geq |H| + \tau_\chi(H)$. Let $\{\nu_1, \dots, \nu_{\varphi(m)}\}$ be the full reduced residue system $\pmod m$. Then the elements $g^{\nu_i}$ ($i = 1, \dots, \varphi(m)$) are pairwise distinct. It suffices to show that these elements are contained in $\mathrm{T}_\chi$. Let $\epsilon$ be a primitive $|G|$-th root of 1 and $\mathcal{G} = \mathrm{Gal}(Q(\epsilon)/Q)$. There exists $\sigma_i \in \mathcal{G}$ such that $\sigma_i(\epsilon) = \epsilon^{\nu_i}$ ($i = 1, \dots, \varphi(m)$). Since $\chi(g) = 0$, then $\chi(g^{\nu_i}) = \sigma_i(\chi(g)) = 0$ (recall that $\chi(g)$ is a sum of powers of $\epsilon$) so that $g^{\nu_i} \in \mathrm{T}_\chi \cap H$, and we conclude that $\tau_\chi(H) = |H \cap \mathrm{T}_\chi| \geq \varphi(m)$. Next, $|H| \geq o(g) = m$ since $g \in H$. It follows from the inequality in the second sentence of the proof that $\mathrm{n}_\chi \geq m + \varphi(m)$. ∎

Proof of Theorem B. $1^o$. Let $G$ be an $\mathcal{A}$-group, i.e., for some $\chi \in \mathrm{Irr}_1(G)$ we have $\mathrm{n}_\chi = m + \varphi(m)$, where $m$ is the order of a suitable $g \in \mathrm{T}_\chi$ (note that $m > 1$ since $g \neq 1$). It follows from the proof of Theorem A that for $H = \mathrm{C}_G(g)$, we have

$$m \leq |H| \leq \mathrm{n}_\chi - \tau_\chi(H) \leq \mathrm{n}_\chi - \varphi(m) = m.$$

Therefore, $|H| = m$, i.e., $\langle g \rangle = H (= \mathrm{C}_G(g))$. We also have $\tau_\chi(H) = \varphi(m)$ and $|H| = \mathrm{n}_\chi - \tau_\chi(H)$, i.e., $H$ is weakly $\chi$-maximal. Since $H$ is cyclic, we have $\chi(1) = 2$ (Lemma 9(b)). Besides, by the paragraph preceding Lemma 7, we have $\mathrm{Z}(\chi) \leq H$, and again, by Lemma 7, $g^2 \in \mathrm{Z}(\chi)$. Let $x \mapsto \bar{x}$ be the natural homomorphism of $H$ to $\bar{H} = H/\mathrm{Z}(\chi)$. We have $|H : \langle g^2 \rangle| = 2$. Since $g^2 \in \mathrm{Z}(\chi) \neq H = \langle g \rangle$, we get

$$\mathrm{Z}(\chi) = \langle g^2 \rangle,\ |H : \mathrm{Z}(\chi)| = 2. \tag{2}$$

$2^o$. Let us prove that $H - \mathrm{Z}(\chi) \subseteq \mathrm{T}_\chi$. To this end, rewrite the equality $\langle \chi_H, \chi_H \rangle = 2$ in the following form:

$$\sum_{x \in H} |\chi(x)|^2 = 2|H| = 2m.$$

On the other hand, we have

$$2m = \sum_{x \in H} |\chi(x)|^2 = \sum_{x \in \mathrm{Z}(\chi)} |\chi(x)|^2 + \sum_{x \in H - \mathrm{Z}(\chi)} |\chi(x)|^2$$

$$= \chi(1)^2|\mathrm{Z}(\chi)| + \sum_{x\in H-\mathrm{Z}(\chi)} |\chi(x)|^2.$$

Since $H$ is abelian then, by Lemma 7, $\chi(1) = 2$, and we obtain

$$\chi(1)^2|\mathrm{Z}(\chi) = 4 \cdot \frac{m}{2} = 2m.$$

Therefore, $\sum_{x\in H-\mathrm{Z}(\chi)} |\chi(x)|^2 = 0$, i.e., $H - \mathrm{Z}(\chi) \subseteq \mathrm{T}_\chi$, as desired.

$3^o$. Since $G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$ (Lemma 3), it follows from $2^o$ that

$$G - \mathrm{Z}(\chi) \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi. \tag{3}$$

By inequality (7) in [BZ, Chapter 21], we have $\mathrm{n}_\chi \geq |\mathrm{Z}(\chi)|(\chi(1)^2 - 1)$. Therefore, by hypothesis and (2), we obtain

$$m + \varphi(m) = \mathrm{n}_\chi \geq |\mathrm{Z}(\chi)|(\chi(1)^2 - 1) = \frac{3}{2}m. \tag{4}$$

It follows that $\varphi(m) \geq \frac{1}{2}m$. In this case, since $m$ is even, we get $m = 2^\lambda$ for some positive integer $\lambda$. Thus,

$$H = \langle g\rangle,\ \mathrm{Z}(\chi) = \langle g^2\rangle,\ |H| = 2^\lambda,\ |\mathrm{Z}(\chi)| = 2^{\lambda-1}, \tag{5}$$

$$\mathrm{n}_\chi = 2^\lambda + 2^{\lambda-1} = 3 \cdot 2^{\lambda-1}. \tag{6}$$

We consider the following three possibilities for $\mathrm{N}_\chi$ and $H$. Recall that $H$ is weakly $\chi$-maximal.

Case 1. Suppose that $\mathrm{N}_\chi = G$ and $H \triangleleft G$.

In this case, $|G : H| = 2$ (Lemma 11). Therefore, by (5), $|G| = 2^{\lambda+1}$, where $\lambda \geq 2$ since $G$ is nonabelian. Thus, $G$ is one of the following four nonabelian 2-groups with cyclic subgroup $H$ of index 2 (see [B, Theorem 1.2]):

$$G_\lambda^{(1)} = \langle a, b \mid a^{2^\lambda} = 1,\ b^2 = 1,\ a^b = a^{-1}\rangle \cong \mathrm{D}_{2^{\lambda+1}},$$
$$G_\lambda^{(2)} = \langle a, b \mid a^{2^\lambda} = 1,\ b^2 = a^{2^{\lambda-1}},\ a^b = a^{-1}\rangle \cong \mathrm{Q}_{2^{\lambda+1}},$$
$$G_\lambda^{(3)} = \langle a, b \mid a^{2^\lambda} = 1,\ b^2 = 1,\ a^b = a^{-1+2^{\lambda-1}}\rangle \cong \mathrm{SD}_{2^{\lambda+1}},\ \lambda \geq 3,$$
$$G_\lambda^{(4)} = \langle a, b \mid a^{2^\lambda} = 1,\ b^2 = 1,\ a^b = a^{1+2^{\lambda-1}}\rangle \cong \mathrm{M}_{2^{\lambda+1}},\ \lambda \geq 3.$$

Let us prove that all these four groups are $\mathcal{A}$-groups.

Let $i \leq 3$ and $G = G_\lambda^{(i)}$. Take in the Frattini subgroup $\Phi(G)$ a subgroup $L$ of index 2 and let $\chi \in \mathrm{Irr}_1(\bar{G})$, where $\bar{G} = G/L$ is nonabelian of order 8. Then $\chi_{\mathrm{Z}(\bar{G})} = 2\bar{\psi}$, where $\bar{\psi}$ is the faithful linear character of $\mathrm{Z}(\bar{G})$ (Clifford) so $\chi_{\Phi(G)} = 2\psi$. By reciprocity, $\psi^G = 2\chi$ so $G - \Phi(G) \subseteq \mathrm{T}_\chi$. Since $\psi$ is linear, we get $\mathrm{T}_\chi \cap \Phi(G) = \emptyset$, and we conclude that $\mathrm{T}_\chi = G - \Phi(G)$. Thus,

$$\mathrm{n}_\chi = |G - \Phi(G)| = 2^{\lambda+1} - 2^{\lambda-1} = 3 \cdot 2^{\lambda-1} = m + \varphi(m),$$

where $m = 2^\lambda$ is the order of a generator of a cyclic subgroup of index 2 in $G$. Thus, all three groups are $\mathcal{A}$-groups.

Now let $G = G_\lambda^{(4)} \cong \mathrm{M}_{2^{\lambda+1}}$. In this case, $G'$ is the unique minimal normal subgroup of $G$ and $\mathrm{Z}(G) = \Phi(G)$ is cyclic of index 4 in $G$. Let $\chi \in \mathrm{Irr}_1(G)$. Then $\chi_{\mathrm{Z}(G)} = 2\psi$, where $\psi$ is linear (Clifford). Therefore, by reciprocity, $\psi^G = 2\chi$ so $\mathrm{T}_\chi \supseteq G - \mathrm{Z}(G)$ and, since $\mathrm{T}_\chi \cap \mathrm{Z}(G) = \emptyset$, we conclude, as in the previous paragraph, that $G$ is an $\mathcal{A}$-group.

Case 2. Suppose that $\mathrm{N}_\chi = G$ and $H$ is not normal in $G$.

We claim that

$$\mathrm{Z}(\chi) = \mathrm{Z}(G). \tag{7}$$

Set $K = \mathrm{C}_G(g^2)$. Since $\mathrm{Z}(\chi) = \langle g^2 \rangle$ (see (2)), it follows that $K = \mathrm{C}_G(\mathrm{Z}(\chi))$ is normal in $G$. Since $H \leq K$ and $H$ is maximal in $G$ (Lemma 8), we have $K \in \{H, G\}$. Since $H$ is not normal in $G$, we obtain $K = G$, and so $\mathrm{Z}(\chi) = \mathrm{Z}(G)$.

Since $H$ is maximal in $G$ and nonnormal, we get

$$\mathrm{N}_G(H) = H. \tag{8}$$

Take $x \in H - \mathrm{Z}(G)$. Then $\langle x \rangle = H$, by (5), and so, by (8), we have

$$\mathrm{C}_G(x) = H \text{ for all } x \in H - \mathrm{Z}(G). \tag{9}$$

It follows from (2) and (8) that

$$H \cap H^t = \mathrm{Z}(G) \text{ for all } t \in G - H. \tag{10}$$

Set $\bar{G} = G/\mathrm{Z}(G)$. Then $\bar{H}$ is a nonnormal maximal subgroup of order 2 in $\bar{G}$ so $\bar{G} = \bar{H} \cdot \bar{F}$ is a Frobenius group with complement $\bar{H}$ of order 2 (see also (8) and (10)). Since the involution in $\bar{H}$ inverts $\bar{F}$, the subgroup $\bar{F}$ is abelian and all subgroups of $\bar{F}$ are normal in $\bar{G}$ (Burnside). Since $\bar{H}$ is maximal in $\bar{G}$, it follows that $|\bar{F}| = p > 2$, a prime. Thus,

$$\bar{G} \cong \mathrm{D}_{2p}, \text{ the dihedral group.} \tag{11}$$

Set $D = \bigcup_{t \in G} H^t$. Since $D - \mathrm{Z}(G) \subseteq \mathrm{T}_\chi$ and $\mathrm{T}_\chi$ is a normal subset of $G$, we get $D - \mathrm{Z}(G) \subseteq \mathrm{T}_\chi$. By (6), (10), (11) and assumption, we have

$$3 \cdot 2^{\lambda-1} = \mathrm{n}_\chi \geq |D - \mathrm{Z}(G)| = 2^{\lambda-1}|G : H| = 2^{\lambda-1} \cdot p,$$

and we conclude that $p = 3$ since $p > 2$. Thus, $G = H \cdot G'$, where $|G'| = 3$. Write $C = \mathrm{C}_G(G')$; then $C = \mathrm{Z}(G) \times G'$ is cyclic of index 2 in $G$. In this case, $\chi = \mu^G$, where $\mu \in \mathrm{Lin}(C)$. Since $\chi$ vanishes on the set $G - C$ of cardinality $3 \cdot 2^{\lambda-1} = \mathrm{n}_\chi$, it follows that $G$ is an $\mathcal{A}$-group with respect to $\chi$ since $m = o(g) = 2^\lambda$, $3 \cdot 2^{\lambda-1} = 2^\lambda + 2^{\lambda-1} = m + \varphi(m)$ and $g \in G - C = \mathrm{T}_\chi$.

Case 3. Suppose that $\mathrm{N}_\chi < G$.

Set $\mathrm{N}_\chi = G_1$, $\chi_1 = \chi_{G_1}$; then $\chi_1 \in \mathrm{Irr}(G_1)$, by [BZ, Exercise 21.3(a)]. We retain the above introduced notation (see (5)):

$$g \in \mathrm{T}_\chi,\ m = o(g),\ H = \langle g \rangle,\ |H| = 2^\lambda.$$

It follows from $\mathrm{T}_\chi = \mathrm{T}_{\chi_1}$ (indeed, $\mathrm{T}_\chi \subset \mathrm{N}_\chi = G_1$) that $\mathrm{n}_{\chi_1} = m + \varphi(m)$ so that $G_1$ is an $\mathcal{A}$-group of one of the types considered in Cases 1 and 2.

By [BZ, Exercise 21.3(b)], $G - G_1 \subseteq \mathrm{U}_\chi$. Since $H$ is abelian, we get $\chi(1) = 2$, by Lemma 7. Then $x^3 \in \mathrm{Z}(\chi)$ for all $x \in G - G_1$ (Lemma 5(b)). By [BZ, Exercise 21.3(c)], we have $\mathrm{Z}(\chi) \le G_1$ so $G/G_1$ is a group of exponent 3:

$$|G/G_1| = 3^a,\ a \ge 1. \tag{13}$$

By [BZ, Lemma 21.4(b)], $|G/G_1|$ divides $\mathrm{n}_\chi$. By (6), $\mathrm{n}_\chi = 3 \cdot 2^{\lambda-1}$ hence, by (13), we get $a = 1$. Thus,

$$|G : G_1| = 3. \tag{14}$$

Since $H = \langle g \rangle$ is a 2-subgroup and $G_1 \lhd G$ is of index 3, we get $H \le G_1$. Since $\mathrm{Aut}(H)$ is abelian 2-group, it follows that $H < G_1$.

Assume that $H$ is not contained in $G_1$. Then $\mathrm{Z}(\chi_1) = \mathrm{Z}(G_1)$ (see equality (7) in Case 2). Since $G_1/\mathrm{Z}(G_1) \cong \mathrm{S}_3$ (see equality (11) in Case 2 and take into account that $p = 3$), we obtain

$$|G_1| = 6|\mathrm{Z}(G_1)| = 6|\mathrm{Z}(\chi)| = 6 \cdot 2^{\lambda-1} = 2^\lambda \cdot 3.$$

Therefore, in view of (14), we have

$$|G| = 3|G_1| = 2^\lambda \cdot 9. \tag{15}$$

Let $P \in \mathrm{Syl}_3(G)$; then $|P| = 9$ so $P$ is abelian.

Set $K = P \cdot \mathrm{Z}(\chi)$; then $K$ is abelian since $\mathrm{Z}(\chi) \in \mathrm{Syl}_2(K)$ is cyclic and normal in $K$ so $P \lhd G$ (Burnside), and $|G : K| = 2$, by (15) and (5). It follows that $\chi_K$ is reducible so $\chi$ is induced from $K$, and we conclude that $G - K \subseteq \mathrm{T}_\chi$. Then $\mathrm{N}_\chi = \langle \mathrm{T}_\chi \rangle \ge \langle G - K \rangle = G$, contrary to the assumption.

Thus, $H \lhd G_1$. In this case, $G_1$ is one of groups $G_\lambda^{(i)}$, $i = 1, 2, 3, 4$ (see Case 1). We have (see (14))

$$|G_1| = 2^{\lambda+1},\ |G_1 : H| = 2,\ |G| = 2^{\lambda+1} \cdot 3,\ |G : G_1| = 3. \tag{16}$$

Let $G_1 \not\cong \mathrm{Q}_8$. Then $G = G_1 \times P$, where $|P| = 3$ (by [B, Theorem 34.8], $\mathrm{Aut}(G_1)$ is a 2-group). Since $P < \mathrm{Z}(\chi)$, it follows that $\mathrm{Z}(\chi) \not\le G_1$, contrary to [BZ, Exercise 21.3(c)].

It remains to consider the case $G_1 \cong \mathrm{Q}_8$. In this case, $G/\mathrm{Z}(G_1) \cong \mathrm{A}_4$. Since $\mathrm{Z}(G_1) = \mathrm{Z}(G) < G'$, it follows that $\mathrm{Z}(G_1)$ is the Schur multiplier of $\mathrm{A}_4$, and we conclude that $G \cong \mathrm{SL}(2, 3)$, the group of part (c). ∎

Let $\chi \in \mathrm{Irr}_1(G)$, $g \in \mathrm{T}_\chi$ of order $m = 2^\lambda$ and $H = \langle g \rangle$ be as in the proof of Theorem B; recall then $\mathrm{n}_\chi = 3 \cdot 2^{\lambda-1}$. Since $\chi(1) = 2$, we have $\langle \chi_H, \chi_H \rangle = 2$.

In part (a) of Theorem B, the character $\chi$ has on $H$ exactly $|H - \Phi(G)| = 2^{\lambda-1}$ zeros. Therefore

$$1 + \frac{|\mathrm{T}_\chi - H|}{|H|} = 1 + \frac{3 \cdot 2^{\lambda-1} - 2^{\lambda-1}}{2^\lambda} = 2 \tag{17}$$

so that in $(*)$ we have equality. By Veitsblit's theorem, $G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$.

In part (b) of theorem B, the character $\chi$ has on $H$ exactly $|H - \mathrm{C}_G(G')| = 2^{\lambda-1}$ zeros, so (17) holds, and we have $G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$ again.

In part (c) of Theorem B, the character $\chi$ has on $H$ exactly two zeros. Therefore,

$$1 + \frac{|\mathrm{T}_\chi - H|}{|H|} = 1 + \frac{6-2}{4} = 2$$

so, as above, $G - H \subseteq \mathrm{T}_\chi \cup \mathrm{U}_\chi$.

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