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AN IMPLICIT DIVISION OF BOUNDED AND UNBOUNDED LINEAR OPERATORS WHICH PRESERVES THEIR PROPERTIES

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Abstract. We give an answer to the following problem: Given two linear operators A and B such that BA and A verify some property P, then when does B verify the same property P? Of course, we have to assume that B satisfies some condition Q independent of (or weaker than) P. This problem is solved in the setting of both bounded and unbounded operators on a Hilbert space. Some interesting counterexamples are also given.

1. Introduction

A widely known property is that every self-adjoint operator is normal whether it is bounded or not. Conversely, a normal operator is self-adjoint if it has a real spectrum. An unbounded normal symmetric operator is self-adjoint. Another question is if an operator is a normal product of two self-adjoint operators, then when is it self-adjoint? This has been completely solved in the setting of bounded and unbounded operators, see [1, 7, 8, 10].

In this paper we consider in some sense the converse of the previous question for bounded and unbounded operators and its analog for different classes of operators. To be more precise, the problem considered here is the following: given two linear operators (bounded or not) A and B such that BA satisfies a property "P" such that A (or B) also satisfies the same property "P", then when does B (or A) satisfy the property "P"? Of course one will have to assume that B satisfies some property "Q" which is either independent

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of "P" or weaker than it, e.g. if BA and A are self-adjoint, then when is B self-adjoint? Similarly if both BA and A are nonnegative, then when is B nonnegative? ... etc. This last question in particular has obviously a positive answer for positive real numbers (if $ab > 0$ and $b > 0$, then $a > 0$) and this is in fact what has made us choose the title of the paper.

We assume that the reader is familiar with notions and results about linear operators (both bounded and unbounded) and we refer the reader to the references [2, 11] for further details.

The question considered for bounded operators is standard enough so that we include it in the introduction.

In general, if BA and A are self-adjoint, then B need not remain self-adjoint. For consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}. \text{ Then } BA = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Thus BA is self-adjoint while B is not (it is not even normal). Observe that A and B are both invertible.

If, however, A and B are two commuting bounded operators and if BA and A are both self-adjoint such that A is invertible, then B is self-adjoint.

The commutativity of A and B in the previous result can be dropped at the cost of adding the assumption "B normal" and we can say that if BA and A are two self-adjoint operators such that $0 \notin W(A)$ (the numerical range of A). Then B is self-adjoint if it is normal. The proof is a mere consequence of a result by M. R. Embry ([3]).

As for normal operators, one can easily prove that if A and B are two commuting operators and if BA is normal and A is unitary, then B is normal. The proof is elementary and can be easily established by doing some straightforward arithmetic.

We also have a similar result for positive operators. The proof is again standard.

Proposition 1.1. *Assume that A and B are bounded and commute. If BA and A are positive such that A is invertible, then B is positive.*

2. The Mixed Bounded-Unbounded Case

We recall the Fuglede theorem ([4]. Cf. [6]) for the reader's convenience.

Theorem 2.1. *If A is a bounded operator and N is an unbounded normal operator, then*

$$AN \subset NA \Longrightarrow AN^* \subset N^*A.$$

Theorem 2.2. *Let B be a unitary operator. Assume that A is an unbounded closed operator commuting with B^*, i.e., $B^*A \subset AB^*$. If BA is normal, then so is A.*

PROOF. We have

$$B^*A \subset AB^* \Longrightarrow BB^*A = B^*(BA) \subset (BA)B^*$$

and since BA is normal, the Fuglede theorem yields

$$B^*(BA)^* \subset (BA)^*B^* \Longrightarrow B^*A^*B^* \subset A^*B^*B^*. \tag{2.1}$$

Now we have

$$\begin{aligned} AA^* &= B^*BAA^*B^*B = B^*(BA)(BA)^*B \\ &= B^*(BA)^*(BA)B \text{ (by the normality of } BA) \\ &= B^*A^*B^*BAB \\ &\subset A^*B^*\underbrace{B^*B}_{I}AB \text{ (by (2.1))} = A^*B^*AB \\ &\subset A^*AB^*B \text{ (by the commutativity hypothesis)} \\ &= A^*A \end{aligned}$$

Since A is closed, AA^* is self-adjoint. But self-adjoint operators are maximally symmetric (see e.g. [11]). Thus

$$AA^* = A^*A \text{ or } A \text{ is normal.}$$

□

THEOREM 2.3. *Let A be a unitary operator. Assume that B is closed and commutes with A^* (i.e., $A^*B \subset BA^*$). If BA is normal, then so is B.*

PROOF. The idea of proof is similar to the preceding one. However, some details are different and need to be emphasized. We have

$$A^*B \subset BA^* \Longrightarrow A^*BA \subset BA^*A = BAA^*$$

(to simplify the proof a bit we do not replace $AA^* = A^*A$ by I at present). Since BA is normal, the Fuglede theorem then implies that

$$A^*(BA)^* \subset (BA)^*A^*. \tag{2.2}$$

We have on the one hand

$$\begin{aligned} (BA)^*BA &= AA^*(BA)^*BA \\ &\subset A(BA)^*A^*BA \text{ (by (2.2))} \\ &\subset A(BA)^*BA^*A \text{ (by the commutativity hypothesis)} \\ &\subset (BAA^*)^*B = B^*B. \end{aligned}$$

On the other hand

$$\begin{aligned} BB^* &= BAA^*(BAA^*)^* \\ &\subset BAA^*(A^*BA)^* \text{ (by the commutativity hypothesis)} \\ &= BAA^*(BA)^*A \\ &\subset BA(BA)^*A^*A \text{ (by the Fuglede theorem)} \\ &= BA(BA)^*. \end{aligned}$$

But BA is normal and hence

$$BB^* \subset (BA)(BA)^* = (BA)^*BA \subset B^*B \Longrightarrow BB^* \subset B^*B.$$

Since B is closed, similar arguments to those of the foregoing proof apply and yield the normality of B. The proof is over. □

Corollary 2.4. *Assume B is a bounded and self-adjoint operator verifying $B^2 = I$. Assume that A is a closed and symmetric operator commuting with B. If BA is self-adjoint, then so is A.*

Proof. Since BA is self-adjoint, it is normal. Theorem 2.2 then implies that A is normal. But A is assumed to be symmetric and hence A is self-adjoint. □

Theorem 2.3 can be used analogously to establish the following result.

Corollary 2.5. *Assume A is a bounded and self-adjoint operator verifying $A^2 = I$. Assume that B is a closed and symmetric operator commuting with A. If BA is self-adjoint, then B is also self-adjoint.*

As done in the bounded case we can drop the commutativity hypothesis on A and B and the invertibility of A and replace it by different conditions. The proof uses the following result

Theorem 2.6 ([9]). *Assume H is an unbounded normal operator. Also assume that A is a bounded operator for which $0 \notin W(A)$ and such that $AH \subset H^*A$. Then $H = H^*$.*

We have

Corollary 2.7. *Assume B is an unbounded normal operator and A is a bounded operator such that $0 \notin W(A)$. If BA and A are both self-adjoint, then so is B.*

Proof. As A and BA are self-adjoint, then

$$AB^* \subset (BA)^* = BA.$$

Since B is normal and $0 \notin W(A)$, Theorem 2.6 applies and yields the self-adjointness of B. □

3. The Unbounded Case

Theorem 3.1. *There exist an unbounded closed (self-adjoint or normal) operator A and a non-closed (non self-adjoint or non-normal) unbounded operator B such that BA is closed (self-adjoint or normal). Hence "BA closed (self-adjoint or normal), A closed (self-adjoint or normal) $\not\Rightarrow$ B closed (self-adjoint or normal)".*

Proof. We give an explicit counterexample. Consider the following two operators defined by

$$Af(x) = e^{2x}f(x) \text{ and } Bf(x) = (e^{-x} + 1)f(x)$$

on their respective domains

$$D(A) = \{f \in L^2(\mathbb{R}) : \; e^{2x}f \in L^2(\mathbb{R})\}$$

and

$$D(B) = \{f \in L^2(\mathbb{R}) : \; e^{-2x}f, e^{-x}f \in L^2(\mathbb{R})\}$$

(think of B as a sum of two operators $S = e^{-2x} + e^{-x} + 1$ and $T = -e^{-2x}$, say, on their maximal domains). Then obviously A is closed as it is self-adjoint while B is not closed since it has a closure $\overline{B}$ defined by $\overline{B}f(x) = (e^{-x}+1)f(x)$ on $D(\overline{B}) = \{f \in L^2(\mathbb{R}) : e^{-x}f \in L^2(\mathbb{R})\}$. Now the operator BA is defined by $BAf(x) = (e^{2x} + e^{x})f(x)$ on

$$D(BA) = \{f \in D(A) : \; Af \in D(B)\} = \{f \in L^2(\mathbb{R}) : \; e^{2x}f, e^{x}f \in L^2(\mathbb{R})\}$$

and one can see that BA is closed on this domain as it is self-adjoint. ∎

One can give some positive results using known results from the literature. An instance of that can be found in [7] where it is proved, among other things, that if HK is a normal product of two self-adjoint operators (H is unbounded and $\sigma(K)\cap\sigma(-K) \subseteq \{0\}$), then HK is self-adjoint. This can be used to prove the following

Corollary 3.2. *Let B be a densely defined normal operator. Assume that A is a self-adjoint, densely defined (or bounded) and boundedly invertible operator with inverse A^{-1}. One assumes further that $\sigma(A^{-1}) \cap \sigma(-A^{-1}) \subseteq \{0\}$. If BA is self-adjoint, then B is self-adjoint.*

Proof. Since A is invertible, $AA^{-1} = I$ and hence

$$B = B(AA^{-1}) = (BA)A^{-1}.$$

Now both BA and A^{-1} are self-adjoint and hence B is self-adjoint as it is normal and since $\sigma(A^{-1}) \cap \sigma(-A^{-1}) \subseteq \{0\}$. ∎

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