GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 483 – 487

CORRECTING TAYLOR'S CELL-LIKE MAP

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Abstract. J. L. Taylor constructed a cell-like map of a compactum X onto the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ such that X is not cell-like. In this note, we point out a defect in the construction and show how to fix it.

Using Adams' example in [1], J. L. Taylor constructed in [7] a cell-like map of a compactum X onto the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ such that X is not cell-like. This Taylor's example is very important in Shape Theory and its related theories. In fact, it was widely used by many authors for various counterexamples. In this note, we point out a defect in the construction and show how to fix it.

In [1], J. F. Adams constructed a compact polyhedron A with a map $\alpha : \Sigma^r A \to A$ from the r-fold suspension of A onto A such that every composition

$$\alpha \circ \Sigma^r \alpha \circ \cdots \circ \Sigma^{(i-1)r} \alpha : \Sigma^{ir} A \to A$$

is not null-homotopic. Taylor defined the compactum X as the inverse limit of the inverse sequence:

$$A \xleftarrow{\alpha} \Sigma^r A \xleftarrow{\Sigma^r \alpha} \Sigma^{2r} A \xleftarrow{\Sigma^{2r} \alpha} \cdots .$$

Then, the inverse limit projection of X to A is not null-homotopic, which implies X is not cell-like. Observe that $\Sigma^{ir} A$ is homeomorphic to ($\approx$) the quotient space

$$\mathbf{I}^{ir} \times A \big/ \big\{ \{z\} \times A \bigm| z \in \partial \mathbf{I}^{ir} \big\}.$$

The Hilbert cube $\mathbf{I}^{\mathbb{N}}$ can be regarded as the inverse limit of the sequence:

$$\mathbf{I}^r \xleftarrow{p_1} \mathbf{I}^{2r} \xleftarrow{p_2} \mathbf{I}^{3r} \xleftarrow{p_3} \cdots ,$$

2010 *Mathematics Subject Classification.* 54C10, 54C56, 55P40, 57N25.
Key words and phrases. Taylor's cell-like map, Hilbert cube, n-fold suspension.

where $p_i : \mathbf{I}^{(i+1)r} = \mathbf{I}^{ir} \times \mathbf{I}^r \to \mathbf{I}^{ir}$ is the projection. For each space Y and $n \in \mathbb{N}$, the following definition is adopted:

$$\Sigma^n Y = \mathbf{I}^n \times Y \big/ \big\{\{z\} \times Y \;\big|\; z \in \partial \mathbf{I}^n\big\}. \tag{1}$$

Let $q_Y^n : \mathbf{I}^n \times Y \to \Sigma^n Y$ be the quotient map. As is easily observed, the projection $\mathrm{pr}_{\mathbf{I}^{ir}} : \mathbf{I}^{ir} \times A \to \mathbf{I}^{ir}$ induces the map $f_i : \Sigma^{ir} A \to \mathbf{I}^{ir}$. Taylor's map $f : X \to \mathbf{I}^{\mathbb{N}}$ is defined by the following <u>commutative</u> diagram:

$$(\star)\qquad \begin{array}{ccccccc} A & \xleftarrow{\ \alpha\ } & \Sigma^r A & \xleftarrow{\ \Sigma^r\alpha\ } & \Sigma^{2r} A & \xleftarrow{\ \Sigma^{2r}\alpha\ } & \Sigma^{3r} A & \xleftarrow{\ \Sigma^{3r}\alpha\ } \cdots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & \\ & & \mathbf{I}^r & \xleftarrow[p_1]{} & \mathbf{I}^{2r} & \xleftarrow[p_2]{} & \mathbf{I}^{3r} & \xleftarrow[p_3]{} \cdots . \end{array}$$

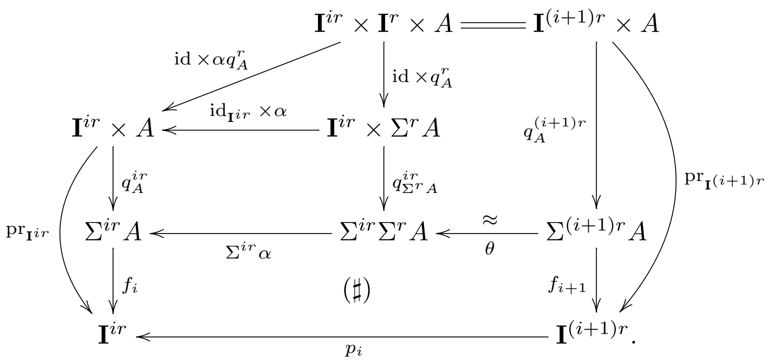
In the above diagram $(\star)$, the map $\Sigma^{ir}\alpha : \Sigma^{ir}\Sigma^r A \to \Sigma^{ir} A$ is induced by the map $\mathrm{id} \times \alpha : \mathbf{I}^{ir} \times \Sigma^r A \to \mathbf{I}^{ir} \times A$, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{I}^{ir} \times A & \xleftarrow{\ \mathrm{id}_{\mathbf{I}^{ir}} \times \alpha\ } & \mathbf{I}^{ir} \times \Sigma^r A \\ \downarrow{\scriptstyle q_A^{ir}} & & \downarrow{\scriptstyle q_{\Sigma^r A}^{ir}} \\ \Sigma^{ir} A & \xleftarrow[\Sigma^{ir}\alpha]{} & \Sigma^{ir}\Sigma^r A. \end{array}$$

It should be remarked that $\Sigma^{ir}\Sigma^r A \approx \Sigma^{(i+1)r} A$ but $\Sigma^{ir}\Sigma^r A \neq \Sigma^{(i+1)r} A$ with respect to our definition of n-fold suspension (1). In fact, observe

$$\begin{aligned} \Sigma^{ir}\Sigma^r A &= \mathbf{I}^{ir} \times \Sigma^r A \big/ \big\{\{z\} \times \Sigma^r A \;\big|\; z \in \partial \mathbf{I}^{ir}\big\} \\ &= \mathbf{I}^{ir} \times \mathbf{I}^r \times A \big/ \big\{\{z\} \times \mathbf{I}^r \times A,\ \{y\} \times A \;\big|\; z \in \partial \mathbf{I}^{ir}, y \in (0,1)^{ir} \times \partial \mathbf{I}^r\big\} \end{aligned}$$

Thus, the commutativity of the diagram $(\star)$ depends on how to identify $\Sigma^{ir}\Sigma^r A$ with $\Sigma^{(i+1)r} A$. In the following diagram, the outside pentagon is commutative but we have to find a homeomorphism $\theta : \Sigma^{(i+1)r} A \to \Sigma^{ir}\Sigma^r A$ making the bottom rectangle $(\sharp)$ commutative:



Assume that such a homeomorphism θ exists and take a point $y \in \partial \mathbf{I}^{ir}$. Since $q^{ir}_{\Sigma^r A}(\{y\} \times \Sigma^r A)$ is a singleton in $\Sigma^{ir}\Sigma^r A$, it follows that

$$\theta\big(q_A^{(i+1)r}(\{(y,z)\} \times A)\big) = q^{ir}_{\Sigma^r A}(\{y\} \times \Sigma^r A) \ \text{ for each } z \in \mathbf{I}^r.$$

If $z \neq z' \in (0,1)^r$ then $q_A^{(i+1)r}(\{(y,z)\} \times A)$ and $q_A^{(i+1)r}(\{(y,z')\} \times A)$ are distinct singletons in $\Sigma^{(i+1)r}A$. This is a contradiction because θ is a bijection. Therefore, we cannot identify $\Sigma^{ir}\Sigma^r A$ with $\Sigma^{(i+1)r}A$ so that the diagram ($\star$) is commutative.

In the rest of this note, we shall show how to fix this defect. To this end, we now adopt the following definition:

$$\Sigma^n Y = \mathbf{B}^n \times Y \big/ \big\{\{z\} \times Y \ \big|\ z \in \mathbf{S}^{n-1}\big\}, \tag{2}$$

where $\mathbf{B}^n$ is the unit closed ball of $\mathbb{R}^n$ and $\mathbf{S}^{n-1}$ ($= \partial \mathbf{B}^n$) is the unit sphere. Let $q^n_Y : \mathbf{B}^n \times Y \to \Sigma^n Y$ be the quotient map. In this definition (2), we have

$$\begin{aligned}\Sigma^{ir}\Sigma^r A &= \mathbf{B}^{ir} \times \Sigma^r A \big/ \big\{\{z\} \times \Sigma^r A \ \big|\ z \in \mathbf{S}^{ir-1}\big\} \\ &= \mathbf{B}^{ir} \times \mathbf{B}^r \times A \big/ \big\{\{z\} \times \mathbf{B}^r \times A,\ \{y\} \times A \ \big|\ z \in \mathbf{S}^{ir-1}, \\ &\qquad\qquad y \in (\mathbf{B}^{ir} \setminus \mathbf{S}^{ir-1}) \times \mathbf{S}^{r-1}\big\}.\end{aligned}$$

Of course, $\Sigma^{(i+1)r}A \neq \Sigma^{ir}\Sigma^r A$. For each $i \in \mathbb{N}$, let $p_i : \mathbf{B}^{(i+1)r} \to \mathbf{B}^{ir}$ be the restriction of the projection of $\mathbb{R}^{(i+1)r} = \mathbb{R}^{ir} \times \mathbb{R}^r$ onto $\mathbb{R}^{ir}$, and define a map $\varphi_i : \mathbf{B}^{ir} \times \mathbf{B}^r \to \mathbf{B}^{(i+1)r}$ as follows:

$$\varphi_i(y,z) = \begin{cases} (y, \sqrt{1 - \|y\|^2} z) & \text{if } z \neq 0, \\ (y, 0) & \text{if } z \neq 0. \end{cases}$$

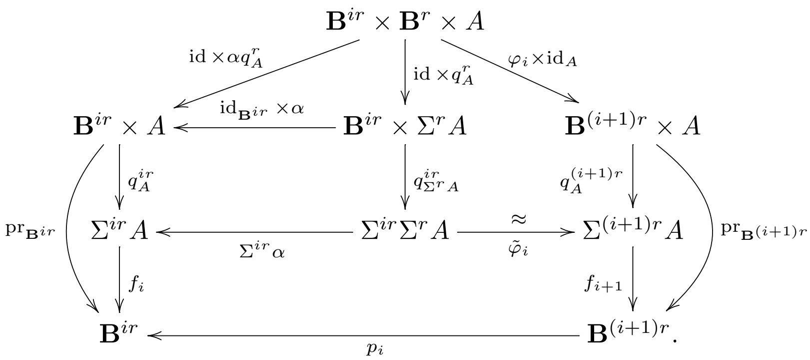
Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{B}^{ir} \times \mathbf{B}^r & & \\ {\scriptstyle \mathrm{pr}_{\mathbf{B}^{ir}}} \downarrow & \searrow^{\varphi_i} & \\ \mathbf{B}^{ir} & \xleftarrow[p_i]{} & \mathbf{B}^{(i+1)r}. \end{array}$$

For each $z \in \mathbf{S}^{ir-1}$, $\varphi(\{z\} \times \mathbf{B}^r)$ is a singleton with $\varphi^{-1}(\varphi(\{z\} \times \mathbf{B}^r)) = \{z\} \times \mathbf{B}^r$. The restriction $\varphi_i | (\mathbf{B}^{ir} \setminus \mathbf{S}^{ir-1}) \times \mathbf{B}^r$ is injective and

$$\varphi^{-1}(\mathbf{S}^{(i+1)r-1}) = (\mathbf{S}^{ir-1} \times \mathbf{B}^r) \cup ((\mathbf{B}^{ir} \setminus \mathbf{S}^{ir-1}) \times \mathbf{S}^{r-1}).$$

As is easily observed, $\varphi_i \times \mathrm{id}_A$ induces the homeomorphism $\tilde{\varphi}_i : \Sigma^{ir}\Sigma^r A \to \Sigma^{(i+1)r} A$ which makes the following diagram commutative:



Identifying each $\Sigma^{ir}\Sigma^r A$ with $\Sigma^{(i+1)r} A$ by this homeomorphism $\tilde{\varphi}_i$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} A & \xleftarrow{\alpha} & \Sigma^r A & \xleftarrow{\Sigma^r \alpha} & \Sigma^{2r} A & \xleftarrow{\Sigma^{2r}\alpha} & \Sigma^{3r} A & \xleftarrow{\Sigma^{3r}\alpha} & \cdots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ & & \mathbf{B}^r & \xleftarrow[p_1]{} & \mathbf{B}^{2r} & \xleftarrow[p_2]{} & \mathbf{B}^{3r} & \xleftarrow[p_3]{} & \cdots . \end{array}$$

Let Y be the inverse limit of the bottom sequence above. Then, we can obtain the map $f : X \to Y$ induced by maps f_i, $i \in \mathbb{N}$. Just as in the proof given in [7], it can be proved that f is a cell-like map. It should be noticed that Y can be regarded as an infinite-dimensional compact convex set in the Fréchet space[1] $\mathbb{R}^{\mathbb{N}}$. Indeed, $\mathbb{R}^{\mathbb{N}}$ can be regarded the inverse limit of the top sequence in the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{R}^r & \xleftarrow{\tilde{p}_1} & \mathbb{R}^{2r} & \xleftarrow{\tilde{p}_2} & \mathbb{R}^{3r} & \xleftarrow{\tilde{p}_3} & \cdots \\ \cup & & \cup & & \cup & & \\ \mathbf{B}^r & \xleftarrow[p_1]{} & \mathbf{B}^{2r} & \xleftarrow[p_2]{} & \mathbf{B}^{3r} & \xleftarrow[p_3]{} & \cdots , \end{array}$$

where $\tilde{p}_i : \mathbb{R}^{(i+1)r} = \mathbb{R}^{ir} \times \mathbb{R}^r \to \mathbb{R}^{ir}$ is the projection. Then, we can apply the classical result of Keller ([4]) to show $Y \approx \mathbf{I}^{\mathbb{N}}$. Indeed, every infinite-dimensional compact convex set in a Fréchet space is affinely homeomorphic to an infinite-dimensional compact convex set in Hilbert space ℓ_2 ([2, Chapter III, Proposition 3.1]), which is homeomorphic to the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ by Keller's Theorem [4] (cf. [2, Chapter III, Theorem 3.1]).

[1] A completely metrizable locally convex topological linear space is called a Fréchet space.

REMARK 1. To show $Y \approx \mathbf{I}^{\mathbb{N}}$, we can also apply Toruńczyk's characterization of the Hilbert cube [8] (cf. [6], [9]). In fact, it is easy to verify that Y has the disjoint cells property.

REMARK 2. To see that Y is an AR, since every p_i is a fine homotopy equivalence, we can also apply Theorem 6.3 in [3].

ACKNOWLEDGEMENTS.
The author would like to express his thanks to Katsuhisa Koshino who found the gap of the proof in [7].

REFERENCES

[1] J. F. Adams, *On the groups $J(X)$. IV,* Topology **5** (1966), 21–71; *errata,* ibid. **7** (1968), 331.
[2] C. Bessaga and A. Pełczyński, Selected topics in infinite-dimensional topology, Monografie Matematyczne **58**, PWN Polish Scientific Publishers, Warsaw, 1975.
[3] Z. Čerin, *Characterizing global properties in inverse limits,* Pacific J. Math. **112** (1984), 49–68.
[4] O.-H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum,* Math. Ann. **105** (1931), 748–758.
[5] S. Mardešić and J. Segal, Shape theory. The inverse system approach, North-Holland Mathematical Library **26**, North-Holland Publishing Co., Amsterdam-New York, 1982.
[6] J. van Mill, Infinite-dimensional topology. Prerequisites and introduction, North-Holland Mathematical Library **43**, North-Holland Publishing Co., Amsterdam, 1989.
[7] J. L. Taylor, *A counterexample in shape theory,* Bull. Amer. Math. Soc. **81** (1975), 629–632.
[8] H. Toruńczyk, *On CE–images of the Hilbert cube and characterization of Q-manifolds,* Fund. Math. **106** (1980), 31–40.
[9] J. J. Walsh, *Characterization of Hilbert cube manifolds: an alternate proof,* in: Geometric and algebraic topology, Banach Center Publ. **18**, PWN Polish Scientific Publishers, Warsaw, 1986, 153–160.

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Received: 31.8.2010.
Revised: 11.1.2011.