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# AN ALTERNATE PROOF THAT THE FUNDAMENTAL GROUP OF A PEANO CONTINUUM IS FINITELY PRESENTED IF THE GROUP IS COUNTABLE

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Abstract. We give an alternate proof, using coarse geometry, that if the fundamental group of a compact, connected, locally connected metric space is countable, then the fundamental group is finitely presented. This result was first proved by Katsuya Eda and the argument can be found in [5].

## 1. Introduction

This paper is motivated by a question posed to the second author by Mladen Bestvina during his talk at the Spring Topology and Dynamics Conference in Gainesville (March 7-9, 2009):

Question 1.1. *Is the fundamental group of a Peano continuum finitely presented if it is countable?*

It turns out that question was also posed by de la Harpe ([10, p. 48]) and it is relevant in view of the following:

Theorem 1.1 (Shelah [16]). *If $X$ is a Peano continuum and $\pi_1(X)$ is countable, then $\pi_1(X)$ is finitely generated.*

Pawlikowski ([13]) presented another proof of 1.1 from which we extract the following (see the paragraph preceding Lemma 2 in [13] or Theorems 2 and 8 in [8]):

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THEOREM 1.2 (Pawlikowski [13]). *If $X$ is a Peano continuum and $\pi_1(X)$ is countable, then $X$ is semi-locally simply connected.*

Notice that the second author constructed (see [17]), for each countable group $G$, a 2-dimensional path-connected subcontinuum $X_G$ of $R^4$ whose fundamental group is $G$ (see [12] and [14] for earlier constructions of compact spaces with a given fundamental group).

Our solution to 1.1 is based on an application of methods of geometric group theory: we construct a geometric action of $\pi_1(X)$ on a coarsely 1-connected proper geodesic space $\tilde{X}$ and we use Švarc-Milnor Lemma ([2, page 140]) plus the fact $G$ is finitely presented if and only if it is coarsely 1-connected.

Let a group $G$ act on a topological space $X$ by homeomorphisms. Consider a subgroup $H \subset G$. One then says that a set $Y$ is precisely invariant under $H$ in $G$ if

$$\forall h \in H, \quad h(Y) = Y \text{ and } \forall g \in G - H, \quad gY \cap Y = \varnothing.$$

Then let $G_x$ be the stabilizer of $x$ in $G$. One says that $G$ acts *discontinuously* at $x$ in $X$ if the stabilizer $G_x$ is finite and there exists a neighborhood $U$ of $x$ that is precisely invariant under $G_x$ in $G$. If $G$ acts discontinuously at every point $x$ in $X$, then one says that $G$ acts *properly discontinuously* on $X$.

In geometric group theory, a geometry is any proper, geodesic metric space. An action of a finitely-generated group $G$ on a geometry $X$ is *geometric* if it satisfies the following conditions:

1. Each element of $G$ acts as an isometry of $X$.
2. The action is cocompact, i.e., the quotient space $X/G$ is a compact space.
3. The action is properly discontinuous, with each point having a finite stabilizer.

As noted, we shall employ the following:

THEOREM 1.3 (Švarc-Milnor [2] or [3]). *A group $G$ acting properly discontinuously and cocompactly via isometries on a length space $X$ is finitely generated and induces a quasi-isometry equivalence $g \to g \cdot x_0$ for any $x_0 \in X$.*

We were informed by Greg Conner ([7]) that Katsuya Eda answered 1.1 about 5 years ago (unpublished). The argument is contained in [5]: by 1.2 the space is semi-locally simply-connected and homotopically Hausdorff. Corollary 5.7 says such space has finitely presented fundamental group.

Alternatively, it is pointed out in Lemma A.3 in [6] that this shows that such a space has finitely presented fundamental group. Here is the argument from Lemma A.3. Semilocally simply-connected implies that the space is two-set simple – see [4]. This implies that the fundamental group is group is the fundamental group of the nerve of a finite cover which implies that it's finitely presented, again from [4].

## 2. Coarse 1-connectivity of uniformly path connected spaces

In order to complete the proof of our main result 3.5, we need to relate coarse $1-$connectedness to simple connectedness.

Recall $(X,d)$ is $t$-chain connected for $t>0$ if every pair of points $x,y\in X$ can be connected by a $t-$chain, i.e., if there exist points $x_0=x,x_1,\dots,x_k=y$ in $X$ so that $d(x_i,x_{i+1})\le t,\forall i$. Space $(X,d)$ is *coarsely 1-connected* if it is $t$-chain connected for some $t>0$ (that is equivalent to $(X,d)$ being coarsely 0-connected - see [11]) and for each $r>0$ there is $R>r$ such that the induced map $\mathrm{Rips}_r(X)\to\mathrm{Rips}_R(X)$ induces the trivial homomorphism of the fundamental groups (see Definition 42 on p.19 of [11]). Here $\mathrm{Rips}_r(X)$ is the *Rips complex* of $X$, i.e., the complex whose vertices are the points of $X$ and whose simplices are given by finite subsets $A$ of $X$ of diameter at most $r$.

Definition 2.1. *A path connected metric space $(X,d)$ is* uniformly path connected *if there is a function $\alpha\colon(0,\infty)\to(0,\infty)$ so that every two points $x,y\in X$ can be connected by a path of diameter at most $\alpha(d(x,y))$.*

*The fundamental group $\pi_1(X,x_0)$ of a path connected metric space $X$ is* uniformly generated *(see [9]) if it has a generating set of loops of diameter at most $R$ for some $R>0$. Equivalently, every map $f\colon(S^1,0)\to(X,x_0)$ can be extended over the $1-$skeleton of some subdivision $\tau$ of $(B^2,0)$ to a map $F$ so that the diameter $F(\partial\Delta)$ is at most $R$, for every simplex $\Delta$ of $\tau$.*

Theorem 2.2. *Suppose $X$ is a uniformly path connected space. $X$ is coarsely $1-$connected if and only if $\pi_1(X,x_0)$ is uniformly generated.*

Proof. Assume $X$ is coarsely $1-$connected. Fix positive numbers $r,R$ so that $\pi_1$ applied to $\mathrm{Rips}_r(X)\to\mathrm{Rips}_R(X)$ is trivial. Furthermore, let $l$ be a positive number so that every two points of $X$ that are at most $R$ apart can be connected by a path of diameter at most $l$. Let $\alpha\colon(S^1,0)\to(X,x_0)$ be a loop. Subdivide $(S^1,0)$ to obtain a subdivision $\tau$ (notation: $S^1_\tau$) so that the diameter of $\alpha(\Delta)$ is at most $r$ for every edge $\Delta$ of $\tau$. The map $\alpha|_{(S^1_\tau)^{(0)}}$ induces a simplicial map $\tilde\alpha\colon(S^1_\tau,0)\to(\mathrm{Rips}_r(X),x_0)$, which extends to a map $\tilde\beta\colon(B^2,0)\to\mathrm{Rips}_R(X)$. We may assume $\sigma$ is a subdivision of $(B^2,0)$ so that $\tilde\beta$ is simplicial and $\sigma|_{S^1}$ is a subdivision of $\tau$. Then $\tilde\beta$ induces a map $\beta\colon((B^2_\sigma)^{(1)},0)\to(X,x_0)$ as follows: $\beta$ equals $\tilde\beta$ on vertices and $S^1$ and for every edge $E$ of $B^2_\sigma\setminus S^1$ we can connect two boundary points $\beta(\partial E)$ by a path of diameter at most $l$. Hence we obtain an extension $\beta\colon((B^2_\sigma)^{(1)},0)\to(X,x_0)$ of $\alpha$ so that diameter of $\beta(\Delta)$ is at most $2\cdot l$ for every simplex $\Delta$ of $B^2_\sigma$. This means that $\pi_1(X,x_0)$ is $2\cdot l$-generated.

Assume $\pi_1(X,x_0)$ is uniformly generated by loops of diameter at most $D$. Fix $r>0,l>0$ so that every two points of distance at most $r$ can be connected by a path of diameter at most $l$. We can assume $D > l$. Pick any simplicial map $\alpha\colon (S^1_\tau, 0) \to (\text{Rips}_r(X), x_0)$. It induces a map $\tilde{\alpha}\colon ((S^1_\tau)^{(0)}, 0) \to (X, x_0)$ as follows: for every edge $E$ of $S^1_\tau$ we connect two boundary points $\tilde{\alpha}(\partial E)$ by a path of diameter at most $l$ to obtain a map $\tilde{\alpha}\colon (S^1_\tau, 0) \to (X, x_0)$. Such map extends over 1−skeleton of some subdivision $\sigma$ (containing $\tau$) of $(B^2, 0)$ to a map $\tilde{\beta}$ so that diameter $\tilde{\beta}(\partial\Delta)$ is at most $D$, for every simplex $\Delta$ of $\sigma$. Then $\tilde{\beta}$ induces a map $\beta\colon ((B^2_\sigma)^{(0)}, 0) \to \text{Rips}_D(X)$ which extends over $B^2_\sigma$. Note that $\beta|_{(\partial B^2, 0)} \simeq \alpha$: for every edge $E$ of $\tau$ the set $\beta(E) \cup \alpha(E)$ is contained in a simplex of $\text{Rips}_D(X)$ because of uniform path connectedness and $D > l$. ☐

## 3. Main result

Given a Peano continuum $X$ we assume it has a geodesic metric $d_X$ (see [1]). Pick a base point $x_0$ of $X$ and consider the space $\tilde{X}$ of homotopy (rel. endpoints) classes of paths in $X$ originating at $x_0$.

In this section we assume $X$ is semi-locally simply connected.

Definition 3.1. *Given $[\alpha] \in \tilde{X}$ and a path $\beta$ in $X$ originating at $\alpha(1)$, the canonical lift $\tilde{\beta}$ of $\beta$ is a path in $\tilde{X}$ defined by $\tilde{\beta}(t) = [\alpha * (\beta|_{[0,t]})]$, the concatenation of $\alpha$ and $\beta$ restricted to interval $[0, t]$.*

Given two elements $[\alpha]$ and $[\beta]$ of $\tilde{X}$ we define the distance $d([\alpha], [\beta])$ as the infimum of lengths $l(\gamma)$ of all paths $\gamma$ from $\alpha(1)$ to $\beta(1)$ such that $\gamma$ is homotopic rel.endpoints to $\alpha^{-1} * \beta$.

Proposition 3.2. *$(\tilde{X}, d)$ is a proper geodesic space such that the endpoint projection $p\colon \tilde{X} \to X$ is 1-Lipschitz and canonical lifts of geodesics in $X$ are geodesics in $\tilde{X}$.*

Proof. Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$. Notice that any two paths at distance less than $\delta$ are homotopic rel.endpoints if they join the same two points.

Given two elements $[\alpha], [\beta]$ of $\tilde{X}$ the path $\alpha^{-1} * \beta$ can be approximated by a piecewise-geodesic path $\gamma$. As $l(\gamma)$ is finite, so is $d([\alpha], [\beta])$. If $d([\alpha], [\beta]) = 0$, then $\alpha(1) = \beta(1)$. As $d([\alpha], [\beta]) = 0$ there is a loop $\gamma$ at $x_1$ of length less than $\delta$ satisfying $\gamma \sim \alpha^{-1} * \beta$. That means $\alpha \sim \beta$ as $\gamma$ is null-homotopic in $X$. Thus $[\alpha] = [\beta]$ if $d([\alpha], [\beta]) = 0$. It is easy to see $d$ is symmetric and satisfies the Triangle Inequality.

Notice $d([\alpha], [\beta]) \geq d_X(\alpha(1), \beta(1))$, so $p$ is 1-Lipschitz. Also, it is clear that canonical lifts of geodesics in $X$ are geodesics in $\tilde{X}$.

Suppose $\gamma_n$ is a sequence of paths in $X$ joining $\alpha(1)$ and $\beta(1)$ such that $l(\gamma_n)$ converges to $M = d([\alpha], [\beta])$ and $\gamma_n \sim \alpha^{-1} * \beta$ for all $n \geq 1$. We may assume each $\gamma_n$ is parameterized so that the length of $\gamma_n|_{[0,t]}$ is $t \cdot l(\gamma_n)$. Subdivide the interval $[0, 1]$ into points $y_0 = 0, y_1, \ldots, y_k = 1$ such that $0 < y_{i+1} - y_i < \frac{\delta}{2 \cdot M}$ for all $0 \leq i < k$. We may assume $\gamma_n(y_i)$ converges to $z_i \in X$

for each $0 \leq i \leq k$. The piecewise-geodesic path $\omega$ from $\alpha(1)$ to $\beta(1)$ obtained by connecting points $z_0, z_1, \ldots, z_k$ is homotopic to $\gamma_n$ for $n$ large enough. Also, $l(\omega)$ equals the limit of $l(\gamma_n)$, so $l(\omega) = d([\alpha], [\beta])$. Notice the canonical lift of $\omega$ is a geodesic from $[\alpha]$ to $[\beta]$ in $\tilde{X}$.

To show $(\tilde{X}, d)$ is a proper metric space assume $\{[\alpha_n]\}_{n \geq 1}$ is a bounded sequence in $\tilde{X}$. We may assume $\alpha_n(1)$ converges to $x_1$ and then alter each $\alpha_n$ by concatenating it with a geodesic from $\alpha_n(1)$ to $x_1$. It suffices to show that the resulting sequence of elements $[\beta_n]$ of $\tilde{X}$ has a convergent subsequence. First of all, we may assume the sequence of lengths $l(\beta_n)$ converges to $M > 0$ (if $M = 0$, then $[\beta_n]$ converge to $[x_0]$), each $\beta_n$ is piecewise-geodesic and the length of $\beta_n|_{[0,t]}$ is $t \cdot l(\beta_n)$. Subdivide the interval $[0, 1]$ into points $y_0 = 0, y_1, \ldots, y_k = 1$ such that $0 < y_{i+1} - y_i < \frac{\delta}{2 \cdot M}$ for all $0 \leq i < k$. We may assume $\beta_n(y_i)$ converges to $z_i \in X$ for each $0 \leq i \leq k$. The piecewise-geodesic path $\omega$ from $x_0$ to $x_1$ obtained by connecting points $z_0, z_1, \ldots, z_k$ is homotopic to $\beta_n$ for $n$ large enough. That means $[\beta_n]$ is constant starting from a sufficiently large $n$. □

Proposition 3.3. *$(\tilde{X}, d)$ is simply connected and the endpoint projection $p\colon \tilde{X} \to X$ is a covering map.*

Proof. Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$.

Claim 1. *For any $[\alpha] \in \tilde{X}$ the restriction of $p$ to the ball $B([\alpha], \delta)$ is an isometry onto $B(\alpha(1), \delta)$.*

Proof of Claim. Given $\beta, \omega \in B([\alpha], \delta)$ let $\gamma$ be a geodesic path from $\beta(1)$ to $\omega(1)$. As $d([\beta], [\omega]) < 2 \cdot \delta$ there is a path $\lambda$ from $\beta(1)$ to $\omega(1)$ of length less than $2 \cdot \delta$ such that $\lambda \sim \beta^{-1} * \omega$. Observe $\lambda \sim \gamma$ as both paths are of diameter less than $2 \cdot \delta$. That means $d([\beta], [\omega]) = d_X(\beta(1), \omega(1))$ as the length of $\gamma$ equals $d_X(\beta(1), \omega(1))$ and $d([\beta], [\omega]) \geq d_X(\beta(1), \omega(1))$. □

Given $[\beta] \in \tilde{X}$ with $\beta(1) \in B(x_1, \delta)$ let $\gamma$ be a geodesic path from $\beta(1)$ to $x_1$. Observe $d([\beta], [\beta * \gamma]) < \delta$ and $p([\beta * \gamma]) = x_1$. That means $p^{-1}(B(x_1, \delta))$ is the union of balls $B([\alpha], \delta)$ with $\alpha$ ranging over all paths in $p^{-1}(x_1)$. By Claim we conclude $p$ is a covering projection.

To show $\tilde{X}$ is simply connected suppose $\alpha$ is a loop in $\tilde{X}$ based at the trivial path. Since $p(\alpha)$ can be homotoped to a piecewise-geodesic loop and canonical lifts of piecewise-geodesic loops are paths in $\tilde{X}$, we may assume $\alpha$ is the canonical lift of a piecewise-geodesic loop $\beta$ based at $x_0$. The canonical lift of $\beta$ is a loop if and only if $\beta$ is null-homotopic. As $p$ is a covering projection, $\alpha$ is null-homotopic as well. □

Proposition 3.4. *The action of $G = \pi_1(X, x_0)$ on $\tilde{X}$ ($g \cdot [\alpha]$ being $[\beta * \alpha]$, where $[\beta] = g$) is geometric.*

PROOF. $G$ acts by isometries as $d(g \cdot [\alpha], g \cdot [\beta]) = d([\alpha], [\beta])$ for all $\alpha, \beta \in \tilde{X}$.

Let $\delta > 0$ be a number such that any loop in $X$ of diameter less than $4 \cdot \delta$ is null-homotopic in $X$. Given $[\alpha] \in \tilde{X}$ let $U$ be the $\delta$-ball around $[\alpha]$ in $\tilde{X}$. If $[\beta] \in U \cap (g \cdot U)$ there are paths $\gamma_i$, $i = 1, 2$, such that $\beta \sim \alpha * \gamma_1$, $\beta \sim g \cdot \alpha * \gamma_2$ and $l(\gamma_i) < \delta$ for $i = 1, 2$. Thus $g = [\alpha * \gamma_1 * \gamma_2^{-1} * \alpha^{-1}]$ equals 1 in $G$ as $\gamma_1 \sim \gamma_2$ (both are paths of diameter less than $\delta$ and join the same points). That proves the action of $G$ on $\tilde{X}$ is properly discontinuous.

Since $p\colon \tilde{X} \to X$ is open and, set-theoretically, equals $\tilde{X} \to \tilde{X}/G$, $\tilde{X}/G$ is homeomorphic to $X$ proving that the action of $G$ on $\tilde{X}$ is cocompact. ∎

THEOREM 3.5. *The fundamental group of a Peano continuum $X$ is finitely presented if it is countable.*

PROOF. $\tilde{X}$ is uniformly path connected as is it geodesic according to 3.2. By the Švarc-Milnor Lemma and 3.4 the group $G = \pi_1(X, x_0)$ is finitely generated and is quasi-isometric to $\tilde{X}$. As $\tilde{X}$ is coarsely 1-connected (see 2.2) and coarse 1-connectivity is an invariant of quasi-isometries (see Corollary 47 in [11]), $G$ is also coarsely 1-connected. As $G$ is a finitely generated group, $G$ is necessarily finitely presented (see the proof of Corollary 51 in [11] on p.22 or Proposition 8.24 in [2]). Alternatively, the fundamental group of the Cayley graph $\Gamma(G)$ of $G$ must be uniformly generated by 2.2 which means $G$ is finitely presented. ∎

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