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STUDY OF A QUASISTATIC CONTACT PROBLEM IN VISCOELASTICITY

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Abstract. We study a quasistatic frictional contact of a viscoelastic body with a foundation. The contact is modelled with a normal compliance condition such that the penetration is restricted with unilateral constraints and the associated version of Coulomb's law of dry friction. We establish the existence of a weak solution if the coefficient of friction is small enough. The proof is based on arguments of time-discretization, compactness and lower semicontinuity.

1. Introduction

Contact mechanics is the branch of solid mechanics which typically involves two bodies instead of one and focuses its objective on their common interface rather their interiors. Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. A first attempt to study frictional contact problems within the framework of variational inequalities was made in [6]. The mathematical, mechanical and numerical state of the art can be found in [13]. In [9] we find a detailed analysis of the contact problems in linear elasticity with the mathematical and numerical studies. In the present paper we consider a quasistatic contact problem between a viscoelastic body and an obstacle say a foundation. The contact is modelled with a normal compliance condition similar to the one in [8] such that the penetration is restricted with unilateral constraints and the associated version of Coulomb's law of dry friction. Under this compliance condition the interpenetration of the body's surface into the foundation is allowed and may be justified by considering the

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interpenetration and deformation of surface asperities. However according to [8], the method presented here considers a compliance model in which the compliance term doesn't represent necessarily an important perturbation of the original problem without contact. This will help us to study the models, where a strictly limited penetration is performed with the limit procedure to the Signorini contact problem. In the last years a considerable attention has been paid to the analysis of quasistatic frictional contact problems. Indeed, in linear elasticity the quasistatic frictional contact problem using a normal compliance law has been studied in [2] by considering incremental problems and in [10] by a different method, based on a time-regularization. The quasistatic contact problem with local or nonlocal friction has been solved respectively in [11] and in [4] by using a time-discretization. A similar technique was used in [5] in order to study a quasistatic unilateral contact problem with friction and adhesion. In [3] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. In viscoelasticity, the quasistatic contact problem with normal compliance and friction has been solved in [12] by using arguments of fixed point theorem. Also, in [7] quasistatic contact problems in viscoelasticity and viscoplasticity were studied. Carrying out the variational analysis, the authors systymatically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators. In [1] a quasistatic unilateral contact problem with nonlocal friction in viscoelasticity was studied and an existence result of a weak solution was established for a coefficient of friction sufficiently small. In this work, as in [1] we extend the existence result obtained in [14], for a quasistatic unilateral contact problem with normal compliance and finite penetration between an elastic body and a foundation, to the contact between a viscoelastic body and a foundation. As in [4], we propose a variational formulation written in the form of two variational inequalities. By means of Euler's implicit scheme, the quasistatic contact problem leads us to solve a well-posed variational inequality at each time step. Finally under a smallness assumption on the coefficient of friction we prove by using lower semicontinuity and compactness arguments that the limit of the discrete solution is a solution to the continuous problem.

2. Variational formulation

Let $\Omega \subset \mathbf{R}^d$; $(d = 2, 3)$, be a domain, with a Lipschitz boundary Γ, initially occupied by a viscoelastic body. Γ is divided into three measurable parts such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where Γ_1, Γ_2, Γ_3 are disjoint open sets and $meas\,(\Gamma_1) > 0$. The body is subjected to volume forces of density φ_1, prescribed zero displacements and tractions φ_2 on the part Γ_1 and Γ_2, respectively. On Γ_3 the body is in unilateral and frictional contact with finite penetration with a foundation.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the viscoelastic body is the following.

Problem P$_1$. *Find a displacement field* $u : \Omega \times [0,T] \to \mathbf{R}^d$ *such that*

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + G(\varepsilon(u)) \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$div\sigma + \varphi_1 = 0 \text{ in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$u_\nu \le g,\ \sigma_\nu + p(u_\nu) \le 0,\ (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \ \text{ on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\begin{cases} |\sigma_\tau| \le \mu p(u_\nu) \\ |\sigma_\tau| < \mu p(u_\nu) \Longrightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu p(u_\nu) \Longrightarrow \exists \lambda \ge 0 \text{ s.t. } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$u(0) = u_0 \ \text{ in } \Omega. \tag{2.7}$$

Here (2.1) is the viscoelastic constitutive law in which σ denotes the stress tensor, $\mathcal{A}$ the fourth order tensor of viscosity coefficients and G the tensor of elasticity; (2.2) represents the equilibrium equation, (2.3) and (2.4) are the displacement-tractions boundary conditions and, finally, the function u_0 denotes the initial displacement. We make some comments on the contact conditions (2.5) and (2.6) in which σ_ν denotes the normal stress, p is a prescribed nonnegative function, u_ν is the normal displacement, g is a positive constant which denotes the maximum value of the penetration, σ_τ represents the tangential traction and $\dot{u}_\tau$ represents the tangential velocity. Indeed, when $u_\nu < 0$ i.e., when there is separation between the body and the obstacle then the condition (2.5) combined with assumptions (2.14) shows that the reaction of the foundation vanishes (since $\sigma_\nu = 0$). When $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement. When $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. We note then when $g = 0$, the condition (2.5) becomes the classical Signorini contact condition without a gap

$$u_\nu \le 0,\ \sigma_\nu \le 0,\ \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p = 0$, condition (2.5) becomes the classical Signorini contact condition with a gap:

$$u_\nu \le g,\ \sigma_\nu \le 0,\ \sigma_\nu(u_\nu - g) = 0.$$

The last two conditions are used to model the unilateral conditions with a rigid foundation. Conditions (2.6) represent a version of Coulomb's law of dry friction. Examples of normal compliance functions can be found in [2, 7, 8, 12, 13].

Next, in the study of the mechanical problem P_1 we use the following notations and assumptions.

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and S_d denotes the space of second order symmetric tensors in $\mathbf{R}^d$. In (2.6) and below, a dot above a variable represents its derivative with respect to time.

To proceed with the variational formulation, we need some function spaces:

$$H = \left(L^2(\Omega)\right)^d, \; Q = \left\{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$H_1 = \left(H^1(\Omega)\right)^d, \; Q_1 = \{\tau \in Q : div\tau \in H\}.$$

H, Q are Hilbert spaces equipped with the respective inner products:

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \;\; \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

Now, let V be the closed subspace of H_1 given by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds ([6]),

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where a constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and let $\|.\|_V$ be the associated norm. It follows from (2.8) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent and $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For every $v \in H_1$, we denote by v_ν and v_τ the normal and the tangential components of v on Γ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu,$$

where ν is a unit outward normal vector to Γ. We also denote by σ_ν and σ_τ the normal and tangential component of a function $\sigma \in Q_1$ defined by

$\sigma_\nu = \sigma\nu.\nu$, $\sigma_\tau = \sigma - \sigma_\nu \nu$, and we recall that when σ is a regular function, the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + \langle div\sigma, v \rangle_H = \int_\Gamma \sigma\nu.vda \;\forall v \in H_1.$$

We assume that the tensor of viscosity $\mathcal{A} = (\mathcal{A}_{ijkh}) : \Omega \times S_d \to S_d$ is a bounded symmetric positive definite fourth order tensor, i.e.,

$$\text{(2.10)} \qquad \begin{cases} \cdot\ \mathcal{A}_{ijkh} \in L^\infty(\Gamma_3), 1 \le i,j,k,h \le d. \\ \cdot\ \mathcal{A}\sigma.\tau = \sigma.\mathcal{A}\tau,\ \forall \sigma, \tau \in S_d,\ a.e. \text{ in } \Omega. \\ \cdot\ \text{There exists } \alpha > 0 \text{ such that} \\ \mathcal{A}\tau.\tau \ge \alpha |\tau|^2 \;\; \forall \tau \in S_d,\ a.e. \text{ in } \Omega. \end{cases}$$

We define the bilinear form $a(\cdot,\cdot)$ on $V \times V$ by

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u).\varepsilon(v)\,dx.$$

It follows from (2.10) that a is continuous and coercive, that is,

$$\text{(2.11)} \qquad \begin{cases} (a)\ \text{there exists } \beta > 0 \text{ such that} \\ |a(u,v)| \le \beta \|u\|_V \|v\|_V \quad \forall u, v \in V, \\ (b)\ a(v,v) \ge \alpha \|v\|_V^2 \quad \forall v \in V. \end{cases}$$

Hypotheses on the tensor of elasticity G.

$$\text{(2.12)} \qquad \left.\begin{array}{l} (a)\ G : \Omega \times S_d \to S_d; \\ (b)\ \text{there exists } M_G > 0 \text{ such that} \\ \qquad |G(x,\varepsilon_1) - G(x,\varepsilon_2)| \le M_G |\varepsilon_1 - \varepsilon_2|, \\ \qquad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega; \\ (c)\ \text{the mapping } x \to G(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \varepsilon \text{ in } S_d; \\ (e)\ x \to G(x,0) \in Q. \end{array}\right\}$$

We denote by $b : V \times V \to \mathbf{R}$ the map linear with respect to the second argument, defined by

$$b(u,v) = \langle G\varepsilon(u), \varepsilon(v) \rangle_Q.$$

Next, for every real Banach space $(X,\ \|.\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

For $p \in [1,\infty]$ we use the standard notation of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T:V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

The forces are assumed to satisfy

$$\varphi_1 \in L^\infty(0,T;H), \qquad \varphi_2 \in L^\infty\left(0,T;\left(L^2(\Gamma_2)\right)^d\right). \tag{2.13}$$

Next, for *a.e.* $t \in (0,T)$, we denote by $f(t)$ the element of V' defined by

$$\langle f(t), v\rangle_{V',V} = \int_\Omega \varphi_1.vdx + \int_{\Gamma_2} \varphi_2.vda \quad \forall\, v \in V.$$

The conditions (2.13) imply

$$f \in L^\infty\left(0,T;V'\right).$$

We assume that the contact function p satisfies

$$\begin{cases} (a)\;\; p :]-\infty, g] \to \mathbf{R}_+; \\ (b)\;\; \text{there exists } L_p > 0 \text{ such that} \\ \qquad |p(u) - p(v)| \le L_p |u - v|, \;\text{ for all } u,\, v \le g; \\ (c)\;\; p(v) = 0 \text{ for all } v \le 0. \end{cases} \tag{2.14}$$

We define the functional

$$j : V \times V \to \mathbf{R}$$

by

$$j(v,w) = \int_{\Gamma_3} p(v_\nu)\, w_\nu da + \int_{\Gamma_3} \mu p(v_\nu)\, |w_\tau|\, da$$

where the coefficient of friction μ is assumed to satisfy

$$\mu \in L^\infty(\Gamma_3) \;\text{ and } \mu \ge 0 \;\; a.e. \text{ on } \Gamma_3. \tag{2.15}$$

We define the space $H^{\frac{1}{2}}(\Gamma_3) = \left\{\mu\,|_{\Gamma_3}\,;\; \mu \in H^{\frac{1}{2}}(\Gamma),\, \mu = 0 \text{ on } \Gamma_1\right\}$ and by $\langle\cdot,\cdot\rangle$ the duality pairing on $H^{\frac{1}{2}}(\Gamma_3)$, $H^{-\frac{1}{2}}(\Gamma_3)$. For $\sigma \in Q_1$ such that $\sigma\nu = h$ on Γ_2 where $h \in \left(L^2(\Gamma_2)\right)^d$, the normal stress $\sigma_\nu(u) \in H^{-\frac{1}{2}}(\Gamma_3)$ is given by

$$\begin{cases} \forall w \in H^{\frac{1}{2}}(\Gamma_3) \\ \langle \sigma_\nu(u), w\rangle = \langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma(u), v\rangle_H - (h,v)_{(L^2(\Gamma_2))^d} \\ \forall v \in V \text{ such that } v_\nu = w \text{ and } v_\tau = 0 \text{ on } \Gamma_3. \end{cases}$$

Also, in the study of Problem P_1 we need the set of admissible displacements field

$$U = \{v \in V : v_\nu \le g \;\; a.e. \text{ on } \Gamma_3\}$$

and we assume that the initial data u_0 satisfies

$$u_0 \in U. \tag{2.16}$$

In the sequel, everywhere below c will denote a positive constant which does not depend on $n \in \mathbf{N}^*$ and $t \in [0,T]$ and whose value may change from line to line.

Finally, with these notations using the same techniques to those in [14] we obtain a variational formulation of the problem P_1 as follows.

PROBLEM P_2. *Find a displacement field* $u \in W^{1,\infty}(0,T;V)$ *such that* $u(0) = u_0$, $u(t) \in U$, *for all* $t \in [0,T]$, *and for almost all* $t \in (0,T)$,

(2.17)
$$\begin{aligned} & a(\dot{u}(t), v - \dot{u}(t)) + b(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \\ & \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} + \langle \sigma_\nu(u(t)) + p(u_\nu(t)), v_\nu - \dot{u}_\nu(t)\rangle \quad \forall v \in V, \end{aligned}$$

and

$$\langle \sigma_\nu(u(t)) + p(u_\nu(t)), z_\nu - u_\nu(t)\rangle \geq 0 \quad \forall z \in U. \tag{2.18}$$

One has the following theorem

THEOREM 2.1. *Let (2.11)–(2.16) hold. Then Problem P_2 has at least one solution if*

$$d_\Omega^2 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < \alpha.$$

REMARK 2.2. *We note that we have (see [4, Remark 2.1])*

$$\langle \sigma_\nu(u(t)) + p(u_\nu(t)), \dot{u}_\nu(t)\rangle = 0 \;\; a.e.\ t \in (0,T). \tag{2.19}$$

3. Time-discretized formulation

The proof of Theorem 2.1 is based on a time-discretization. For $n \in \mathbf{N}^*$, we consider a partition of the time interval $[0,T]$, $0 = t_0 < t_1 < ... < t_n = T$, where $t_i = i\Delta t$, $i = 0, ..., n$, with step size $k = \Delta t = T/n$. We denote by u^i the approximation of u at time t_i and $\delta u^{i+1} = \left(u^{i+1} - u^i\right)/k$. For a function $w \in C([0,T];X)$ where X is a Banach space we set $w^i = w(t_i)$. We use an implicit scheme and obtain the following sequence (P_n^i) $i = 0, ..., n-1$ of time-discretized problems defined for $u^0 = u_0$ by:

PROBLEM P_n^i. *For* $u^i \in U$, find $u^{i+1} \in U$ *such that*

(3.1)
$$\begin{cases} a\left(\delta u^{i+1}, w - \delta u^{i+1}\right) + b\left(u^i, w - \delta u^{i+1}\right) + j\left(u^{i+1}, w\right) - j\left(u^{i+1}, \delta u^{i+1}\right) \\ \geq \left\langle f^{i+1}, w - \delta u^{i+1}\right\rangle_{V',V} + \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), w_\nu - \delta u_\nu^{i+1}\right\rangle \; \forall\, w \in V, \\ \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), w_\nu - u_\nu^{i+1}\right\rangle \geq 0 \;\forall\, w \in U, \end{cases}$$

where

$$f^{i+1} = \frac{1}{k}\int_{ik}^{(i+1)k} f(s)\, ds.$$

Now as in [1] in order to solve the problem P_n^i we define the convex sets U_k^{i+1} as $U_k^{i+1} = (U - u^i)/k$. It is easy to see that

$$\left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), w_\nu - u_\nu^{i+1}\right\rangle \geq 0 \qquad \forall w \in U$$

is equivalent to

$$\left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), w_\nu - \delta u_\nu^{i+1} \right\rangle \geq 0 \qquad \forall w \in U_k^{i+1}.$$

Also as in [1] the problem P_n^i is equivalent to the following problem Q_n^i.

Problem Q_n^i. *For $u^i \in U$, find $\delta u^{i+1} \in U_k^{i+1}$ such that*

$$\left\{ \begin{array}{l} a\left(\delta u^{i+1}, w - \delta u^{i+1}\right) + b\left(u^i, w - \delta u^{i+1}\right) + j\left(u^{i+1}, w\right) \\ -\, j\left(u^{i+1}, \delta u^{i+1}\right) \geq \left\langle f^{i+1}, w - \delta u^{i+1} \right\rangle_{V',V} \qquad \forall w \in U_k^{i+1}. \end{array} \right. \tag{3.2}$$

We have the following result.

Proposition 3.1. *Problem Q_n^i has a unique solution if*

$$d_\Omega^2 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < \alpha.$$

To prove this proposition, for $\eta \in K$, we define the following auxiliary problem.

Problem $\mathrm{Q}_{n\eta}^i$. *For $u^i \in U$, find $u_\eta^{i+1} \in U_k^{i+1}$ such that*

$$\left\{ \begin{array}{l} a\left(\dfrac{u_\eta^{i+1} - u^i}{k}, w - \dfrac{u_\eta^{i+1} - u^i}{k}\right) + b\left(u^i, w - \dfrac{u_\eta^{i+1} - u^i}{k}\right) + j\left(\eta, w\right) \\ \quad -\, j\left(\eta, \dfrac{u_\eta^{i+1} - u^i}{k}\right) \geq \left\langle f^{i+1}, w - \dfrac{u_\eta^{i+1} - u^i}{k} \right\rangle_{V',V} \qquad \forall\ w \in U_k^{i+1}. \end{array} \right.$$

Next, we denote $\left(u_\eta^{i+1} - u^i\right)/k = \theta u_\eta^{i+1}$; then Problem $\mathrm{Q}_{n\eta}^i$ is equivalent to the following problem.

Problem $\mathrm{R}_{n\eta}^i$. *For $u^i \in U$, find $\theta u_\eta^{i+1} \in U_k^{i+1}$ such that*

$$\left\{ \begin{array}{l} a\left(\theta u_\eta^{i+1}, w - \theta u_\eta^{i+1}\right) + b\left(u^i, w - \theta u_\eta^{i+1}\right) + j\left(\eta, w\right) - j\left(\eta, \theta u_\eta^{i+1}\right) \\ \geq \left\langle f^{i+1}, w - \theta u_\eta^{i+1} \right\rangle_{V',V} \quad \forall\ w \in U_k^{i+1}. \end{array} \right.$$

We can prove the following lemma.

Lemma 3.2. *Problem $R_{n\eta}^i$ has a unique solution.*

Proof. The problem $\mathrm{R}_{n\eta}^i$ is equivalent to the following optimization problem: find $\theta u_\eta^{i+1} \in U_k^{i+1}$ such that $J_i\left(\theta u_\eta^{i+1}\right) = \min_{v \in U_k^{i+1}} J_i(v) = \frac{1}{2} a(v,v) + b\left(u^i, v\right) + j(\eta, v) - \left\langle f^{i+1}, v \right\rangle_{V',V}$. The functional J_i is proper, continuous, strictly convex, and coercive on the closed convex set U_k^{i+1}. Then there exists a unique element $\theta u_\eta^{i+1} \in U_k^{i+1}$ which minimizes the functional J_i. $\square$

Now to prove Proposition 3.1, we define the following mapping

$$\Phi : K \to K,$$

as

$$\eta \to \Phi(\eta) = u_{\eta}^{i+1}.$$

The following lemma holds.

Lemma 3.3. *Φ has a unique fixed point η^* and $u_{\eta^*}^{i+1}$ is a unique solution of Problem P_n^i.*

Proof. We set $w = \theta u_{\eta_2}^{i+1}$ in inequality of Problem $\mathrm{R}_{n\eta_1}^i$ and $w = \theta u_{\eta_1}^{i+1}$ in inequality of Problem $\mathrm{R}_{n\eta_2}^i$. After adding the resulting inequalities, we obtain that

$$\begin{aligned} a(\theta u_{\eta_2}^{i+1} - \theta u_{\eta_1}^{i+1}, \theta u_{\eta_2}^{i+1} - \theta u_{\eta_1}^{i+1}) \\ &\le j\left(\eta_1, \theta u_{\eta_2}^{i+1} - u^i\right) - j\left(\eta_1, \theta u_{\eta_1}^{i+1} - u^i\right) \\ &+ j\left(\eta_2, \theta u_{\eta_1}^{i+1} - u^i\right) - j\left(\eta_2, \theta u_{\eta_2}^{i+1} - u^i\right). \end{aligned}$$

Whence using (2.9) and (2.11)(b), we get

$$\left\|\theta u_{\eta_2}^{i+1} - \theta u_{\eta_1}^{i+1}\right\|_V \le L_p \frac{d_\Omega}{\alpha}\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right)\|\eta_2 - \eta_1\|_V.$$

On the other hand we have

$$\|\Phi(\eta_2) - \Phi(\eta_1)\|_V = k\left\|\theta u_{\eta_2}^{i+1} - \theta u_{\eta_1}^{i+1}\right\|_V.$$

Then for $k < 1$, i.e., for $n > T$, we deduce

$$\|\Phi(\eta_2) - \Phi(\eta_1)\|_V \le \frac{d_\Omega^2}{\alpha} L_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right)\|\eta_2 - \eta_1\|_V.$$

Hence it follows that if $d_\Omega^2 L_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < \alpha$, Φ is contractive. Thus it admits a unique fixed point η^*, $\theta u_{\eta^*}^{i+1}$ is a unique solution of Problem Q_n^i and consequently $u_{\eta^*}^{i+1}$ is a unique solution of Problem P_n^i. ◻

4. Existence of a solution for problem P_2

The main result of this section is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution.

Indeed, we define the following sequences of functions:

$$\begin{aligned} &u^n(t) = u^i + (t - t_i)\,\delta u^i \text{ on } [t_i, t_{i+1}], \\ &\widetilde{u}^n(t) = u^{i+1}, \quad f^n(t) = f^{i+1}, \ \forall\ t \in (t_i, t_{i+1}],\ i = 0, ..., n-1. \end{aligned}$$

We have the following lemma.

Lemma 4.1. *There exists a function u, such that passing to a subsequence still denoted (u^n) we have*

$$u^n \to u \text{ weak } * \text{ in } W^{1,\infty}(0, T; V).$$

PROOF. With these notations from inequality (3.2) we deduce the following inequality:

(4.1)
$$\begin{cases} a\left(\dot{u}^n(t), w - \dot{u}^n(t)\right) + b\left(\widetilde{u}^n(t), w - \dot{u}^n(t)\right) + j\left(\widetilde{u}^n(t), w\right) \\ -j\left(\widetilde{u}^n(t), \dot{u}^n(t)\right) \geq \left\langle f^n(t), w - \dot{u}^n(t)\right\rangle_{V',V} \quad \forall\, w \in U_k^{i+1},\, t_i \leq t \leq t_{i+1}. \end{cases}$$

Taking now $w = 0$ as test function in (4.1), we derive

$$a\left(\dot{u}^n(t), \dot{u}^n(t)\right) \leq b\left(\widetilde{u}^n(t), \dot{u}^n(t)\right) + j\left(\widetilde{u}^n(t), \dot{u}^n(t)\right) + \left\langle f^n(t), \dot{u}^n(t)\right\rangle_{V',V}.$$

This inequality implies

$$\alpha \left\|\dot{u}^n(t)\right\|_V \leq M_G \left\|\widetilde{u}^n(t)\right\|_V + \left\|G(0)\right\|_Q + d_\Omega^2 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) + \left\|f^n(t)\right\|_{V'}.$$

Moreover as

$$d_\Omega^2 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < \alpha,$$

we obtain, almost everywhere in time

$$\alpha \left\|\dot{u}^n(t)\right\|_V \leq \left(M_G \left\|\widetilde{u}^n(t)\right\|_V + \left\|G(0)\right\|_Q\right) + \alpha + \|f\|_{L^\infty\left(0,T;V'\right)}.$$

So it follows that

(4.2)
$$\begin{aligned} &\alpha \left\|\dot{u}^n(t)\right\|_V \\ &\leq c\left(\int_0^t \left\|\dot{u}^n(s)\right\|_V ds + M_G \left\|u_0\right\|_V + \left\|G(0)\right\|_Q\right) + \alpha + \|f\|_{L^\infty\left(0,T;V'\right)}. \end{aligned}$$

Therefore, using Gronwall's inequality, (4.2) yields

$$\left\|\dot{u}^n(t)\right\|_V \leq c\left(\left\|u_0\right\|_V + \left\|G(0)\right\|_Q + \alpha + \|f\|_{L^\infty\left(0,T;V'\right)}\right). \tag{4.3}$$

Then we deduce that

$$\begin{aligned} \left\|u^n(t)\right\|_V &\leq \textstyle\int_0^t \left\|\dot{u}^n(s)\right\|_V ds + \left\|u_0\right\|_V \\ &\leq c(\left\|u_0\right\|_V + \left\|G(0)\right\|_Q + \alpha + \|f\|_{L^\infty\left(0,T;V'\right)}). \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) it results that (u^n) is bounded in $W^{1,\infty}(0,T;V)$. Then there exists a function $u \in W^{1,\infty}(0,T;V)$ such that passing to a subsequence still denoted (u^n) we have $u^n \to u$ weak $*$ in $W^{1,\infty}(0,T;V)$. ☐

Next as in [4] we have

LEMMA 4.2. *There exists a subsequence of* $(\widetilde{u}^n)$ *still denoted* $(\widetilde{u}^n)$ *such that the following results on convergence hold*

$$\begin{aligned} &(i)\ \ \widetilde{u}^n \to u \text{ weak } * \text{ in } L^\infty(0,T;V), \\ &(ii)\ \ \widetilde{u}^n(t) \to u(t) \text{ weakly in } V \text{ a.e. } t \in [0,T]. \end{aligned} \tag{4.5}$$

REMARK 4.3. *As in* [1] *we have* $u(t) \in U$ *for all* $t \in [0,T]$.

Now we need to prove the following result.

LEMMA 4.4. *The following convergence result holds:*

$$u^n \to u \text{ strongly in } C([0,T];V). \tag{4.6}$$

PROOF. In inequality (4.1) we take $w = \dot{u}^m(t)$ and in the same inequality at the order m we take $w = \dot{u}^n(t)$. After adding the resulting inequalities we find

$$\begin{aligned} & a(\dot{u}^m(t) - \dot{u}^n(t), u^m(t) - u^n(t)) \\ & \quad \le b(\tilde{u}^n(t), u^m(t) - u^n(t)) + b(\tilde{u}^m(t), u^n(t) - u^m(t)) \\ & \qquad + j(\tilde{u}^n(t), u^m(t)) + j(\tilde{u}^m(t), u^n(t)) - j(\tilde{u}^n(t), u^n(t)) \\ & \qquad - j(\tilde{u}^m(t), u^m(t)) + \langle f^m(t) - f^n(t), u^m(t) - u^n(t)\rangle_{V',V}. \end{aligned}$$

Integrating this previous inequality, it follows by using Young's inequality that

$$\begin{aligned} & \|u^m(t) - u^n(t)\|_V^2 \le \\ & c\left(T^2/n^2 + T^2/m^2 + \int_0^t \|u^m(s) - u^n(s)\|_V^2\, ds + \|f^m - f^n\|^2_{L^2(0,T;V')}\right). \end{aligned} \tag{4.7}$$

On the other hand for all $\varepsilon > 0$, $\exists N_1 \in \mathbf{N}$ such that

$$\forall m, n \ge N_1 : \|f^m - f^n\|^2_{L^2(0,T;V')} \le \varepsilon^2$$

and

$$\exists N_2 \in \mathbf{N} \text{ such that } \forall m, n \ge N_2 : T^2/n^2 + T^2/m^2 \le \varepsilon^2.$$

Then from (4.7) it follows that $\forall \varepsilon > 0$, $\exists N_0 = \max(N_1, N_2)$ such that $\forall m, n \ge N_0$:

$$\|u^m(t) - u^n(t)\|_V^2 \le c\left(2\varepsilon^2 + \textstyle\int_0^t \|u^m(s) - u^n(s)\|_V^2\, ds\right).$$

Using Gronwall inequality this yields

$$\|u^m(t) - u^n(t)\|_V^2 \le c\varepsilon^2.$$

Hence, we deduce: $\forall \varepsilon > 0$, $\exists N_0 \in \mathbf{N}$ such that $\forall m, n \ge N_0$:

$$\|u^m(t) - u^n(t)\|_V \le c\varepsilon,$$

and so the lemma is proved. ∎

Now we have all the ingredients to prove Theorem 2.1. To this end, we shall prove the following proposition.

Proposition 4.5. *For all $z \in L^2(0,T;V)$ the weak limit u of u^n satisfies the following inequality:*

$$\tag{4.8} \begin{cases} \displaystyle\int_0^T \Big(a(\dot u(t), z(t)-\dot u(t)) + b(u(t), z(t)-\dot u(t)) + j(u(t), z(t)) \\ \qquad - j(u(t), \dot u(t))\Big)dt \\ \quad \geq \displaystyle\int_0^T \langle f(t), z(t)-\dot u(t)\rangle_{V',V}\, dt \\ \qquad + \displaystyle\int_0^T \langle \sigma_\nu(u(t)) + p(u_\nu(t)), z_\nu(t) - \dot u_\nu(t)\rangle\, dt \end{cases}$$

and satisfies the unilateral condition

$$\tag{4.9} \langle \sigma_\nu(u(t)) + p(u_\nu(t)), z_\nu - u_\nu(t))\rangle \geq 0,\ \forall t \in [0,T] \quad \forall z \in U.$$

Proof. From the first inequality (3.1) it follows that for any $z \in L^2(0,T;V)$

$$\begin{cases} a(\dot u^n(t), z(t)-\dot u^n(t)) + b(\widetilde u^n(t), z(t)-\dot u^n(t)) + j(\widetilde u^n(t), z(t)) \\ \quad - j(\widetilde u^n(t), \dot u^n(t)) \\ \geq \langle f^n(t), z(t)-\dot u^n(t)\rangle_{V',V} \ + \langle \sigma_\nu(\tilde u^n(t)) + p(\tilde u^n_\nu(t)), z_\nu(t)\rangle \\ \qquad \text{for } a.a.\ t \in (0,T). \end{cases}$$

Integrating both sides of the previous inequality on $(0,T)$ we obtain the following inequality:

$$\tag{4.10} \begin{aligned} &\int_0^T a(\dot u^n(t), z(t)-\dot u^n(t))\,dt + \int_0^T b(\widetilde u^n(t), z(t)-\dot u^n(t))\,dt \\ &\quad + \int_0^T j(\widetilde u^n(t), z(t))\,dt - \int_0^T j(\widetilde u^n(t), \dot u^n(t))\,dt \\ &\geq \int_0^T \langle f^n(t), z(t)-\dot u^n(t)\rangle_{V',V}\,dt + \int_0^T \langle \sigma_\nu(\tilde u^n(t)) + p(\tilde u^n_\nu(t)), z_\nu(t)\rangle\,dt. \end{aligned}$$

Firstly, we start with the proof of the following lemmas which enable us to pass to the limit in (4.10).

Lemma 4.6. *We have the following relations:*

$$\tag{4.11} \liminf_{n\to\infty} \int_0^T a(\dot u^n(t), \dot u^n(t))\,dt \geq \int_0^T a(\dot u(t), \dot u(t))\,dt,$$

$$\tag{4.12} \liminf_{n\to\infty} \int_0^T j(\widetilde u^n(t), \dot u^n(t))\,dt \geq \int_0^T j(u(t), \dot u(t))\,dt.$$

PROOF. The functional $v \to \int_0^T a(v(t), v(t))\,dt$ is convex and continuous on $L^2(0,T;V)$, so it is sequentially weakly lower semicontinuous, then it suffices to use Lemma 4.1 to prove (4.11). For the proof of (4.12) we refer the reader to [14]. $\square$

LEMMA 4.7. *For all $z \in L^2(0,T;V)$ we have:*

$$\lim_{n\to\infty} \int_0^T a(\dot{u}^n(t), z(t))\,dt = \int_0^T a(\dot{u}(t), z(t))\,dt, \tag{4.13}$$

$$\lim_{n\to\infty} \int_0^T b(\widetilde{u}^n(t), z(t) - \dot{u}^n(t))\,dt = \int_0^T b(u(t), z(t) - \dot{u}(t))\,dt, \tag{4.14}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), z(t))\,dt = \int_0^T j(u(t), z(t))\,dt, \tag{4.15}$$

(4.16)
$$\lim_{n\to\infty} \int_0^T \langle f^n(t), z(t) - \dot{u}^n(t)\rangle_{V',V}\,dt = \int_0^T \langle f(t), z(t) - \dot{u}(t)\rangle_{V',V}\,dt,$$

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \langle \sigma_\nu(\tilde{u}^n(t)) + p(\tilde{u}^n_\nu(t)), z_\nu(t)\rangle\,dt \\ &\qquad = \int_0^T \langle \sigma_\nu(u(t)) + p(u_\nu(t)), z_\nu(t)\rangle\,dt. \end{aligned} \tag{4.17}$$

PROOF. For the proof of (4.13) see [4]. To prove (4.14), we have

$$\begin{aligned} b(\widetilde{u}^n(t), z(t) - \dot{u}^n(t)) = {} & b(\widetilde{u}^n(t), z(t) - \dot{u}^n(t)) - b(u^n(t), z(t) - \dot{u}^n(t)) \\ & + b(u^n(t), z(t) - \dot{u}^n(t)) - b(u(t), z(t) - \dot{u}^n(t)) \\ & + b(u(t), z(t) - \dot{u}^n(t)). \end{aligned}$$

Then it suffices to use (4.6), (2.12)(b), Lemma 4.1 and that

$$\|\widetilde{u}^n(t) - u^n(t)\|_V \le k\,\|\dot{u}^n(t)\|_V \quad a.e. t \in (0,T)$$

to conclude that

$$\begin{aligned} &\int_0^T (b(\widetilde{u}^n(t), z(t) - \dot{u}^n(t)) - b(u^n(t), z(t) - \dot{u}^n(t)))\,dt \to 0, \\ &\int_0^T (b(u^n(t), z(t) - \dot{u}^n(t)) - b(u(t), z(t) - \dot{u}^n(t)))\,dt \to 0, \\ &\int_0^T b(u(t), z(t) - \dot{u}^n(t))\,dt \to \int_0^T b(u(t), z(t) - \dot{u}(t))\,dt. \end{aligned}$$

To show (4.15), (4.16) and (4.17), it suffices to invoke [14]. $\square$

Now, to end the proof of Proposition 4.5 we begin by proving inequality (4.9). Indeed from inequality (3.2) we deduce the discrete inequality

$$\begin{cases} a\left(\delta u^{i+1}, w - u^{i+1}\right) + b\left(u^{i}, w - u^{i+1}\right) + j\left(u^{i+1}, w - u^{i+1}\right) \\ \geq \left\langle f^{i+1}, w - u^{i+1}\right\rangle_{V',V} \quad \forall\, w \in U \end{cases}$$

which implies that for all $w \in L^2(0,T;V)$ such that $w(t) \in U$, *a.e.* $t \in [0,T]$:

(4.18)

$$\begin{aligned} &a\left(\dot{u}^n(t), w(t) - \widetilde{u}^n(t)\right) + b\left(\widetilde{u}^n(t), w(t) - \widetilde{u}^n(t)\right) + j(\widetilde{u}^n(t), w(t) - \widetilde{u}^n(t)) \\ &\geq \left\langle f^n(t), w(t) - \widetilde{u}^n(t)\right\rangle_{V',V}. \end{aligned}$$

Integrating (4.18) with respect to time in $[0,T]$, we get

(4.19)

$$\begin{cases} \displaystyle\int_0^T a\left(\dot{u}^n(t), w(t) - \widetilde{u}^n(t)\right) dt + \int_0^T b\left(\widetilde{u}^n(t), w(t) - \widetilde{u}^n(t)\right) dt \\ \displaystyle + \int_0^T j(\widetilde{u}^n(t), w(t) - \widetilde{u}^n(t)) dt \geq \int_0^T \left\langle f^n(t), w(t) - \widetilde{u}^n(t)\right\rangle_{V',V} dt, \\ \qquad \forall w \in L^2(0,T;V) \text{ such that } w(t) \in U,\ a.e.\ t \in [0,T]. \end{cases}$$

Using (4.5), (4.6) and

$$\left\|\widetilde{u}^n(t) - u^n(t)\right\|_V \leq k\left\|\dot{u}^n(t)\right\|_V \quad a.e.\ t \in (0,T),$$

we pass to the limit as $n \to \infty$ in (4.19) to get

$$(4.20)\quad \begin{cases} \displaystyle\int_0^T a\left(\dot{u}(t), w(t) - u(t)\right) dt + \int_0^T b\left(u(t), w(t) - u(t)\right) dt \\ \displaystyle + \int_0^T j\left(u(t), w(t) - u(t)\right) dt \geq \int_0^T \left\langle f(t), w(t) - u(t)\right\rangle_{V',V} dt, \\ \qquad \forall w \in L^2(0,T;V) \text{ such that } w(t) \in U,\ a.e.\ t \in [0,T]. \end{cases}$$

Then by a classical argument, one obtains from (4.20) the following inequality:

$$\begin{aligned} &a\left(\dot{u}(t), v - u(t)\right) + b\left(u(t), v - u(t)\right) + j\left(u(t), v - u(t)\right) \\ &\geq \left\langle f(t), v - u(t)\right\rangle_{V',V} \quad \forall v \in U,\ a.e.\ t \in [0,T]. \end{aligned}$$

Finally, by using Green's formula, one obtains inequality (4.9). ◻

PROPOSITION 4.8. *The function u satisfies Problem P_2.*

Proof. By passing to the limit as $n \to +\infty$ in inequality (4.10) using Lemmas 4.6 and 4.7, one obtains the following inequalty:

$$\begin{cases} \displaystyle\int_0^T \Big(a(\dot u(t), z(t) - \dot u(t)) + b(u(t), z(t) - \dot u(t)) + j(u(t), z(t)) \\ \quad - j(u(t), \dot u(t)) \Big) dt \\ \displaystyle\geq \int_0^T \langle f(t), z(t) - \dot u(t) \rangle_{V',V}\, dt + \int_0^T \langle \sigma_\nu(u(t)) + p(u_\nu(t)), z_\nu(t) \rangle\, dt \end{cases}$$

and then keeping in mind (2.19), one obtains inequality (4.8). If we set in (4.8) $z \in L^2(0,T;V)$ defined by

$$z(s) = \begin{cases} v & \text{for } s \in [t, t+\lambda] \\ \dot u(s) & \text{elsewhere} \end{cases},$$

we obtain the following inequality:

$$\begin{aligned} &\frac{1}{\lambda} \int_t^{t+\lambda} \big(a(\dot u(s), v - \dot u(s)) + b(u(s), v - \dot u(s)) + j(u(s), v) \\ &\qquad - j(u(s), \dot u(s)) \big) ds \\ &\qquad \geq \frac{1}{\lambda} \int_t^{t+\lambda} \langle f(s), v - \dot u(s) \rangle_{V',V}\, ds \\ &\qquad\quad + \frac{1}{\lambda} \int_t^{t+\lambda} \langle \sigma_\nu(u(s)) + p(u_\nu(s)), v_\nu - \dot u_\nu(s) \rangle\, ds. \end{aligned}$$

Passing to the limit as $\lambda \to 0^+$, we obtain that inequality (2.17) is satisfied for almost all $t \in (0,T)$. Thus we conclude that the function u is a solution of Problem P_2. ☐

5. Conclusion

In this problem we have established an existence result of a weak solution under a smallness assumption on the coefficient of friction for a quasistatic unilateral contact problem with finite penetration in viscoelasticity. The question of the uniqueness of the solution remains still open.

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