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THE COARSE SHAPE PATH CONNECTEDNESS

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Abstract. The bi-pointed coarse shape category of topological spaces is constructed and the notions of a coarse shape path and coarse shape connectedness of a space are naturally introduced. It is proven that the shape path connectedness strictly implies the coarse shape path connectedness even on metrizable compacta. Furthermore, the coarse shape path connectedness on metrizable compacta reduces to ordinary connectedness.

1. Introduction

The coarse shape theory, abstract and standard (for topological spaces), was recently founded by the authors ([6]). It functorially generalizes shape theory in that there exist spaces (metrizable continua) having the same coarse shape type and different shape types (see [4], [9] and [6]). However, many of the well-known shape invariants are coarse shape invariants as well (connectedness, triviality of shape, shape dimension $sd \leq n$, n-shape connectedness, movability, n-movability, being an $FANR$, strong movability, stability, the Mittag-Leffler property; see [13], [5] and [12]).

Long ago J. Krasinkiewicz and P. Minc introduced in [7] the notions of joinability and weak joinability for metrizable continua. They proved, for instance, that solenoids are not weakly joinable. Afterwards Š. Ungar introduced the notions of shape path and of the shape path connectedness for topological spaces ([14]). He noticed that joinability and shape path connectedness coincide on the class of all metrizable continua, and that this property is a shape invariant on that class.

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In this paper we introduce the notions of a coarse shape path and of the coarse shape path connectedness of a topological space, quite analogously to the shape case, and we compare them to the former case as well as to ordinary connectedness. We proved that shape path connectedness strictly implies the coarse shape path connectedness (even on metrizable continua), and that coarse shape path connectedness implies connectedness (Theorem 3.3). Moreover, on metrizable compacta, connectedness and coarse shape path connectedness coincide (Theorem 3.5). It is an open problem whether coarse shape path connectedness strictly implies connectedness (Remark 3.10). A possible (counter)example does not admit a countable polyhedral expansion (Lemma 3.6).

Since in this setting one has to deal with bi-pointed morphisms, we have firstly clarified in details how a polyhedral (ANR-) resolution of a space induces a bi-pointed resolution (Lemma 2.3). Then, by applying the corresponding homotopy functor, one obtains a polyhedral (ANR-) expansion of a bi-pointed space (Theorem 2.1). The rest of Section 2 follows the general rule of constructing a (coarse) shape category ([8, 6]) of (bi-)pointed spaces, and at the end we point out that the classification by (bi-)pointed coarse shape types is strictly coarser than that by (bi-)pointed shape types (Theorem 2.5).

2. The bi-pointed coarse shape category

Let Top denote the category of all topological spaces X and all (continuous) mappings $f : X \to Y$. By considering all pairs of topological spaces (X, A) and all mappings of pairs $f : (X, A) \to (Y, B)$, one obtains the category of topological pairs, denoted by Top^2. In the special case of all singleton subspaces, Top^2 reduces to the corresponding *pointed* category, denoted by Top_0. Finally, by distinguishing and fixing an ordered pair of points of a space, (X, x_0, x_1), and considering all the mappings $f : (X, x_0, x_1) \to (Y, y_0, y_1)$ satisfying $f(x_i) = y_i$, $i = 0, 1$, one obtains the *bi-pointed* category of topological spaces, denoted by Top_{00}.

By reducing the object classes to all polyhedra (weak topology), polyhedral pairs, pointed polyhedra and bi-pointed polyhedra, one obtains the full subcategories $Pol \subseteq Top$, $Pol^2 \subseteq Top^2$, $Pol_0 \subseteq Top_0$ and $Pol_{00} \subseteq Top_{00}$, respectively. In the same way, we consider the subcategories determined by the class of all absolute neighbourhood retracts - ANR's (for metric spaces).

The corresponding homotopy (quotient) categories and subcategories are denoted by

$$HPol, HANR \subseteq HTop, \quad HPol^2, HANR^2 \subseteq HTop^2,$$

$$HPol_0, HANR_0 \subseteq HTop_0 \text{ and } HPol_{00}, HANR_{00} \subseteq HTop_{00}.$$

Of course, in the case of pairs, pointed case and bi-pointed case, the homotopies are morphisms of the corresponding categories, and thus, in the pointed and bi-pointed case they preserve the distinguished points. The definitions of a resolution and an expansion, as well as their main properties including mutual relationship, one can find in [8, I.2 and I.6]. A well known fact is that $HPol$, $HPol^2$ and $HPol_0$ (as well as $HANR$, $HANR^2$ and $HANR_0$) are pro-reflective subcategories of $HTop$, $HTop^2$ and $HTop_0$, respectively ([8, Theorem I.6.7, Theorem I.6.2; Theorem I.6.10, Theorem I.6.8]) (recall that the term "dense", used in [8], has been replaced, according to [11], by "pro-reflective").

THEOREM 2.1. *$HPol_{00}$ and $HANR_{00}$ are pro-reflective subcategories of $HTop_{00}$.*

First of all, let us prove the following simple facts (see [8, I.6.1 and III.A1.3.1]).

LEMMA 2.2. *Let A be a nondiscrete topological space consisting of two points. Then,*

(i) *every mapping of A to a T_1-space is a constant mapping, and consequently, every morphism*

$$\boldsymbol{q} = (q_\mu) : A \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$$

of pro-Top, where all Y_μ are T_1-spaces, consists of constant mappings $q_\mu : A \to Y_\mu$, $\mu \in M$;

(ii) *the trivial covering $\{A\}$ of A is a normal refinement of every open covering of A;*

(iii) *every morphism*

$$\boldsymbol{p} = (p_\lambda) : A \to \boldsymbol{A} = (A_\lambda = \{a_\lambda\}, p_{\lambda\lambda'}, \Lambda)$$

of pro-Top is a (polyhedral and ANR-) resolution of A, especially the trivial one

$$\boldsymbol{p}' = (p'_1 \equiv p') : A \to \boldsymbol{A}' = (A'_1 = \{a'\}, 1_{\{a'\}}, \{1\}),$$

which is not an inverse limit.

PROOF. Let $A = \{a, b\}$ be a nondiscrete space and let $f : A \to Y$ be a mapping to a T_1-space Y. Suppose that $f(a) \neq f(b)$. Then there exist open V and V' in Y such that $f(a) \in V \setminus V'$, $f(b) \in V' \setminus V$. Since A is not discrete, at least for one of the points a, b, A is the minimal neighbourhood. It follows that $f(A) \subseteq V$ or $f(A) \subseteq V'$ - a contradiction. Thus, (i) holds. To prove (ii), observe first that, since A is not discrete, A must be a member of any open covering of A. Moreover, $\{A\}$ is a normal covering of A because the single mapping (ϕ), $\phi = c_1 : A \to [0, 1]$ (the constant mapping to 1), is obviously a partition of unity on A subordinated to $\{A\}$ (notice that, if $\mathcal{U} = (U_j)_{j \in J}$ is a normal covering of A, and $U_{j'} \in \mathcal{U}$ such that $\varnothing \neq U_{j'} \neq A$, then, for every subordinated partition of unity (ϕ_j), it must be $\phi_{j'} = c_0$ - the constant mapping to 0). Finally, by (ii), $\{A\}$ refines every normal covering of A. Thus, the singleton $\{\mathcal{U}_0 = \{A\}\}$ is cofinal in the partially ordered set $Cov(A)$ of all normal coverings of A. Now, the conclusion follows by [8, Theorem I.6.7] and its proof, and by the obvious fact that $\lim \boldsymbol{A} \approx \lim \boldsymbol{A}'$ is a singleton. $\square$

Since a polyhedral (*ANR*-) expansion of a space is usually obtained via a polyhedral (*ANR*-) resolution, we want to clarify some related elementary facts in the bi-pointed case. First, we define a resolution of a bi-pointed space in the most natural way: A morphism

$$\boldsymbol{p} = (p_\lambda) : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) = ((X_\lambda, x_\lambda, x'_\lambda), p_{\lambda\lambda'}, \Lambda)$$

of *pro-*Top_{00} is said to be a *resolution of a bi-pointed space* (X, x_0, x_1) provided it has properties (R1) and (R2) (see [8, I.6.1 and I.6.5]) for bi-pointed ANR's (Q, q_0, q_1). We say that such a $\boldsymbol{p}$ is a *polyhedral* (ANR-) *resolution* of (X, x_0, x_1) if all $(X_\lambda, x_\lambda, x'_\lambda)$ are bi-pointed polyhedra (ANR's).

LEMMA 2.3. *Let X be a topological space and let $x_0, x_1 \in X$. Let*

$$\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$$

be a polyhedral (ANR-) resolution of X and let, for each $\lambda \in \Lambda$, $x_\lambda = p_\lambda(x_0), x'_\lambda = p_\lambda(x_1) \in X_\lambda$. Then,

(i) *the morphism*

$$\boldsymbol{p} = (p_\lambda) : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) = ((X_\lambda, x_\lambda, x'_\lambda), p_{\lambda\lambda'}, \Lambda)$$

(of pro-Top_{00}) is a polyhedral (ANR-) resolution of the bi-pointed space (X, x_0, x_1);

(ii) *the restriction*

$$\boldsymbol{p}' = (p'_\lambda) : \{x_0, x_1\} \to \{\boldsymbol{x}_0, \boldsymbol{x}_1\} = (\{x_\lambda, x'_\lambda\}, p'_{\lambda\lambda'}, \Lambda)$$

of $\boldsymbol{p}$ (morphism of pro-Top) is a polyhedral (ANR-) resolution of the subspace $\{x_0, x_1\} \subseteq X$;

(iii) *the morphism*

$$\boldsymbol{p} = (p_\lambda) : (X, \{x_0, x_1\}) \to (\boldsymbol{X}, \{\boldsymbol{x}_0, \boldsymbol{x}_1\}) = ((X_\lambda, \{x_\lambda, x'_\lambda\}), p_{\lambda\lambda'}, \Lambda)$$

(of pro-Top^2) is a polyhedral (ANR-) resolution of the topological pair $(X, \{x_0, x_1\})$;

(iv) *$\{x_0, x_1\} \subseteq X$ is normally embedded.*

PROOF. Let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a polyhedral (ANR-) resolution of a space X, and let $x_0, x_1 \in X$. Denote, for each $\lambda \in \Lambda$, $x_\lambda = p_\lambda(x_0)$ and $x'_\lambda = p_\lambda(x_1)$. First notice that, for every related pair $\lambda \leq \lambda'$, $p_{\lambda\lambda'}(x_{\lambda'}) = x_\lambda$ and $p_{\lambda\lambda'}(x'_{\lambda'}) = x'_\lambda$. Thus, $(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) \equiv ((X_\lambda, x_\lambda, x'_\lambda), p_{\lambda\lambda'}, \Lambda)$ is indeed an object of *pro-*Pol_{00} (*pro-*ANR_{00}) and $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$ is a morphism of *pro-*Top_{00}. If $x_0 = x_1$, then the statements reduce to the well known pointed case. Hence, without loss of generality, we may assume that $x_0 \neq x_1$. We are to show that $\boldsymbol{p}$ is a resolution of the bi-pointed space (X, x_0, x_1).

CASE 1. $\{x_0, x_1\} \subseteq X$ is not a discrete subspace.

Let (Q, q_0, q_1) be a bi-pointed ANR, let $\mathcal{V}$ be an open covering of Q and let $h : (X, x_0, x_1) \to (Q, q_0, q_1)$ be a bi-pointed mapping. By Lemma 2.2(i), for every $\lambda \in \Lambda$, $x_\lambda = x'_\lambda$. If $q_0 \neq q_1$, there is no such bi-pointed mapping h, and thus condition (R1) for $\boldsymbol{p}$ is trivially fulfilled. Similarly, then there is no bi-pointed mapping of any $(X_\lambda, x_\lambda, x'_\lambda = x_\lambda)$ to (Q, q_0, q_1), and hence, condition (R2) for $\boldsymbol{p}$ is trivially fulfilled. Let $q_0 = q_1$. However, the conclusion now follows by the ordinary pointed case. Therefore, statement (i) in this case is proven. Further, since $x_\lambda = x'_\lambda$ for every $\lambda \in \Lambda$, statement (ii) follows by Lemma 2.2(iii). Now, the assumption, (ii) and Corollary I.6.6 of [8] imply that $\boldsymbol{p} : (X, \{x_0, x_1\}) \to ((X_\lambda, \{x_\lambda, x'_\lambda\}), p_{\lambda\lambda'}, \Lambda)$ is a resolution of the pair $(X, \{x_0, x_1\})$, which proves statement (iii). Finally, according to (iii) and (ii), statement (iv) follows by Theorem I.6.12 of [8].

Case 2. $\{x_0, x_1\} \subseteq X$ is a discrete subspace.

Let (Q, q_0, q_1) be a bi-pointed ANR, let $\mathcal{V}$ be an open covering of Q and let $h : (X, x_0, x_1) \to (Q, q_0, q_1)$ be a bi-pointed mapping. Let us first consider the subcase $q_0 = q_1$. Choose an open covering $\mathcal{V}'$ of Q such that $Cl\mathcal{V}'$ refines $\mathcal{V}$ and that member $V'_0 \in \mathcal{V}'$ containing $q_0 = q_1$ is the only one and contractible (every ANR is normal and locally contractible). In the same way, let $\mathcal{V}''$ be chosen with respect to $\mathcal{V}'$. By property (R1) of the resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, there exist a $\lambda \in \Lambda$ and a mapping $g : X_\lambda \to Q$ such that $(gp_\lambda, h) \leq \mathcal{V}''$. Then, by the uniqueness of V''_0, it holds that $g(x_\lambda), g(x'_\lambda), h(x_0) = h(x_1)(= q_0 = q_1) \in V''_0 \subseteq ClV''_0 \subseteq V'_0$. Now, by means of a contraction of V''_0 to q_0 (respectively the complement $Q \setminus V'_0$), the mapping g can be replaced by a bi-pointed mapping $g' : (X_\lambda, x_\lambda, x'_\lambda) \to (Q, q_0, q_1 = q_0)$ such that $(g'p_\lambda, h) \leq \mathcal{V}'' \leq \mathcal{V}' \leq \mathcal{V}$. This proves that $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$ has property (R1). In order to prove (R2) for $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$, let $(Q, q_0, q_1 = q_0)$ be a bi-pointed ANR and let $\mathcal{V}$ be an open covering of Q. By property (R2) of the resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, there exists an open covering $\mathcal{V}'$ of Q such that, for every $\lambda \in \Lambda$ and every pair of mappings $g, g' : X_\lambda \to Q$ satisfying $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}'$, there exists a $\lambda' \geq \lambda$ such that $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}$. Clearly, this also holds in the special case of a pair of bi-pointed mappings $g, g' : (X_\lambda, x_\lambda, x'_\lambda) \to (Q, q_0, q_1 = q_0)$. Thus, $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$ has property (R2) as well, and the statement (i), in the case of a discrete $\{x_0, x_1\}$ and $q_0 = q_1$, is proven. In the subcase $q_0 \neq q_1$, choose an open covering $\mathcal{V}'$ of $\mathcal{V}$ such that $Cl(\mathcal{V}') \leq \mathcal{V}$ and $\mathcal{V}'$ has a unique $V'_i \in \mathcal{V}'$ containing q_i, $i = 0, 1$, and moreover, $Cl(V'_0) \cap Cl(V'_1) = \varnothing$ and V'_i is contractible to q_i, $i = 0, 1$. Further, choose an open covering $\mathcal{V}''$ of Q such that $Cl(\mathcal{V}'') \leq \mathcal{V}'$ and $\mathcal{V}''$ has a unique V''_i containg q_i, and V''_i is contractible to q_i, $i = 1, 2$. By property (R1) of the resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, there exist a $\lambda \in \Lambda$ and a mapping $g : X_\lambda \to Q$ such that $(gp_\lambda, h) \leq \mathcal{V}''$. Then $g(x_\lambda), h(x_0)(= q_0) \in V''_0 \subseteq ClV''_0 \subseteq V'_0$ and $g(x'_\lambda), h(x_1)(= q_1) \in V''_1 \subseteq ClV''_1 \subseteq V'_1$. Now, by means of contractions of V''_i to q_i, $i = 1, 2$, (respectively the complements $Q \setminus V'_i$, $i = 1, 2$), the mapping g can be replaced by a bi-pointed mapping $g' : (X_\lambda, x_\lambda, x'_\lambda) \to (Q, q_0, q_1)$ such that $(g'p_\lambda, h) \leq \mathcal{V}'' \leq \mathcal{V}' \leq \mathcal{V}$. This proves that $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$ has property (R1). In order to prove (R2) for $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$, let (Q, q_0, q_1) be a bi-pointed ANR and let $\mathcal{V}$ be an open covering of Q. By property (R2) of the resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, there exists an open covering $\mathcal{V}'$ of Q such that, for every $\lambda \in \Lambda$ and every pair of mappings $g, g' : X_\lambda \to Q$ satisfying $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}'$, there exists a $\lambda' \geq \lambda$ such that $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}$. Clearly, this also holds in the special case of a pair of bi-pointed mappings $g, g' : (X_\lambda, x_\lambda, x'_\lambda) \to (Q, q_0, q_1)$. Thus, $\boldsymbol{p} : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$ has property (R2) as well, and the statement (i) is proven.

Statement (ii) is obviously true by applying the characterization in terms of conditions (B1) and (B2) ([8, Theorem I.6.5 and Corollary I.6.1]). Further, observe that, for each $\lambda \in \Lambda$, the subpolyhedron (sub-ANR) $\{x_\lambda, x'_\lambda\}$ $(\subseteq X_\lambda)$ is a normal space. Thus, by (ii) and Corollary I.6.6 of [8], $\boldsymbol{p} : (X, \{x_0, x_1\}) \to ((X_\lambda, \{x_\lambda, x'_\lambda\}), p_{\lambda\lambda'}, \Lambda)$ is a resolution of the pair $(X, \{x_0, x_1\})$, which proves statement (iii). Finally, according to (iii) and (ii), statement (iv) follows by Theorem I.6.12 of [8]. Hence, all the statements in the subcase $q_0 \neq q_1$ are proven. □

Proof of Theorem 2.1. Let (X, x_0, x_1) be a bi-pointed space. By Lemma 2.3(iii), for every polyhedral (ANR-) resolution $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X (existing by Theorem I.6.7 and Remark I.6.7 of [8]), the same morphism $\boldsymbol{p}$, reinterpreted as a morphism

$$\boldsymbol{p} = (p_\lambda) : (X, \{x_0, x_1\}) \to (\boldsymbol{X}, \{\boldsymbol{x}_0, \boldsymbol{x}_1\}) = ((X_\lambda, \{x_\lambda, x'_\lambda\}), p_{\lambda\lambda'}, \Lambda)$$

of *pro*-Top^2, is a polyhedral (ANR-) resolution of the pair $(X, \{x_0, x_1\})$. By applying the homotopy functor, Theorem I.6.8 of [8] implies that the morphism

$$H\boldsymbol{p} = ([p_\lambda]) : (X, \{x_0, x_1\}) \to H(\boldsymbol{X}, \{\boldsymbol{x}_0, \boldsymbol{x}_1\}) = ((X_\lambda, \{x_\lambda, x'_\lambda\}), [p_{\lambda\lambda'}], \Lambda)$$

of *pro*-$HTop^2$ is a polyhedral (ANR-) expansion of the pair $(X, \{x_0, x_1\})$. Since a homotopy which is rel $\{a, b\}$ is the homotopy that is rel $\{a\}$ and rel $\{b\}$ as well, it follows by our construction that the same morphism $H\boldsymbol{p}$, reinterpreted as a morphism

$$H\boldsymbol{p} = ([p_\lambda]) : (X, x_0, x_1) \to H(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) = ((X_\lambda, x_\lambda, x'_\lambda), [p_{\lambda\lambda'}], \Lambda)$$

of *pro*-$HTop_{00}$, is a polyhedral (ANR-) expansion of the bi-pointed space (X, x_0, x_1). This completes the proof of the theorem. □

The construction of the *bi-pointed coarse shape category* Sh^*_{00} follows now the general rule, i.e., it is the category $Sh^*_{(HTop_{00}, HPol_{00})}$, or equivalently, $Sh^*_{(HTop_{00}, HANR_{00})}$ (see [8, I.2-3]; [6, Sections 3-4]). Briefly, the objects of Sh^*_{00} are all bi-pointed topological spaces (X, x_0, x_1), while a morphism set $Sh^*_{00}((X, x_0, x_1), (Y, y_0, y_1))$ consists of all equivalence classes $F^* = \langle \boldsymbol{f}^* \rangle$ of morphisms $\boldsymbol{f}^* : (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) \to (\boldsymbol{Y}, \boldsymbol{y}_0, \boldsymbol{y}_1)$ of *pro**-$HPol_{00}$ (equivalently, *pro**-$HANR_{00}$) ranging over the corresponding expansions. Hereby, each morphism $\boldsymbol{f}^*$ is the equivalence class of a $*$-morphism $(f,[f_\mu^n]):(\boldsymbol{X},\boldsymbol{x}_0,\boldsymbol{x}_1)\to(\boldsymbol{Y},\boldsymbol{y}_0,\boldsymbol{y}_1)$ of *inv**-$HPol_{00}$ (equivalently, *inv**-$HANR_{00}$). The category *pro**-$HPol_{00}$ (equivalently, *pro**-$HANR_{00}$) is a *realizing category* for Sh_{00}^*, i.e.,

$$\begin{aligned}Sh_{00}^*((X,x_0,x_1),(Y,y_0,y_1)) &\approx pro^*\text{-}HPol_{00}((\boldsymbol{X},\boldsymbol{x}_0,\boldsymbol{x}_1),(\boldsymbol{Y},\boldsymbol{y}_0,\boldsymbol{y}_1))\\ &\approx pro^*\text{-}HANR_{00}((\boldsymbol{X},\boldsymbol{x}_0,\boldsymbol{x}_1),(\boldsymbol{Y},\boldsymbol{y}_0,\boldsymbol{y}_1)),\end{aligned}$$

and every bi-pointed coarse shape morphism $F^*:(X,x_0,x_1)\to(Y,y_0,y_1)$ is represented by a diagram (in *pro**-$HTop_{00}$)

$$\begin{array}{ccc}(\boldsymbol{X},\boldsymbol{x}_0,\boldsymbol{x}_1) & \overset{\boldsymbol{p}}{\leftarrow} & (X,x_0,x_1)\\ \boldsymbol{f}^*\downarrow & & \\ (\boldsymbol{Y},\boldsymbol{y}_0,\boldsymbol{y}_1) & \underset{\boldsymbol{q}}{\leftarrow} & (Y,y_0,y_1)\end{array}.$$

For the sake of completeness, let us briefly recall the definitions of a $*$-morphism and the corresponding equivalence relation (in the absolute case). A $*$-*morphism* $(f,[f_\mu^n]):\boldsymbol{X}\to\boldsymbol{Y}$ of *inv**-*HTop* consists of a function (the *index function*) $f:M\to\Lambda$ and, for each $\mu\in M$, of a sequence $([f_\mu^n])$ of the homotopy classes of mappings $f_\mu^n:X_{f(\mu)}\to Y_\mu$, $n\in\mathbb{N}$, such that

$$(\forall\mu\le\mu')(\exists\lambda\ge f(\mu),f(\mu'))(\exists n\in\mathbb{N})(\forall n'\ge n)f_\mu^{n'}p_{f(\mu)\lambda}\simeq q_{\mu\mu'}f_{\mu'}^{n'}p_{f(\mu')\lambda}.$$

Given a pair of $*$-morphisms $(f,[f_\mu^n]),(f',[f_\mu'^n]):\boldsymbol{X}\to\boldsymbol{Y}$, then $(f,[f_\mu^n])$ is said to be *equivalent to* $(f',[f_\mu'^n])$, denoted by $(f,[f_\mu^n])\sim(f',[f_\mu'^n])$, provided

$$(\forall\mu\in M)(\exists\lambda\ge f(\mu),f'(\mu))(\exists n\in\mathbb{N})(\forall n'\ge n)f_\mu^{n'}p_{f(\mu)\lambda}\simeq f_\mu'^{n'}p_{f'(\mu)\lambda}.$$

Further, there exists a functor $S^*:HTop_{00}\to Sh_{00}^*$ - the *bi-pointed coarse shape functor*, keeping the objects fixed (as well as in the bi-pointed shape case, $S:HTop_{00}\to Sh_{00}$). More precisely, by general theory ([8, I.2.3]), for every bi-pointed mapping $f:(X,x_0,x_1)\to(Y,y_0,y_1)$ and every pair of polyhedral (*ANR*-) expansions $\boldsymbol{p}$ and $\boldsymbol{q}$ of (X,x_0,x_1) and (Y,y_0,y_1) respectively, there exists a unique $\boldsymbol{f}:(\boldsymbol{X},\boldsymbol{x}_0,\boldsymbol{x}_1)\to(\boldsymbol{Y},\boldsymbol{y}_0,\boldsymbol{y}_1)$ of *pro*-$HPol_{00}$ (*pro*-$HANR_{00}$) such that $\boldsymbol{fp}=\boldsymbol{q}\lfloor[f]\rfloor$, where $\lfloor[f]\rfloor$ denotes the rudimentary embedding of $[f]$ into *pro*-$HTop_{00}$. Then $\boldsymbol{f}$ represents the bi-pointed shape morphism $F\equiv S([f]):(X,x_0,x_1)\to(Y,y_0,y_1)$ of Sh_{00}. Further, by applying the embedding functor $\underline{J}$: *pro*-$HTop_{00}\to$ *pro**-$HTop_{00}$ which keeps the objects fixed (compare Proposition 3.24 of [6]), one obtains $\underline{J}(\boldsymbol{f})\underline{J}(\boldsymbol{p})=\underline{J}(\boldsymbol{q})\underline{J}(\lfloor[f]\rfloor)$ in *pro**-$HTop_{00}$, with $\underline{J}(\boldsymbol{f})$ unique. Then $\underline{J}(\boldsymbol{f})\equiv\boldsymbol{f}^*$ represents the bi-pointed coarse shape morphism $F^*\equiv S^*([f]):(X,x_0,x_1)\to(Y,y_0,y_1)$ of Sh_{00}^*. Further, the functor $\underline{J}$ of the "pro-categories" induces the embedding functor $J:Sh_{00}\to Sh_{00}^*$ of the "shape" categories such that $S^*=JS$. Thus, every bi-pointed shape morphism may be considered as a special bi-pointed coarse shape morphism.

Remark 2.4. Similarly, and much simpler, one constructs the *pointed coarse shape category* Sh_0^* of pointed topological spaces and related functors. Further, by Theorem 2.1 and following the general rule (see [8, I.2-3], and [13, Sections 5-6]), one can construct the *bi-pointed* (*pointed*) *weak shape category* Sh_{*00} (Sh_{*0}), of bi-pointed (pointed) topological spaces as well as the corresponding functors.

Theorem 2.5. *There exist a metrizable continuum X and a pair of points $x_0, x_1 \in X$ such that $Sh(X, x_0) \neq Sh(X, x_1)$ and $Sh^*(X, x_0) = Sh^*(X, x_1)$. In addition, there exists an $x_0' \in X$, $x_0 \neq x_0' \neq x_1$, such that $Sh(X, x_0, x_0') \neq Sh(X, x_0, x_1)$ and $Sh^*(X, x_0, x_0') = Sh^*(X, x_0, x_1)$.*

Proof. Let Σ_2 be the dyadic solenoid and let $z_0 \in \Sigma_2$. Further, let S^1 be a 1-sphere and let $x_0 \in S^1$. We describe S^1 by using complex numbers $z = e^{2\pi i\theta}$, i.e.,

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}.$$

Take X to be the wedge $\Sigma_2 \vee S^1$, where z_0 and x_0 are identified. Consider the polyhedral resolution (which is the inverse limit)

$$\boldsymbol{p} = (p_j) : X \to \boldsymbol{X} = (X_j = S_1 \vee S_2, p_{jj'}, \mathbb{N}),$$

where $S_1 = S_2 = S^1$ and $p_{jj+1} = p \vee 1_{S_2}$ with $p(z) = z^2$. Notice that, for every $j \in \mathbb{N}$, $p_j(x_0) = x_0$ is the wedge point of $X_j = S_1 \vee S_2$. Choose $z_1 \in S_1 \subseteq X_1$ determined by its argument $\theta_1 = \pi$, and for every $j \geq 2$, choose inductively $z_j = e^{2\pi i\theta_j} \in S_1 \subseteq X_j$ by

$$\theta_j = \begin{cases} \frac{\theta_{j-1}}{2}, \ j \text{ even} \\ \pi + \frac{\theta_{j-1}}{2}, \ j \text{ odd} \end{cases}.$$

Since $p(z) = z^2$, one readily sees that (z_j) is a thread of $\boldsymbol{X}$. Thus, there is a unique point $x_1 \in X$ (more precisely, $x_1 \in \Sigma_2 \setminus S^1$) such that $p_j(x_1) = z_j$, $j \in \mathbb{N}$. By Lemma 2.3(i) and the proof of Theorem2.1,

$$\boldsymbol{p}_0 = (p_j) : (X, x_0) \to (\boldsymbol{X}, \boldsymbol{x}_0) = ((X_j, x_0), p_{jj'}, \mathbb{N}) \quad \text{and}$$
$$\boldsymbol{p}_1 = (p_j) : (X, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_1) = ((X_j, z_j), p_{jj'}, \mathbb{N})$$

are polyhedral resolutions, while

$$H\boldsymbol{p}_0 = ([p_j]) : (X, x_0) \to H(\boldsymbol{X}, \boldsymbol{x}_0) = ((X_j, x_0), [p_{jj'}], \mathbb{N}) \quad \text{and}$$
$$H\boldsymbol{p}_1 = ([p_j]) : (X, x_1) \to H(\boldsymbol{X}, \boldsymbol{x}_1) = ((X_j, z_j), [p_{jj'}], \mathbb{N})$$

are the corresponding $HPol_0$-expansions of the pointed spaces (X, x_0) and (X, x_1), respectively. One can show, in the same way as in Example II.3.4 of [8], that $\check{\pi}_1(X, x_0) = \mathbb{Z}$ and $\check{\pi}_1(X, x_1) = \{0\}$, and thus, $Sh(X, x_0) \neq Sh(X, x_1)$. We are to prove that $Sh^*(X, x_0) = Sh^*(X, x_1)$. Let $j \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n < j$, put $f_j^n : (X_j, x_0) \to (X_j, z_j)$ to be any pointed mapping. If $n = j$, consider an arc $x_0 z_j \subseteq S_1$ (shorter one) in the codomain space, and the closed half circle C_1^+ of S_1 containing x_0 and the closed half circle C_2^- of S_2 containing x_0 (in the domain space). Then define $f_j^j : (X_j, x_0) \to (X_j, z_j)$ to be (pointed) continuous, sending $C_1^+ \cup C_2^-$ onto the arc $x_0 z_j$ with the ends of $C_1^+ \cup C_2^-$ to x_0, $S_1 \setminus C_1^+$ homeomorphically onto $S_1 \setminus \{x_0\}$ and $S_2 \setminus C_2^-$ homeomorphically onto $S_2 \setminus \{x_0\}$. Since $C_1^+ \cup C_2^- \subseteq S_1 \vee S_2$ is contractible (through $S_1 \vee S_2$) to the wedge point x_0 relative to x_0, f_j^j is a pointed homotopy equivalence (see [10, Chap. I, Sec. 4]). Then, for every $k = j-1, \ldots, 1$, choose $f_k^j : (X_k, x_0) \to (X_k, z_k)$ inductively according to $p_{kk+1} f_{k+1}^j$ and the commutativity (homotopy factorization through p_{kk+1} of $\boldsymbol{p}_0$). As for f_j^j, we infer that that all f_k^j, $k = 1, \ldots, j$, are pointed homotopy equivalences. Therefore, this construction yields a pointed $*$-morphism

$$(1_{\mathbb{N}}, [f_j^n]) : H(\boldsymbol{X}, \boldsymbol{x}_0) \to H(\boldsymbol{X}, \boldsymbol{x}_1)$$

of *pro-HPol*$_0$, that represents an isomorphism $F^* : (X, x_0) \to (X, x_1)$ of Sh_0^*.

To prove the second assertion, choose any $x_0' \in S_2 \subseteq X$, $x_0' \neq x_0$. It is obvious that $Sh(X, x_0) = Sh(X, x_0')$. However, $Sh(X, x_0, x_0') \neq Sh(X, x_0, x_1)$ because of $Sh(X, x_0') = Sh(X, x_0) \neq Sh(X, x_1)$. Let us show that $Sh^*(X, x_0, x_0') = Sh^*(X, x_0, x_1)$. By Lemma 2.3(i) and the proof of Theorem 2.1,

$$\begin{aligned} &\boldsymbol{p} = (p_j) : (X, x_0, x_0') \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_0') = ((X_j, x_0, x_0'), p_{jj'}, \mathbb{N}) \quad \text{and} \\ &\boldsymbol{p}' = (p_j) : (X, x_0, x_1) \to (\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) = ((X_j, x_0, z_j), p_{jj'}, \mathbb{N}) \end{aligned}$$

are polyhedral resolutions, while

$$\begin{aligned} &H\boldsymbol{p} = ([p_j]) : (X, x_0, x_0') \to H(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_0') = ((X_j, x_0, x_0'), [p_{jj'}], \mathbb{N}) \quad \text{and} \\ &H\boldsymbol{p}' = ([p_j]) : (X, x_0, x_1) \to H(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1) = ((X_j, x_0, z_j), [p_{jj'}], \mathbb{N}) \end{aligned}$$

are the $HPol_{00}$-expansions of the bi-pointed spaces (X, x_0, x_0') and (X, x_0, x_1), respectively. Let $j \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n < j$, put $f_j^n : (X_j, x_0, x_0') \to (X_j, x_0, z_j)$ to be any bi-pointed mapping. If $n = j$, consider an arc $A = x_0 z_j \subseteq S_1$ (shorter one) in the codomain space and an arc B on S_2 in the domain space such that x_0 is its end point and $x_0' \in Int(B)$. Then define $f_j^j : (X_j, x_0, x_0') \to (X_j, x_0, z_j)$ to be (bi-pointed) continuous, sending S_1 onto S_1 by the identity, B onto A with the ends of B to the point x_0, and the open arc $S_2 \setminus B$ homeomorphically onto $S_2 \setminus x_0$. Since $B \subseteq S_2 \subseteq S_1 \vee S_2$ is deformable (through S_2) onto the subarc $B' = x_0 x_0' \subseteq B$ relative to B', the mapping f_j^j is a bi-pointed homotopy equivalence (see also [10, Chap. I, Sec. 4]). Then, for every $k = j-1, \ldots, 1$, choose $f_k^j : (X_k, x_0, x_0') \to (X_k, x_0, z_k)$ inductively according to $p_{kk+1} f_{k+1}^j$ and the commutativity (homotopy factorization through p_{kk+1} of $\boldsymbol{p}$). Similarly to f_j^j, all f_k^j, $k = 1, \ldots, j$, are bi-pointed homotopy equivalences. Therefore, this construction yields a bi-pointed $*$-morphism

$$(1_{\mathbb{N}}, [f_j^n]) : H(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_0') \to H(\boldsymbol{X}, \boldsymbol{x}_0, \boldsymbol{x}_1)$$

of *pro-*$HPol_{00}$, that represents an isomorphism $F^* : (X, x_0, x_0') \to (X, x_0, x_1)$ of Sh_{00}^*. □

3. The coarse shape path connectedness

Recall the notion of a shape path ([14], see also [7]). Let X be a topological space and let $x_0, x_1 \in X$. A *shape path* in X from x_0 to x_1 is a bi-pointed shape morphism $\Omega : (I, 0, 1) \to (X, x_0, x_1)$, where I is the unit segment of $\mathbb{R}$. X is said to be *shape path connected* if, for every pair $x, x' \in X$, there exists a shape path from x to x'. In the case of metric continua, the shape path connectedness coincides with the approximate path connectedness (i.e., joinability, [7, 8]). In [7] is also introduced the notion of weak joinability - strictly coarser than joinability. The shape path connectedness is a property lying between the path connectedness and ordinary connectedness of a space, and it is stronger than connectedness. We shall see that its full analogue in the coarse shape theory is much closer to connectedness, while in the weak shape theory the difference vanishes.

Definition 3.1. *Let X be a topological space and let $x_0, x_1 \in X$. A* coarse shape path in X from x_0 to x_1 *is a bi-pointed coarse shape morphism $\Omega^* : (I, 0, 1) \to (X, x_0, x_1)$. X is said to be* coarse shape path connected *if, for every pair $x, x' \in X$, there exists a coarse shape path in X from x to x'. The notions of a* weak shape path *and the* weak shape path connectedness *are defined analogously.*

Similarly to the ordinary paths and shape paths, one can "multiply" the coarse shape paths as well. So, if there are coarse shape paths in X from x_0 to x_1 and from x_1 to x_2, say Ω_0^* and Ω_1^* respectively, then $\Omega^* = \Omega_0^* \cdot \Omega_1^*$, obtained by "double speeding" (that is possible because of Lemma 2.3(i)), is a coarse shape path in X from x_0 to x_2. Clearly, this yields an equivalence relation on X.

Observe that the composite of a coarse shape path and an appropriate bi-pointed coarse shape morphism is a coarse shape path. More precisely, if there exists a coarse shape path Ω^* in X from x_0 to x_1, then, for every bi-pointed coarse shape morphism $F^* : (X, x_0, x_1) \to (Y, y_0, y_1)$, the composite $F^*\Omega^*$ is a coarse shape path in Y from y_0 to y_1 (clearly, a full analogue of that holds also in the bi-pointed shape theory ([14])). Hence, the next facts are almost obvious.

Lemma 3.2. *Let X and Y be topological spaces such that, for every pair $y, y' \in Y$, there exist a pair $x, x' \in X$ and a morphism $F : (X, x, x') \to (Y, y, y')$ of Sh_{00} ($F^* : (X, x, x') \to (Y, y, y')$ of Sh_{00}^*). If X is (coarse) shape* *path connected, then so is Y. Especially, every continuous image $f(X) \subseteq Y$ of a (coarse) shape path connected space X is (coarse) shape path connected, and thus, the (coarse) shape path connectedness is a topological property.*

THEOREM 3.3. *Consider the following topological properties:*

(i) *path connectedness;*
(ii) *shape path connectedness;*
(iii) *coarse shape path connectedness;*
(iv) *weak shape path connectedness;*
(v) *connectedness.*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v). Moreover, the first and second implication are strict.

PROOF. First of all, it is almost trivial to show that weak shape path connectedness is equivalent to connectedness, (iv) $\Leftrightarrow$ (v), so we omit the details. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow by the very definitions. Let X be coarse shape path connected. Assume to the contrary, i.e., that X is not connected. Then there exist $x_0, x_1 \in X$ lying in different components of X, and there exists a mapping $f : X \to \{0,1\} \subseteq \mathbb{R}$ ($\{0,1\}$ discrete) such that $f(x_0) = 0$ and $f(x_1) = 1$. Then $f : (X, x_0, x_1) \to (\{0,1\}, 0, 1)$ is a bi-pointed mapping. It induces the bi-pointed coarse shape morphism $F^* : (X, x_0, x_1) \to (\{0,1\}, 0, 1)$. Consider a coarse shape path $\Omega^* : (I, 0, 1) \to (X, x_0, x_1)$ from x_0 to x_1. Then the composite bi-pointed coarse shape morphism $F^*\Omega^* : (I, 0, 1) \to (\{0,1\}, 0, 1)$ is a coarse shape path in $\{0,1\}$ from 0 to 1. It is clear that $F^*\Omega^* : I \to \{0,1\}$ is an ordinary coarse shape morphism as well. According to Theorem 7 of [6] and Theorem 1 of [9] (bi-pointed analogues), $F^*\Omega^*$ is represented in inv^*-$HPol_{00}$ by a $*$-morphism that is a sequence $([g^n])$ of rel-homotopy classes $[g^n]$ of bi-pointed mappings $g^n : (I, 0, 1) \to (\{0,1\}, 0, 1)$. It implies that the discrete space $\{0,1\}$ is path connected - a contradiction. To see that (ii) does not imply (i), a counterexample is the (metric) continuum

$$X = (\{0\} \times I) \cup \{(x, \sin\frac{1}{x}) \mid x \in \left\langle 0, \frac{2}{\pi}\right]\} \subseteq \mathbb{R}^2.$$

Finally, (ii) strictly implies (iii) because of Example 3.4 below. ∎

EXAMPLE 3.4. Solenoids are coarse shape path connected but not shape path connected.

To verify Example 3.4, recall the well known fact that solenoids are (metric) continua which are not weakly joinable, and thus they are not shape path connected ([7], 1.5. Example). In order to show that they are coarse shape path connected, it suffices to prove, for instance, that the dyadic solenoid Σ_2 is such a space. Consider its sequential polyhedral resolution

(which is the inverse limit)

$$\boldsymbol{p} = (p_j) : \Sigma_2 \to \boldsymbol{\Sigma}_2 = (S_j = \mathbb{S}^1, p_{jj'}, \mathbb{N}),$$

where $p_{jj+1}(z) = z^2$. Let $x, x' \in \Sigma_2$, and denote $x_j = p_j(x), x'_j = p_j(x') \in S_j$, $j \in \mathbb{N}$. By Lemma 2.3(i) and the proof of Theorem 2.1,

$$\boldsymbol{p} = (p_j) : (\Sigma_2, x, x') \to (\boldsymbol{\Sigma}_2, \boldsymbol{x}, \boldsymbol{x}') = ((S_j, x_j, x'_j), p_{jj'}, \mathbb{N})$$

is a polyhedral resolution of the bi-pointed space (Σ_2, x, x'), and

$$H\boldsymbol{p} = ([p_j]) : (\Sigma_2, x, x') \to H(\boldsymbol{\Sigma}_2, \boldsymbol{x}, \boldsymbol{x}') = ((S_j, x_j, x'_j), [p_{jj'}], \mathbb{N})$$

is the corresponding $HPol_{00}$-expansion. We are going to construct a coarse shape path in Σ_2 from x to x'. Let, for every $j \in \mathbb{N}$ a (bi-pointed) mapping $\omega_j^j : (I, 0, 1) \to (S_j, x_j, x'_j)$ be chosen arbitrarily. Then, for each j and every $n \in \mathbb{N}$, put $\omega_j^n : (I, 0, 1) \to (S_j, x_j, x'_j)$ to be

$$\omega_j^n = \begin{cases} u_j^n, \ n < j \\ p_{jn}\omega_n^n, \ n \geq j \end{cases},$$

where the mappings $u_j^n : (I, 0, 1) \to (S_j, x_j, x'_j)$, $n < j$, are chosen arbitrarily (for instance, all of them may be ω_j^j). One trivially verifies that $([\omega_j^n]) : (I, 0, 1) \to (\boldsymbol{\Sigma}_2, \boldsymbol{x}, \boldsymbol{x}')$ is a $*$-morphism of inv^*-$HPol_{00}$. Thus, $\boldsymbol{\omega}^* = [([\omega_j^n])] : (I, 0, 1) \to (\boldsymbol{\Sigma}_2, \boldsymbol{x}, \boldsymbol{x}')$ is a morphism of pro^*-$HPol_{00}$, and consequently, the coarse shape morphism $\Omega^* = \langle \boldsymbol{\omega}^* \rangle : (I, 0, 1) \to (\Sigma_2, x, x')$ is a desired coarse shape path in Σ_2 from x to x'.

Since the connectedness is a coarse shape invariant ([13, Lemma 11 and Theorem 6]; [5, Theorem 3]), it seems that the implication (iii) $\Rightarrow$ (v) of Theorem 3.3 should be also strict. However, the next result shows that a possible counterexample cannot be a metrizable continuum.

THEOREM 3.5. *A compact metrizable space X is connected if and only if it is coarse shape path connected. Consequently, the weak joinability (trivially) implies the coarse shape path connectedness, but not conversely.*

In order to prove the theorem, we need the following lemma.

LEMMA 3.6. *Let X be a connected topological space. If X admits a countable polyhedral resolution, then it is coarse shape path connected.*

PROOF. According to [2, VIII., Exercises, A.1. (p. 229)] and [13, Lemma 9], we may assume that X admits a sequential polyhedral resolution $\boldsymbol{p} = (p_j) : X \to \boldsymbol{X} = (X_j, p_{jj'}, \mathbb{N})$. Moreover, since X is connected, we may assume that each polyhedron X_j, $j \in \mathbb{N}$, is path connected. By Lemma 2.3(i) and the proof of Theorem 2.1, given a pair $x, x' \in X$, the bi-pointed morphism $\boldsymbol{p} : (X, x, x') \to ((X_j, x_j, x'_j), p_{jj'}, \mathbb{N})$ of *pro*-Top_{00} is a polyhedral resolution of (X, x, x'), and

$$H\boldsymbol{p} = ([p_\lambda]) : (X, x, x') \to ((X_j, x_j, x'_j), [p_{jj'}], \mathbb{N})$$

is a $HPol_{00}$-expansion of (X, x, x'). Now one can construct a desired coarse shape path Ω^* in X from x to x' as we did it in the verification of Example 3.4. □

Proof of Theorem 3.5. The sufficiency follows by (iii) $\Rightarrow$ (iv) of Theorem 2.5. Conversely, let X be a metrizable continuum. According to the main result of [3] and Theorems I.6.1 and I.6.2 of [8], X admits a sequential polyhedral resolution. Hence, the converse follows by Lemma 3.6. Finally, if a metrizable compactum is weakly joinable (see [7]) then it is connected, and, as we just have proven, it is coarse shape path connected. The converse does not hold because a solenoid is not weakly joinable. □

By 3.1. Theorem of [7] and 7.1.6. Corollary of [1], the shape path connectedness is a shape invariant on the class of all metrizable compacta. Here is the coarse shape analogue.

Corollary 3.7. *The coarse shape path connectedness is a coarse shape (and, thus, a shape) invariant on the class of all metrizable compacta.*

Proof. By [13], Lemma 11 and Theorem 6 (or Theorem 3 of [5]), the connectedness is a coarse shape invariant (on Top). Now, the conclusion follows by Theorem 3.5. □

Concerning homotopy domination, the following fact holds.

Theorem 3.8. *Connectedness ((coarse) shape path connectedness) is a hereditary homotopy property, i.e., if Y is homotopy dominated by X and X is connected ((coarse) shape path connected), then Y is connected ((coarse) shape path connected). Consequently, connectedness, shape path connectedness and coarse shape path connectedness are invariants of the homotopy type on Top.*

Proof. Let X and Y be topological spaces such that $Y \leq X$ in $HTop$, i.e., there exist mappings $f : X \to Y$ and $g : Y \to X$ satisfying $fg \simeq 1_Y$, and let X be connected ((coarse) shape path connected). Then the image subspace $f(X) \subseteq Y$ is contained in a unique component ((coarse) shape path component) Y' of Y. Let $y \in Y$ be an arbitrary point. Choose a homotopy $H : Y \times I \to Y$ such that $H_0 = 1_Y$ and $H_1 = fg$. Then the bi-pointed mapping

$$\omega_y : (I, 0, 1) \to (Y, y, fg(y)), \qquad \omega_y(t) = H(y, t),$$

is an ordinary path in Y from y to $fg(y)$. This implies that y and $fg(y)$ belong to the same path component of Y. Then, by Theorem 2.5, y and $fg(y)$ belong to the same component ((coarse) shape path component) of Y. Since $fg(y) \in Y'$, it must be $y \in Y'$ as well. Consequently, $Y' = Y$, which proves the theorem. □

Related to implication (ii) $\Rightarrow$ (iii) of Theorem 3.3, recall that, for every metrizable continuum X, the following assertions are mutually equivalent (see Ch. VII. of [1], [7] and II. 8. of [8]):

(a) There exists a point $x_0 \in X$ such that the pointed space (X, x_0) is 1-movable;
(b) for every point $x \in X$, the pointed space (X, x) is 1-movable;
(c) X is shape path connected.

Thus, the following fact obviously holds.

COROLLARY 3.9. *On the class of all pointed* 1*-movable metrizable compacta, the shape path connectedness, coarse shape path connectedness, weak shape path connectedness and connectedness are equivalent properties.*

REMARK 3.10. It is interesting and important to know whether implication (iii) $\Rightarrow$ (v) of Theorem 3.3 is strict. For (see Theorem 6 of [13]), if it is the case, then the (bi-pointed) coarse shape theory is strictly finer than the (bi-pointed) weak shape theory.

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