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Using non-cofinite resolutions in shape theory. Application to Cartesian products

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Abstract. The strong shape category of topological spaces SSh can be defined using the coherent homotopy category CH, whose objects are inverse systems consisting of topological spaces, indexed by cofinite directed sets. In particular, if X, Y are spaces and $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ is a cofinite HPol-resolution of Y, then there is a bijection between the set $\mathrm{SSh}(X, Y)$ of strong shape morphisms $F\colon X \to Y$ and the set $\mathrm{CH}(X, \boldsymbol{Y})$ of homotopy classes $[\boldsymbol{f}]$ of coherent homotopy mappings $\boldsymbol{f}\colon X \to \boldsymbol{Y}$. In the paper it is shown that such a bijection exists also in the case when $\boldsymbol{Y}$ is not cofinite. This fact makes it possible to study strong shape properties of the Cartesian product $X \times P$ of a compact Hausdorff space X and a polyhedron P using the standard resolution of $X \times P$, which is a non-cofinite HPol-resolution. As an application, one reduces the question whether $X \times P$ is a product of X and P in the category SSh to a question concerning homotopy classes of coherent homotopy mappings. Analogous results also hold for the ordinary shape category of topological spaces Sh and the pro-homotopy category of cofinite inverse systems of spaces.

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1. Introduction

1.1. The strong shape category SSh has topological spaces as objects. Its morphisms can be defined using the coherent homotopy category CH of cofinite inverse systems of topological spaces, i.e., inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, indexed by cofinite directed sets $(\Lambda, \leq)$ (see [8], 1.1 and 8.2). The morphisms of CH are homotopy classes $[\boldsymbol{f}]\colon \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ of coherent homotopy mappings (shorter, coherent mappings) $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$. Cofiniteness of the index set M guarantees that the homotopy relation $\simeq$ between coherent mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an equivalence relation and therefore, the classes $[\boldsymbol{f}]$ are well defined (see Section 2).

We denote by HPol the class of spaces having the homotopy type of polyhedra (see [8], 7.1). If $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ is a cofinite HPol-resolution, i.e., a cofinite resolution which consists of spaces Y_μ, $\mu \in M$, belonging to HPol (see [8], 7.1), then the definition of strong shape morphisms (see [8], 8.2) implies the existence of a bijection $\Gamma_{\boldsymbol{q}}$ between the set $\mathrm{SSh}(X, Y)$ of strong shape morphisms $F\colon X \to Y$ and the set $\mathrm{CH}(X, \boldsymbol{Y})$ of homotopy classes of coherent mappings $[\boldsymbol{f}]\colon X \to \boldsymbol{Y}$ (see more details in 2.6).

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1.2. In the study of Cartesian products $X \times P$, where X is a compact Hausdorff space and P is a polyhedron (CW-topology), it is convenient to use the standard HPol-resolution of $X \times P$, introduced in [9] (see Section 4). Unfortunately, that resolution is not cofinite. Therefore, in this paper we will extend the definition of coherent mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ to include systems which are non-cofinite. Hereby, for coherent mappings, their compositions and the identity mapping $\boldsymbol{1_X}\colon \boldsymbol{X} \to \boldsymbol{X}$, we use the same defining formulae as used in the cofinite case. Moreover, the definition of homotopy $\simeq$ of coherent mappings remains unchanged. However, in this broader setting of non-cofinite systems, in general, homotopy of coherent mappings fails to be an equivalence relation.

Fortunately, there are enough cases of coherent mappings $\boldsymbol{f}$, where homotopy continues to be an equivalence relation and thus, the homotopy classes $[\boldsymbol{f}]$ of $\boldsymbol{f}$ are still well defined. Furthermore, whenever the homotopy classes $[\boldsymbol{f}]$ are well defined, they have their usual properties. E.g., if $[\boldsymbol{f}]$, $[\boldsymbol{g}]$ and $[\boldsymbol{gf}]$ are well defined, we define the composition $[\boldsymbol{g}][\boldsymbol{f}]$ of the classes $[\boldsymbol{f}]$ and $[\boldsymbol{g}]$ as the homotopy class $[\boldsymbol{gf}]$. Similarly, the associative law $[\boldsymbol{h}]([\boldsymbol{g}][\boldsymbol{f}]) = ([\boldsymbol{h}][\boldsymbol{g}])[\boldsymbol{f}]$ remains valid if both sides are well defined (precisely, Lemma 1 holds). In particular, in the case when $\boldsymbol{X}$ is a rudimentary system, i.e., it consists of a single space X, then the homotopy classes $[\boldsymbol{f}]\colon X \to \boldsymbol{Y}$ and the set $\mathrm{CH}(X, \boldsymbol{Y})$ of all such classes are well defined. The other such case is when $\boldsymbol{Y}$ is cofinite.

The main result of the first part of this paper consists of extending the definition of the bijection $\Gamma_{\boldsymbol{q}}\colon \mathrm{SSh}(X, Y) \to \mathrm{CH}(X, \boldsymbol{Y})$ from the case of cofinite HPol-resolutions $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$, considered in 1.1, to the case of non-cofinite HPol-resolutions $\boldsymbol{q}$. To achieve this, with such a resolution $\boldsymbol{q}$ one associates a particular cofinite HPol-resolution $\boldsymbol{q}^*\colon Y \to \boldsymbol{Y}^*$ and proves that there is a bijection $\boldsymbol{\Phi_Y}\colon \mathrm{CH}(X, \boldsymbol{Y}) \to \mathrm{CH}(X, \boldsymbol{Y}^*)$ (Theorem 3). By 1.1, $\Gamma_{\boldsymbol{q}^*}\colon \mathrm{SSh}(X, Y) \to \mathrm{CH}(X, \boldsymbol{Y}^*)$ is a bijection. Therefore,

$$\Gamma_{\boldsymbol{q}} = (\Phi_{\boldsymbol{Y}})^{-1} \Gamma_{\boldsymbol{q}^*} \tag{1}$$

is a well-defined function $\Gamma_{\boldsymbol{q}}\colon \mathrm{SSh}(X, Y) \to \mathrm{CH}(X, \boldsymbol{Y})$. Clearly, one has the following result.

Theorem 1. *If X and Y are topological spaces and $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ is an HPol-resolution of Y, then $\Gamma_{\boldsymbol{q}}\colon \mathrm{SSh}(X, Y) \to \mathrm{CH}(X, \boldsymbol{Y})$is a bijection. If $\Gamma_{\boldsymbol{q}}(F) = [\boldsymbol{f}]$, F and $[\boldsymbol{f}]$ are said to be associated with each other.*

1.3. The fundamental question concerning strong shape (shape) of the Cartesian product $X \times Y$ of two spaces is to determine whether $X \times Y$ is a product in the strong shape category SSh (in the shape category Sh). It is well known that the answer is positive if both spaces X, Y are polyhedra or both spaces are compact Hausdorff spaces [2, 10]. However, a simple example, due to J.E. Keesling [2], shows that the Cartesian product of two (separable) metric spaces need not be a product in Sh. For the strong shape category SSh, no such example is known.

J. Dydak and S. Mardešić [1] showed that the Cartesian product of the dyadic solenoid and the wedge (pointed sum) of a sequence of copies of the 1-sphere S^1 is not a product in Sh. Is there a compact Hausdorff space X and a polyhedron P such that $X \times P$ fails to be a product in SSh is an open question.

In 1972, Y. Kodama proved that the Cartesian product of an FANR and a paracompact space is a product in Sh ([3], Theorem 3′). The author proved that the Cartesian product of an FANR and a finitistic space is a product in SSh [10]. An open problem of Kodama, raised in 1977 [3], asks whether the Cartesian product of a movable metric compactum X and a metric space Y is a product in Sh. Even in the simple case, when X is the Hawaiian earring and Y is the wedge of a sequence of copies of the 1-sphere S^1, this author does not know if $X \times Y$ is a product in Sh or SSh.

1.4. In the present paper we are interested in the Cartesian products $X \times P$, where X is a compact Hausdorff space and P is a polyhedron. Recall that the canonical projections $\pi_X : X \times P \to X$, $\pi_P : X \times P \to P$ induce homotopy classes of mappings $[\pi_X] : X \times P \to X$, $[\pi_P] : X \times P \to P$ and the latter induce strong shape morphisms $\overline{S}[\pi_X] : X \times P \to X$, $\overline{S}[\pi_P] : X \times P \to P$, where $\overline{S} : \mathrm{H} \to \mathrm{SSh}$ is the strong shape functor from the homotopy category H to SSh. It keeps spaces fixed and maps morphisms of H to the induced strong shape morphisms (see [8], 8.2). To state precisely what we mean when we say that $X \times P$ is a product in SSh, for a topological space Z, we consider the following two statements $(\mathrm{ESS})_Z$ and $(\mathrm{USS})_Z$ (the abbreviations stand for existence and uniqueness in strong shape):

$(\mathrm{ESS})_Z$ For every strong shape morphism $F : Z \to X$ and every homotopy class of mappings $[g] : Z \to P$, there exists a strong shape morphism $H : Z \to X \times P$ such that $\overline{S}[\pi_X]H = F$ and $\overline{S}[\pi_P]H = \overline{S}[g]$.

$(\mathrm{USS})_Z$ If $H_i : Z \to X \times P$, $i = 1, 2$, are two strong shape morphisms such that $\overline{S}[\pi_X]H_1 = \overline{S}[\pi_X]H_2$ and $\overline{S}[\pi_P]H_1 = \overline{S}[\pi_P]H_2$, $i = 1, 2$, then $H_1 = H_2$.

That $(X \times P, \overline{S}[\pi_X], \overline{S}[\pi_P])$ is a (direct) product of X and P in SSh, shorter, $X \times P$ is a product in SSh, means that, for every topological space Z, the statements $(\mathrm{ESS})_Z$ and $(\mathrm{USS})_Z$ hold.

Analogously, for ordinary shape, we consider the following statements $(\mathrm{ES})_Z$ and $(\mathrm{US})_Z$ (the abbreviations stand for existence and uniqueness in shape):

$(\mathrm{ES})_Z$ For every shape morphism $F : Z \to X$ and every homotopy class of mappings $[g] : Z \to P$, there exists a shape morphism $H : Z \to X \times P$ such that $S[\pi_X]H = F$ and $S[\pi_P]H = S[g]$.

$(\mathrm{US})_Z$ If $H_i : Z \to X \times P$, $i = 1, 2$, are two shape morphisms such that $S[\pi_X]H_1 = S[\pi_X]H_2$ and $S[\pi_P]H_1 = S[\pi_P]H_2$, then $H_1 = H_2$.

Here $S : \mathrm{H} \to \mathrm{Sh}$ denotes the shape functor, which keeps spaces fixed and maps morphisms of the homotopy category H to the corresponding shape morphisms. That $(X \times P, S[\pi_X], S[\pi_P])$ is a product of X and Y in Sh, shorter, $X \times P$ is a product in Sh(Top), means that, for every topological space Z, the statements $(\mathrm{ES})_Z$ and $(\mathrm{US})_Z$ hold.

1.5. The main result of the second part of this paper (Theorem 2) reduces the above stated question concerning strong shape of Cartesian products $X \times P$ to an analogous question of coherent homotopy.

Theorem 2. *Let $\boldsymbol{X}$ be a cofinite inverse system of compact polyhedra with limit $\boldsymbol{p}\colon X \to \boldsymbol{X}$ and let K be a simplicial complex with carrier $P = |K|$. Let $\boldsymbol{q}\colon X \times P \to \boldsymbol{Y}$ be the standard resolution of $X \times P$ associated with $\boldsymbol{p}$ and K and let $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$ be mappings of systems, induced by the canonical projections π_X, π_P. For every topological space Z, the statements (ESS)$_Z$ for X,P and (ECH)$_Z$ for $\boldsymbol{X}$, K and the statements (USS)$_Z$ for X,P and (UCH)$_Z$ for $\boldsymbol{X}$, K are equivalent, respectively.*

Hereby, (ECH)$_Z$ and (UCH)$_Z$ read as follows.

(ECH)$_Z$ For every homotopy class of coherent mappings $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$ and every homotopy class of mappings $[g]\colon Z \to P$, there exists a homotopy class of coherent mappings $[\boldsymbol{h}]\colon Z \to \boldsymbol{Y}$ such that $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}] = [\boldsymbol{f}]$ and $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}] = [C(g)]$.

(UCH)$_Z$ If $[\boldsymbol{h}_i]\colon Z \to \boldsymbol{Y}$, $i = 1, 2$, are two homotopy classes of coherent mappings such that $[\boldsymbol{\pi}_X][\boldsymbol{h}_1] = [\boldsymbol{\pi}_X][\boldsymbol{h}_2]$ and $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}_1] = [C(\boldsymbol{\pi}_P)][\boldsymbol{h}_2]$, then $[\boldsymbol{h}_1] = [\boldsymbol{h}_2]$.

Here $C(\boldsymbol{\pi}_X), C(\boldsymbol{\pi}_P)$ and $C(g)$ are coherent mappings induced by the mappings $\boldsymbol{\pi}_X, \boldsymbol{\pi}_P$ and g, respectively (see Section 2). The abbreviations (ECH)$_Z$ and (UCH)$_Z$ stand for existence and uniqueness in coherent homotopy, respectively.

In a forthcoming paper [12], Theorem 2 is used in an essential way in proving that the statement (ESS$_Z$) holds for every compact Hausdorff space X, every polyhedron P and every metrizable space Z.

2. Preliminaries on resolutions and coherent homotopy

A mapping $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ between inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ (posssibly not cofinite) consists of an increasing function $f\colon M \to \Lambda$ and of a collection of mappings $f_\mu\colon X_{f(\mu_n)} \to Y_\mu, \mu \in M$, such that $f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}$. A coherent mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function $f\colon M \to \Lambda$ and of a collection of mappings $f_{\boldsymbol{\mu}} = f_{\mu_0 \dots \mu_n}\colon X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, where $\Delta^n = [e_0, \dots, e_n]$ is the standard n-simplex and $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ is a multiindex in M of length $n \geq 0$, i.e., an increasing sequence $\mu_0 \leq \dots \leq \mu_n$ of $n + 1$ elements in M. One requires that the following coherence conditions be fulfilled.

$$f_{\boldsymbol{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d^0\boldsymbol{\mu}}(x, t), & j = 0, \\ f_{d^j\boldsymbol{\mu}}(x, t), & 1 \leq j \leq n - 1, \\ f_{d^n\boldsymbol{\mu}}(p_{f(\mu_{n-1})f(\mu_n)} x, t), & j = n, \end{cases} \tag{2}$$

$$f_{\boldsymbol{\mu}}(x, s_j t) = f_{s^j\boldsymbol{\mu}}(x, t), \ 0 \leq j \leq n; \tag{3}$$

here $d_j\colon \Delta^{n-1} \to \Delta^n$ and $s_j\colon \Delta^{n+1} \to \Delta^n$ are the standard boundary and degeneracy operators; d^j omits μ_j from $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$, i.e., $d^j\boldsymbol{\mu} = (\mu_0, \dots, \widehat{\mu_j}, \dots, \mu_n)$, while s^j repeats μ_j, i.e., $d^j\boldsymbol{\mu} = (\mu_0, \dots, \mu_j, \mu_j, \dots, \mu_n)$. Condition (2) makes sense only when $n > 0$.

Coherent mappings can be viewed as generalizations of mappings, because with every mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ one can associate a coherent mapping $C(\boldsymbol{f})$ which consists of the index function f of $\boldsymbol{f}$ and of mappings $f_{\boldsymbol{\mu}}\colon X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, where $f_{\boldsymbol{\mu}}(x,t) = f_{\mu_0} p_{f(\mu_0)f(\mu_n)}(x)$.

If $\boldsymbol{X}$ consists of a single space X, formula (2) assumes a simpler form

$$f_{\boldsymbol{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d^0\boldsymbol{\mu}}(x,t), \ j = 0, \\ f_{d^j\boldsymbol{\mu}}(x,t), \qquad 1 \le j \le n. \end{cases} \tag{4}$$

2.4. If $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$ are mappings, given by index functions f, g and by mappings f_μ, g_ν, the composition $\boldsymbol{gf}\colon \boldsymbol{X} \to \boldsymbol{Z}$ is the mapping, given by the index function fg and by the mappings $g_\nu f_{g(\nu)}$. The composition $\boldsymbol{gf}$ of two coherent mappings is given by a more complicated formula (see Section 1.3 of [8]), not used in this paper.

2.5. Two mappings $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$, given by increasing index functions f, f' and mappings f_μ, f'_μ, $\mu \in M$, are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, if there exists an increasing function $F\colon M \to \Lambda$, $F \ge f, f'$, such that

$$f_\mu p_{f(\mu_n)F(\mu_n)} \simeq f'_\mu p_{f'(\mu_n)F(\mu_n)}. \tag{5}$$

Two coherent mappings $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$, given by index functions f, f' and mappings $f_{\boldsymbol{\mu}}, f'_{\boldsymbol{\mu}}$ are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, provided there exists a coherent mapping $\boldsymbol{F}\colon \boldsymbol{X} \times I \to \boldsymbol{Y}$, given by an increasing function $F \ge f, f'$ and by mappings $F_{\boldsymbol{\mu}}\colon X_{F(\mu_n)} \times I \times \Delta^n \to Y_{\mu_0}$, which satisfy the coherence conditions and

$$F_{\boldsymbol{\mu}}(x,0,t) = f_{\boldsymbol{\mu}}(p_{f(\mu_n)F(\mu_n)}(x), t), \ F_{\boldsymbol{\mu}}(x,1,t) = f'_{\boldsymbol{\mu}}(p_{f'(\mu_n)F(\mu_n)}(x), t). \tag{6}$$

If $\boldsymbol{X}$ is arbitrary and $\boldsymbol{Y}$ is a cofinite system, homotopy of coherent mappings is an equivalence relation (see [8], Lemmas 1.2 and 2.1). The same is true if $\boldsymbol{X}$ consists of a single space X and $\boldsymbol{Y}$ is arbitrary, because in that case the index function is constant and thus, it is increasing. In these cases the corresponding homotopy classes are well defined and are denoted by $[\boldsymbol{f}]$.

Denote by $\mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Y})$ the set of all coherent mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ between two inverse systems $\boldsymbol{X}$ and $\boldsymbol{Y}$. Throughout this paper we will use the following lemma, sometimes without referring to it explicitly.

Lemma 1.

(i) If $\simeq$ is an equivalence relation on the sets Coh($\boldsymbol{X}, \boldsymbol{Y}$), Coh($\boldsymbol{Y}, \boldsymbol{Z}$) and Coh($\boldsymbol{X}, \boldsymbol{Z}$) and $\boldsymbol{f} \in \mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{g} \in \mathrm{Coh}(\boldsymbol{Y}, \boldsymbol{Z})$, then the homotopy classes $[\boldsymbol{f}]$, $[\boldsymbol{g}]$ and $[\boldsymbol{gf}]$ are well defined and $[\boldsymbol{gf}]$ depends only on $[\boldsymbol{f}]$ and $[\boldsymbol{g}]$. Therefore, one defines the composition $[\boldsymbol{g}][\boldsymbol{f}]$ by putting $[\boldsymbol{g}][\boldsymbol{f}] = [\boldsymbol{gf}]$.

(ii) If $\simeq$ is an equivalence relation on the sets Coh($\boldsymbol{X}, \boldsymbol{Y}$), Coh($\boldsymbol{Y}, \boldsymbol{Z}$), Coh($\boldsymbol{Z}, \boldsymbol{W}$), Coh($\boldsymbol{X}, \boldsymbol{Z}$), Coh($\boldsymbol{Y}, \boldsymbol{W}$) and Coh($\boldsymbol{X}, \boldsymbol{W}$) and $\boldsymbol{f} \in \mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{g} \in \mathrm{Coh}(\boldsymbol{Y}, \boldsymbol{Z})$, $\boldsymbol{h} \in \mathrm{Coh}(\boldsymbol{Z}, \boldsymbol{W})$, then $[\boldsymbol{h}(\boldsymbol{gf})] = [(\boldsymbol{hg})\boldsymbol{f}]$ and the corresponding homotopy class depends only on the classes $[\boldsymbol{f}]$, $[\boldsymbol{g}]$, $[\boldsymbol{h}]$. Moreover, $[\boldsymbol{h}]([\boldsymbol{g}][\boldsymbol{f}]) = ([\boldsymbol{h}][\boldsymbol{g}])[\boldsymbol{f}]$.

Proof. (i) Let $\boldsymbol{f},\boldsymbol{f}'\colon \boldsymbol{X}\to\boldsymbol{Y}$ and $\boldsymbol{g},\boldsymbol{g}'\colon \boldsymbol{Y}\to\boldsymbol{Z}$ be coherent mappings. We must prove that $\boldsymbol{f}\simeq\boldsymbol{f}'$ and $\boldsymbol{g}\simeq\boldsymbol{g}'$ imply $\boldsymbol{g}\boldsymbol{f}\simeq\boldsymbol{g}'\boldsymbol{f}'$. Since $\simeq$ is an equivalence relation in $\mathrm{Coh}(\boldsymbol{X},\boldsymbol{Z})$, it suffices to prove that $\boldsymbol{f}\simeq\boldsymbol{f}'$ implies $\boldsymbol{g}\boldsymbol{f}\simeq\boldsymbol{g}\boldsymbol{f}'$ and $\boldsymbol{g}\simeq\boldsymbol{g}'$ implies $\boldsymbol{g}\boldsymbol{f}'\simeq\boldsymbol{g}'\boldsymbol{f}'$. In part (i) of the proof of Lemma 2.4 of [8], a homotopy $\boldsymbol{H}\colon \boldsymbol{X}\times I\to\boldsymbol{Z}$ was constructed, which proves that $\boldsymbol{g}\boldsymbol{f}\simeq\boldsymbol{g}\boldsymbol{f}'$. In part (ii) of the proof of the same lemma a homotopy $\boldsymbol{K}\colon \boldsymbol{X}\times I\to\boldsymbol{Z}$ was constructed, which proves that $\boldsymbol{g}^*\boldsymbol{f}'\simeq\boldsymbol{g}'^*\boldsymbol{f}'$, where $\boldsymbol{g}^*,\boldsymbol{g}'^*$ are certain coherent mappings from $\mathrm{Coh}(\boldsymbol{Y},\boldsymbol{Z})$. More precisely, $\boldsymbol{g}^*,\boldsymbol{g}'^*$ are shifts of the mappings $\boldsymbol{g}$ and $\boldsymbol{g}'$ by an increasing function $G\geq g,g'$. By the proof of Lemma 2.5 of [8], there are homotopies, which show that $\boldsymbol{g}\boldsymbol{f}'\simeq\boldsymbol{g}^*\boldsymbol{f}'$ and $\boldsymbol{g}'\boldsymbol{f}'\simeq\boldsymbol{g}'^*\boldsymbol{f}'$ and thus, by transitivity of $\simeq$ in $\mathrm{Coh}(\boldsymbol{X},\boldsymbol{Z})$, one concludes that $\boldsymbol{g}\boldsymbol{f}'\simeq\boldsymbol{g}'\boldsymbol{f}'$.

(ii) In the proof of Theorem 2.8 of [8], a homotopy $\boldsymbol{H}\colon \boldsymbol{X}\times I\to\boldsymbol{W}$ was constructed, which shows that $\boldsymbol{h}(\boldsymbol{g}\boldsymbol{f})\simeq(\boldsymbol{h}\boldsymbol{g})\boldsymbol{f}$ and thus, $[\boldsymbol{h}(\boldsymbol{g}\boldsymbol{f})]=[(\boldsymbol{h}\boldsymbol{g})\boldsymbol{f}]$. By (i), $[\boldsymbol{h}(\boldsymbol{g}\boldsymbol{f})]$ depends only on $[\boldsymbol{h}]$ and $[\boldsymbol{g}\boldsymbol{f}]$ and the latter class depends only on $[\boldsymbol{g}]$ and $[\boldsymbol{f}]$. Moreover, $[\boldsymbol{h}]([\boldsymbol{g}][\boldsymbol{f}])=[\boldsymbol{h}][\boldsymbol{g}\boldsymbol{f}]=[\boldsymbol{h}(\boldsymbol{g}\boldsymbol{f})]$ and $([\boldsymbol{h}][\boldsymbol{g}])[\boldsymbol{f}]=[\boldsymbol{h}\boldsymbol{g}][\boldsymbol{f}]=[(\boldsymbol{h}\boldsymbol{g})\boldsymbol{f}]$. □

2.6. If $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$, $\boldsymbol{r}\colon Z\to\boldsymbol{Z}$ are cofinite HPol-resolutions, then the definition of strong shape morphisms shows that there is a bijection $\Gamma_{\boldsymbol{rq}}$ between the set $\mathrm{SSh}(Y,Z)$ of strong shape morphisms $G\colon Y\to Z$ and the set $\mathrm{CH}(\boldsymbol{Y},\boldsymbol{Z})$ of homotopy classes of coherent mappings $[\boldsymbol{g}]\colon \boldsymbol{Y}\to\boldsymbol{Z}$ (see [8], 8.2). If $\Gamma_{\boldsymbol{rq}}(G)=[\boldsymbol{g}]$, we say that G and $[\boldsymbol{g}]$ are associated with each other. We also consider the bijection $\Gamma_{\boldsymbol{r}}\colon \mathrm{SSh}(Y,Z)\to \mathrm{CH}(Y,\boldsymbol{Z})$, defined by putting $\Gamma_{\boldsymbol{r}}(G)=[\boldsymbol{g}']$, where $[\boldsymbol{g}']=[\boldsymbol{g}][C(\boldsymbol{q})]\in\mathrm{CH}(Y,\boldsymbol{Z})$ and $[\boldsymbol{g}]=\Gamma_{\boldsymbol{rq}}(G)$. We say that G and $[\boldsymbol{g}']$ are associated with each other.

If $\boldsymbol{p}\colon X\to\boldsymbol{X}$ is another cofinite HPol-resolution and $F\colon X\to Y$ is a strong shape morphism associated with $[\boldsymbol{f}]\colon \boldsymbol{X}\to\boldsymbol{Y}$, then the composition of strong shape morphisms $GF\colon X\to Z$ is associated with the composition $[\boldsymbol{g}][\boldsymbol{f}]\colon \boldsymbol{X}\to\boldsymbol{Z}$, i.e., $\Gamma_{\boldsymbol{rp}}(GF)=[\boldsymbol{g}][\boldsymbol{f}]$ (see [8], 8.2). Therefore, if F and $[\boldsymbol{f}']\colon X\to\boldsymbol{Y}$ are associated with each other, i.e., $\Gamma_{\boldsymbol{q}}(F)=[\boldsymbol{f}']$, then $[\boldsymbol{f}']=[\boldsymbol{f}][C(\boldsymbol{p})]$ and $[\boldsymbol{f}]=\Gamma_{\boldsymbol{qp}}(F)$ is associated with F. Consequently, $[\boldsymbol{g}][\boldsymbol{f}]$ is associated with GF and since $[\boldsymbol{g}][\boldsymbol{f}']=([\boldsymbol{g}][\boldsymbol{f}])[C(\boldsymbol{p})]$, we conclude that $[\boldsymbol{g}][\boldsymbol{f}']$ is associated with GF.

2.7. If $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ and $\boldsymbol{r}\colon Z\to\boldsymbol{Z}$ are cofinite HPol-resolutions, $g\colon Y\to Z$ is a mapping and $\boldsymbol{g}\colon \boldsymbol{Y}\to\boldsymbol{Z}$ is a mapping of systems such that $[C(\boldsymbol{r})][C(g)]=[\boldsymbol{g}][C(\boldsymbol{q})]$, then the definition of the strong shape functor $\overline{S}\colon \mathrm{H}\to\mathrm{SSh}$ shows that the strong shape morphism $G\colon Y\to Z$, which is associated with $[\boldsymbol{g}]$ equals $\overline{S}[g]$ (see [8], 8.2.(12)).

2.8. Let $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ be a resolution and $\boldsymbol{X}$ a cofinite HPol-system. If $[\boldsymbol{f}]\colon Y\to\boldsymbol{X}$ is a homotopy class of coherent mappings, then there exists a unique homotopy class of coherent mappings $[\boldsymbol{h}]\colon \boldsymbol{Y}\to\boldsymbol{X}$ such that $[\boldsymbol{f}]=[\boldsymbol{h}][C(\boldsymbol{q})]$. This is an immediate consequence of [8], Theorems 7.6 and 8.1 and the fact that the defining property of coherent expansions does not assume cofiniteness of $\boldsymbol{q}$.

3. The cofinite resolution $\boldsymbol{q}^*\colon Y\to\boldsymbol{Y}^*$

3.1. With an inverse system $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$, which need not be cofinite, one can associate a cofinite system $\boldsymbol{Y}^*=(Y^*_\beta,q^*_{\beta\beta'},M^*)$ in the following way (see [13]), I.1.2 or [8], 6.4). The index set M^* is the set of all finite subsets $\beta\subseteq M$, which have a terminal element. Since the ordering $\leq$ in M is anti-symmetric, the terminal element of β is uniquely determined and we denote it by β^*. The set M^* is ordered by putting $\beta_1 \leq^* \beta_2$, whenever $\beta_1 \subseteq \beta_2$. Note that $\beta_1 \leq^* \beta_2$ implies $\beta_1^* \leq \beta_2^*$. The set M^* is directed because $\beta = \beta_1 \cup \beta_2 \cup \{\mu\} \supseteq \beta_1, \beta_2$, for every $\mu \in M$, which has the property that $\mu \geq \beta_1^*, \beta_2^*$. Clearly, M^* is cofinite. Y_β^* and $q_{\beta\beta'}^*$ are defined by putting

$$Y_\beta^* = Y_{\beta^*}, \tag{7}$$
$$q_{\beta\beta'}^* = q_{\beta^*\beta'^*}. \tag{8}$$

Note that every term which appears in the system $\boldsymbol{Y}^*$ also appears in the system $\boldsymbol{Y}$. Therefore, if $\boldsymbol{Y}$ is an HPol-system, so is $\boldsymbol{Y}^*$. We will also consider the mapping $\boldsymbol{u_Y}\colon \boldsymbol{Y} \to \boldsymbol{Y}^*$, given by the increasing function $u\colon M^* \to M$, where $u(\beta) = \beta^*$ and by the identity mappings $u_\beta\colon Y_{u(\beta)} = Y_{\beta^*} \to Y_{\beta^*} = Y_\beta^*$.

3.2. With a mapping $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$, which consists of mappings $q_\mu\colon Y \to Y_\mu$, $\mu \in M$, one can associate a mapping $\boldsymbol{q}^*\colon Y \to \boldsymbol{Y}^*$. It consists of mappings $q_\beta^*\colon Y \to Y_\beta^*$, where $q_\beta^* = q_{\beta^*}$. Since $u_\beta q_{u(\beta)} = q_{\beta^*} = q_\beta^*$, we see that $\boldsymbol{u_Y q} = \boldsymbol{q}^*$ and thus, also $C(\boldsymbol{u_Y})C(\boldsymbol{q}) = C(\boldsymbol{u_Y q}) = C(\boldsymbol{q}^*)$. Since (Y) is rudimentary and $\boldsymbol{Y}^*$ is cofinite, the homotopy classes $[C(\boldsymbol{u_Y})], [C(\boldsymbol{q})], [C(\boldsymbol{q}^*)]$ are well defined and, by Lemma 1, $[C(\boldsymbol{u_Y})][C(\boldsymbol{q})] = [C(\boldsymbol{u_Y})C(\boldsymbol{q})]$. Consequently,

$$[C(\boldsymbol{q}^*)] = [C(\boldsymbol{u_Y})][C(\boldsymbol{q})]. \tag{9}$$

Remark 1. *If the system $\boldsymbol{Y}$ is already cofinite, $\boldsymbol{u_Y}\colon \boldsymbol{Y} \to \boldsymbol{Y}^*$ is an isomorphism in the category pro-Top of inverse systems. Indeed, to define an inverse $\boldsymbol{v_Y}\colon \boldsymbol{Y}^* \to \boldsymbol{Y}$ of $\boldsymbol{u_Y}$, one considers an increasing function $v\colon M \to M^*$, which has the property that $v(\mu) \geq \{\mu\}$, for $\mu \in M$. Such a function exists because M is cofinite and M^* is directed. One then defines mappings $v_\mu\colon Y_{v(\mu)}^* = Y_{(v(\mu))^*} \to Y_\mu$ by putting $v_\mu = q_{\mu,(v(\mu))^*}$. Note that $\{\mu\} \leq v(\mu)$ implies $\mu = (\{\mu\})^* \leq (v(\mu))^*$ and thus, $q_{\mu,(v(\mu))^*}$ is well defined. It is readily seen that $v_\mu u_{v(\mu)} = q_{\mu,(v(\mu))^*}$ and thus, $\boldsymbol{v_Y u_Y}$ is equivalent to the identity morphism $\boldsymbol{1_Y}$ in pro-Top. It is also easy to verify that $u_\beta v_{u(\beta)} = q_{\beta,v(\beta^*)}^*$ and thus, $\boldsymbol{u_Y v_Y}$ is equivalent to the identity morphism $\boldsymbol{1_{Y^*}}$ in pro-Top.*

Lemma 2. *If $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ is a resolution, then $\boldsymbol{q}^*\colon Y \to \boldsymbol{Y}^*$ is a cofinite resolution. If $\boldsymbol{Y}$ consists of spaces from the class HPol, then so does $\boldsymbol{Y}^*$.*

For a proof see [8], Lemma 6.31.

Remark 2. *The construction of the cofinite system $\boldsymbol{Y}^*$, associated with a system $\boldsymbol{Y}$, was first used by this author in 1973 (see [6], Theorem 7.1, also see [13], I.§1, Theorem 2)). Since that time it has been used in a number of different situations. In particular, in 1987, the author used it to show that strong homology groups $\overline{H}_n(X,G)$ of spaces, originally defined using cofinite resolutions, can also be calculated using the same formulae and non-cofinite resolutions [7]. Recently, Ju. T. Lisica obtained a similar result for strong cohomology groups (see [4], Remark 3).*

3.3. With every coherent mapping $\boldsymbol{f}\colon X \to \boldsymbol{Y}$, consisting of mappings $f_{\boldsymbol{\mu}} = f_{\mu_0\dots\mu_n}\colon X\times\Delta^n \to Y_{\mu_0}$, one can associate a coherent mapping $\boldsymbol{f}^*\colon X \to \boldsymbol{Y}^*$, which consists of mappings $f^*_{\boldsymbol{\beta}} = f^*_{\beta_0\dots\beta_n}\colon X\times\Delta^n \to Y^*_{\beta_0} = Y_{\beta_0^*}$, given by

$$f^*_{\beta_0\dots\beta_n} = f_{\beta_0^*\dots\beta_n^*}. \tag{10}$$

If $\boldsymbol{\beta} = (\beta_0,\dots,\beta_n) \in M^*$, then $\beta_0 \le^* \dots \le^* \beta_n$. Therefore, $\beta_0^* \le \dots \le \beta_n^*$ and thus, $\boldsymbol{\beta}^* = (\beta_0^*,\dots,\beta_n^*) \in M$. It is readily seen that the mappings $f^*_{\boldsymbol{\beta}}$ satisfy the coherence conditions. Moreover, by the proof of Lemma 2.12 of [8], one concludes that $\boldsymbol{f}^* \simeq C(\boldsymbol{u_Y})\boldsymbol{f}$ and thus,

$$[\boldsymbol{f}^*] = [C(\boldsymbol{u_Y})][\boldsymbol{f}]. \tag{11}$$

Note that for $\boldsymbol{f},\boldsymbol{f}' \in \mathrm{Coh}(X,\boldsymbol{Y})$, one has $\boldsymbol{f}^*,\boldsymbol{f}'^* \in \mathrm{Coh}(X,\boldsymbol{Y}^*)$ and $\boldsymbol{f} \simeq \boldsymbol{f}'$ implies $\boldsymbol{f}^* \simeq \boldsymbol{f}'^*$, because by (11), $[\boldsymbol{f}] = [\boldsymbol{f}']$ implies $[\boldsymbol{f}^*] = [\boldsymbol{f}'^*]$. Consequently, one can define a function $\Phi_{\boldsymbol{Y}}$ from the set $\mathrm{CH}(X,\boldsymbol{Y})$ of homotopy classes of $\mathrm{Coh}(X,\boldsymbol{Y})$ to the set $\mathrm{CH}(X,\boldsymbol{Y}^*)$ of homotopy classes of $\mathrm{Coh}(X,\boldsymbol{Y}^*)$, by putting $\Phi_{\boldsymbol{Y}}[\boldsymbol{f}] = [\boldsymbol{f}^*]$. In view of (11), we see that

$$\Phi_{\boldsymbol{Y}}[\boldsymbol{f}] = [C(\boldsymbol{u_Y})][\boldsymbol{f}]. \tag{12}$$

3.4. Note that a mapping of systems $\boldsymbol{h}\colon \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, given by an increasing function $h\colon N \to M$ and by mappings $h_\nu\colon Y_{h(\nu)} \to Z_\nu$, $\nu \in N$, induces a mapping of systems $\boldsymbol{h}^*\colon \boldsymbol{Y}^* \to \boldsymbol{Z}^*$, given by the increasing function $h^*\colon N^* \to M^*$, where $h^*(\gamma) = \{h(\gamma^*)\}$, $\gamma \in N^*$, and by the mappings $h^*_\gamma\colon Y^*_{h^*(\gamma)} \to Z^*_\gamma$, where $h^*_\gamma = h_{\gamma^*}\colon Y_{h(\gamma^*)} \to Z_{\gamma^*}$. Note that $Y^*_{h^*(\gamma)} = Y^*_{\{h(\gamma^*)\}} = Y_{\{h(\gamma^*)\}^*} = Y_{h(\gamma^*)}$ and $Z^*_\gamma = Z_{\gamma^*}$ and therefore, h^*_γ is well defined. Also note that

$$\boldsymbol{h}^*\boldsymbol{u_Y} = \boldsymbol{u_Z}\boldsymbol{h}, \tag{13}$$

because both sides of (13) consist of mappings $h_{\gamma^*}\colon Y_{h(\gamma^*)} \to Z^*_\gamma$.

3.5. The main result of this section is the following theorem.

Theorem 3. *For a topological space X and an inverse system $\boldsymbol{Y}$, the function $\Phi_{\boldsymbol{Y}}\colon \mathrm{CH}(X,\boldsymbol{Y}) \to \mathrm{CH}(X,\boldsymbol{Y}^*)$, given by $\Phi_{\boldsymbol{Y}}[\boldsymbol{f}] = [C(\boldsymbol{u_Y})][\boldsymbol{f}]$, is a bijection.*

Proof. We will prove the theorem by defining an inverse $\Psi_{\boldsymbol{Y}}$ of $\Phi_{\boldsymbol{Y}}$. We first define a function, which to every coherent mapping $\boldsymbol{g}\colon X \to \boldsymbol{Y}^*$, given by mappings $g_{\boldsymbol{\beta}}\colon X \times \Delta^n \to Y^*_{\beta_0} = Y_{\beta_0^*}$, assigns a coherent mapping $\boldsymbol{g}^\bullet\colon X \to \boldsymbol{Y}$, given by mappings $g^\bullet_{\boldsymbol{\mu}}\colon X\times\Delta^n \to Y_{\mu_0}$. In order to define the mappings $g^\bullet_{\boldsymbol{\mu}}$ we consider the barycentric subdivision $(\Delta^n)'$ of the standard n-simplex $\Delta^n = [e_0,\dots,e_n]$. For every subset $\{j_0,\dots,j_k\} \subseteq \{0,\dots,n\}$ of $k{+}1$ elements, $0 \le k \le n$, the set of vertices $\{e_{j_0},\dots,e_{j_k}\}$ spans a k-dimensional face of Δ^n, denoted by $\Delta^k_{j_0\dots j_k}$. Note that it does not depend on the order of the indices $j_0,\dots,j_k$. Let $e_{j_0\dots j_k}$ denote the barycenter of $\Delta^k_{j_0\dots j_k}$. For an arbitrary permutation $\rho^n\colon \{0,1,\dots,n\} \to \{0,1,\dots,n\}$, let $\Delta^n_{\rho^n} \subseteq \Delta^n$ be the n-simplex spanned by the barycenters $e_{\rho^n(0)}, e_{\rho^n(0)\rho^n(1)},\dots,e_{\rho^n(0)\dots\rho^n(n)} = e_{0\dots n}$ of the simplices $\Delta^0_{\rho^n(0)}$, $\Delta^1_{\rho^n(0)\rho^n(1)},\dots$, $\Delta^n_{\rho^n(0)\dots\rho^n(n)} = \Delta^n$, respectively. Then $(\Delta^n)'$ consists of the n-simplices $\Delta^n_{\rho^n}$, where ρ^n ranges over all permutations of $\{0,\dots,n\}$, and of all faces of these simplices. Now consider another permutation

$\tau^n\colon \{0,1,\dots,n\} \to \{0,1,\dots,n\}$ such that, for some k, $0 \le k \le n$, the barycenters $e_{\rho^n(0)\dots\rho^n(k)}$ and $e_{\tau^n(0)\dots\tau^n(k)}$ coincide. Then also the simplices $\Delta^n_{\rho^n(0)\dots\rho^n(k)}$ and $\Delta^n_{\tau^n(0)\dots\tau^n(k)}$ coincide and therefore, $\rho^n(k) = \tau^n(k)$. Let us also consider the simplicial mapping $\eta^n\colon (\Delta^n)'^n$, which sends the barycenters $e_{j_0\dots j_k}$ of all k-simplices $\Delta^k_{j_0\dots j_k}$ to e_k. Finally, we define $g^\bullet_{\boldsymbol{\mu}} = g^\bullet_{\mu_0\dots\mu_n}$ on $\Delta^n_{\rho^n}$, by putting

$$g^\bullet_{\boldsymbol{\mu}}(x,t) = q_{\mu_0\mu_{\rho^n(0)}} g_{\beta_0\dots\beta_n}(x,\eta^n(t)),\ \text{for } t \in \Delta^n_{\rho^n}, \tag{14}$$

where $\beta_k = \{\mu_{\rho^n(0)},...,\mu_{\rho^n(k)}\}$, $0 \le k \le n$. Note that $\beta_0 \le^* \dots \le^* \beta_n$ and thus, $\boldsymbol{\beta} = (\beta_0,\dots,\beta_n)$ is a multiindex in M^* of length n. Moreover, $g_{\beta_0\dots\beta_n}$ is a mapping with codomain $Y^*_{\beta_0} = Y_{\beta_0^*} = Y_{\{\mu_{\rho^n(0)}\}^*} = Y_{\mu_{\rho^n(0)}}$. Furthermore, $0 \le \rho^n(0)$ implies $\mu_0 \le \mu_{\rho^n(0)}$ and thus, $q_{\mu_0\mu_{\rho^n(0)}}$ is a mapping with domain $Y_{\mu_{\rho^n(0)}}$. Therefore, the composition on the right-hand side of (14) is well defined.

To see that the mappings $g^\bullet_{\boldsymbol{\mu}}\colon X \times \Delta^n_{\rho^n} \to Y_{\mu_0}$, where ρ^n ranges over the permutations of $\{0,1,\dots,n\}$, define a mapping $g^\bullet_{\boldsymbol{\mu}}\colon X \times \Delta^n \to Y_{\mu_0}$, we need to show that, for two different permutations ρ^n, τ^n, formula (14) gives the same values on the intersection $(X \times \Delta^n_{\rho^n}) \cap (X \times \Delta^n_{\tau^n})$. Note that the intersection $\Delta^n_{\rho^n} \cap \Delta^n_{\tau^n}$ is the simplex spanned by all vertices $e_{j_0\dots j_k}$, common to both simplices $\Delta^n_{\rho^n}$ and $\Delta^n_{\tau^n}$. Let these be the vertices $e_{\rho^n(0)\dots\rho^n(l_0)},\dots,e_{\rho^n(0)\dots\rho^n(l_k)}$, where $l_0 < l_1 < \dots < l_k$. Clearly, η^n maps these vertices to the vertices $e_{l_0},\dots,e_{l_k}$, respectively. Therefore, $\eta^n(t) \in [e_{l_0},\dots,e_{l_k}]$, for $t \in \Delta^n_{\rho^n} \cap \Delta^n_{\tau^n}$. Let $u\colon \{0,\dots,k\} \to \{0,\dots,n\}$ be the increasing function, given by $u(i) = l_i$. Consider the induced simplicial mapping $u_*\colon \Delta^k \to \Delta^n$ and note that $u_*(\Delta^k) = [e_{l_0},\dots,e_{l_k}]$. Therefore, there exists a point t'^k such that $\eta^n(t) = u_*(t')$. Consequently, $g_{\beta_0\dots\beta_n}(x,\eta^n(t)) = g_{\beta_0\dots\beta_n}(x,u_*(t'))$. However, it is a consequence of the coherence conditions (see [8], Lemma 1.10) that $g_{\beta_0\dots\beta_n}(x,u_*(t')) = q g_{u^*(\boldsymbol{\beta})}(x,t')$, where $u^*(\boldsymbol{\beta}) = (\beta_{u(0)},\dots,\beta_{u(k)}) = (\beta_{l_0},\dots,\beta_{l_k})$ and $q = q_{\beta_0^*\beta_{l_0}^*}$. Consequently, viewing t as an element of $\Delta^n_{\rho^n}$, formula (14) shows that $g^\bullet_{\boldsymbol{\mu}}(x,t) = q_{\mu_0\beta^*_{l_0}} g_{\beta_{l_0}\dots\beta_{l_k}}(x,t')$, where $\beta_{l_i} = \{\mu_{\rho^n(0)},...,\mu_{\rho^n(l_i)}\}$. Viewing t as an element of $\Delta^n_{\tau^n}$, the same argument shows that $g^\bullet_{\boldsymbol{\mu}}(x,t) = q_{\mu_0\beta'^*_{l_0}} g_{\beta'_{l_0}\dots\beta'_{l_s}}(x,t')$, where $\beta'_{l_i} = \{\mu_{\tau^n(0)},...,\mu_{\tau^n(l_i)}\}$. However, since $e_{\rho^n(0)\dots\rho^n(i)} = e_{\tau^n(0)\dots\tau^n(i)}$, for $0 \le i \le k$, we conclude that also $\{\rho^n(0),...,\rho^n(i)\} = \{\tau^n(0),...,\tau^n(i)\}$, for $0 \le i \le k$, and thus, $\beta_{l_i} = \beta'_{l_i}$, for $0 \le i \le k$, which shows that, for $t \in \Delta^n_{\rho^n} \cap \Delta^n_{\tau^n}$, the two values of $g^\bullet_{\boldsymbol{\mu}}(x,t)$, coincide.

We will now prove that the mappings $g^\bullet_{\boldsymbol{\mu}}$ have the coherence property (2). Let $\rho^{n-1}\colon \{0,\dots,n-1\} \to \{0,\dots,n-1\}$, $n \ge 1$, be a permutation and let $t \in \Delta^{n-1}_{\rho^{n-1}}$. Let $\rho^n\colon \{0,\dots,n\} \to \{0,\dots,n\}$ be the permutation which coincides with ρ^{n-1} on $\{0,\dots,n-1\}$ and maps n to itself. Recall that $d_n\colon \Delta^{n-1} \to \Delta^n$ is the simplicial mapping which sends the vertices $e_0,\dots,e_{n-1}$ of Δ^{n-1} to the vertices $e_0,\dots,e_{n-1}$ of Δ^n, respectively. Therefore, it sends the simplices $\Delta^0_{\rho^{n-1}(0)}, \Delta^1_{\rho^{n-1}(0)\rho^{n-1}(1)},\dots$, $\Delta^{n-1}_{\rho^{n-1}(0)\dots\rho^{n-1}(n-1)}$ and their barycenters to the simplices $\Delta^0_{\rho^n(0)}, \Delta^1_{\rho^n(0)\rho^n(1)},\dots$, $\Delta^n_{\rho^n(0)\dots\rho^n(n-1)}$ and their barycenters, respectively. This implies that $d_n(\Delta^{n-1}_{\rho^{n-1}}) \subseteq \Delta^n_{\rho^n}$ and thus, $d_n t \in \Delta^n_{\rho^n}$. Moreover, $\eta^n d_n(t) = d_n \eta^{n-1}(t)$. Therefore, (14) shows that $g^\bullet_{\boldsymbol{\mu}}(x,d_n t) = q_{\mu_0\beta_0^*} g_{\beta_0\dots\beta_n}(x,\eta^n(d_n t)) = q_{\mu_0\beta_0^*} g_{\beta_0\dots\beta_n}(x,d_n\eta^{n-1}(t))$, where $\beta_k = \{\mu_{\rho^n(0)},...,\mu_{\rho^n(k)}\}$, $0 \le k \le n$. However, $g_{\beta_0\dots\beta_n}(x,d_n\eta^{n-1}(t)) = g_{\beta_0\dots\beta_{n-1}}(x,\eta^{n-1}(t))$ and, for $0 \le k \le n-1$, $\beta_k = \{\mu_{\rho^n(0)},...,\mu_{\rho^n(k)}\} = \{\mu_{\rho^{n-1}(0)},...,\mu_{\rho^{n-1}(k)}\}$ has the value required by (14), for $t \in \Delta^{n-1}_{\rho^{n-1}}$. Consequently, we obtain the desired formula $g^{\bullet}_{\boldsymbol{\mu}}(x, d_n t) = g^{\bullet}_{d^n \boldsymbol{\mu}}(x,t)$. The required coherence formula for d_j, where $0 \le j < n$, is obtained similarly, giving now the role of the vertex e_n to the vertex e_j. Similar arguments can be used to verify the coherence conditions (3).

Now assume that $\boldsymbol{g}, \boldsymbol{g}' \colon X \to \boldsymbol{Y}^*$ are homotopic coherent mappings. Then there is a homotopy $\boldsymbol{G} \colon X \times I \to \boldsymbol{Y}^*$ which connects $\boldsymbol{g}$ and $\boldsymbol{g}'$. If $\boldsymbol{G}$ is formed by homotopies $G_{\boldsymbol{\beta}} \colon X \times I \times \Delta^n \to Y_{\beta_0}$, we consider homotopies $G^{\bullet}_{\boldsymbol{\mu}} \colon X \times I \times \Delta^n \to Y_{\mu_0}$, defined on the sets $\Delta^n_{\rho^n} \times I$ by putting

$$G^{\bullet}_{\boldsymbol{\mu}}(x,s,t) = q_{\mu_0 \mu_{\rho^n(0)}} G_{\beta_0 \dots \beta_n}(x, s, \eta^n(t)), \tag{15}$$

where $\beta_k = \{\mu_{\rho^n(0)}, ..., \mu_{\rho^n(k)}\}$, $0 \le k \le n$. The verification that the homotopies $G^{\bullet}_{\boldsymbol{\mu}}$ are well defined and satisfy the coherence conditions is as in the case of the mappings $g^{\bullet}_{\boldsymbol{\mu}}$. It is also clear that $G^{\bullet}_{\boldsymbol{\mu}}(x,0,t) = g^{\bullet}_{\boldsymbol{\mu}}(x,t)$ and $G^{\bullet}_{\boldsymbol{\mu}}(x,1,t) = {g'}^{\bullet}_{\boldsymbol{\mu}}(x,t)$ and thus, $g^{\bullet} \simeq {g'}^{\bullet}$. We now define $\Psi_{\boldsymbol{Y}}$ by putting $\Psi_{\boldsymbol{Y}}[\boldsymbol{g}] = [\boldsymbol{g}^{\bullet}]$.

We will now show that $\boldsymbol{g}^{\bullet *} \simeq \boldsymbol{g}$, for every coherent mapping $\boldsymbol{g} \colon X \to \boldsymbol{Y}^*$ and thus, $\Phi_{\boldsymbol{Y}} \Psi_{\boldsymbol{Y}}[\boldsymbol{g}] = [\boldsymbol{g}]$. Indeed, $\boldsymbol{g}^{\bullet *}$ consists of mappings $g^{\bullet *}_{\boldsymbol{\beta}} = g^{\bullet *}_{\beta_0 \dots \beta_n} \colon X \times \Delta^n \to Y^*_{\beta_0} = Y_{\mu_0}$. By (10),

$$g^{\bullet *}_{\boldsymbol{\beta}}(x,t) = g^{\bullet}_{\mu_0 \dots \mu_n}(x,t), \tag{16}$$

where $\mu_k = \beta^*_k$, $0 \le k \le n$. By (14), for $t \in \Delta^n_{\rho^n}$, we have

$$g^{\bullet}_{\mu_0 \dots \mu_n}(x,t) = q_{\mu_0 \mu_{\rho^n(0)}} g_{\beta_0 \dots \beta_n}(x, \eta^n(t)), \tag{17}$$

where $\beta_k = \{\mu_{\rho^n(0)}, ..., \mu_{\rho^n(k)}\} = \{(\beta_{\rho^n(0)})^*, ..., (\beta_{\rho^n(k)})^*\}$, $0 \le k \le n$. Consequently, for $t \in \Delta^n_{\rho^n}$,

$$g^{\bullet *}_{\boldsymbol{\beta}}(x,t) = q_{\beta^*_0, (\beta_{\rho^n(0)})^*} g_{\{(\beta_{\rho^n(0)})^*\} \dots \{(\beta_{\rho^n(0)})^*, \dots, (\beta_{\rho^n(n)})^*\}}(x, \eta^n(t)). \tag{18}$$

We will now define a homotopy $\boldsymbol{K} \colon X \times I \to \boldsymbol{Y}^*$, which connects $\boldsymbol{g}^{\bullet *}$ to $\boldsymbol{g}$. It will consist of mappings $K_{\boldsymbol{\beta}} \colon X \times I \times \Delta^n \to Y^*_{\beta_0} = Y_{\beta^*_0}$. To define these mappings we need a triangulation T^{n+1} of the product $I \times \Delta^n$, which on $0 \times \Delta^n$ is the barycentric triangulation of Δ^n and on $1 \times \Delta^n$ coincides with Δ^n. Moreover, all vertices of T^{n+1} belong to the two bases $0 \times \Delta^n$ and $1 \times \Delta^n$, i.e., are of the form $(0, e_{j_0 \dots j_k})$, where $e_{j_0 \dots j_k}$ is the barycenter of the k-simplex $\Delta^k_{j_0 \dots j_k} = [e_{j_0}, \dots, e_{j_k}] \le \Delta^n$, or $(1, e_j)$, $0 \le j \le n$. The $(n+1)$-simplices of T^{n+1} are spanned by the vertices $(0, e_{j_0}), \dots, (0, e_{j_0 \dots j_k}), (1, e_k), \dots, (1, e_n)$, where $0 \le k \le n$. We denote such a simplex by $T^{n+1}_{j_0 \dots j_k}$. If $k = n$, the simplices $T^{n+1}_{j_0 \dots j_k}$ form the cone over $(0 \times \Delta^n)'$ with the vertex e_n and triangulate the simplex $[(0, e_0), \dots, (0, e_n), (1, e_n)]$. If $k = n-1$, the simplices $T^{n+1}_{j_0 \dots j_{n-1}} = [(0, e_{j_0}), \dots, (0, e_{j_0 \dots j_{n-1}}), (1, e_{n-1}), (1, e_n)]$ form the join of the barycentric subdivision $(0 \times \Delta^{n-1})'$ of the face $(0 \times d_n \Delta^n)$ of $0 \times \Delta^n$ with $[1 \times e_{n-1}, 1 \times e_n]$ and thus, triangulate the simplex $[(0, e_0), \dots, (0, e_{n-1}), (1, e_{n-1}), (1, e_n)]$. In general, for a fixed k, the simplices $T^{n+1}_{j_0 \dots j_k}$ triangulate the simplex $[(0, e_0), \dots, (0, e_k), \dots, (1, e_k), (1, e_n)]$. Consequently, T^{n+1} is a subdivision of the standard triangulation of the product $I \times \Delta^n$. We also need the simplicial mapping $\zeta^{n+1} \colon T^{n+1} \to \Delta^{n+1}$, which sends the vertices $(0, e_{j_0 \dots j_k})$ to e_k and $(1, e_j)$ to e_{j+1}.

Finally, we define $K_{\boldsymbol{\beta}}(x,s,t) = K_{\beta_0\ldots\beta_n}(x,s,t)$, for $(s,t) \in T^{n+1}_{\rho^n(0)\ldots\rho^n(k)}$, by putting

$$K_{\beta_0\ldots\beta_n}(x,s,t) \\ = q_{\beta_0^*,(\beta_{\rho^k(0)})^*} g_{\{(\beta_{\rho^k(0)})^*\}\ldots\{(\beta_{\rho^k(0)})^*,\ldots,(\beta_{\rho^k(k)})^*\}\beta_k\ldots\beta_n}(x,\zeta^{n+1}(s,t)). \tag{19}$$

Note that $0 \le \rho^k(0)$ and thus, $\beta_0 \subseteq \beta_{\rho^k(0)}$. This implies $\beta_0^* \le (\beta_{\rho^k(0)})^*$ and shows that $q_{\beta_0^*,(\beta_{\rho^k(0)})^*}$ is well defined. Also $\{(\beta_{\rho^k(0)})^*\} \subseteq \ldots \subseteq \{(\beta_{\rho^k(0)})^*,\ldots,(\beta_{\rho^k(k)})^*\}$ and, by assumption, $\beta_k \le^* \ldots \le^* \beta_n$. Since $\{\rho^k(0),\ldots,\rho^k(k)\} = \{0,\ldots,k\}$, we see that $\rho^k(0),\ldots,\rho^k(k) \le k$ and thus, $\beta_{\rho^k(0)},\ldots,\beta_{\rho^k(k)} \subseteq \beta_k$. It follows that $\{(\beta_{\rho^k(0)})^*,\ldots,(\beta_{\rho^k(k)})^*\} \subseteq \beta_k$. This shows that the index of g in (19) is an increasing sequence of length $n+1$ of elements of M^*. Moreover, the composition on the right-hand side of (19) is well defined, because the domain of $q_{\beta_0^*(\beta_{\rho^n(0)})^*}$ is $Y_{(\beta_{\rho^n(0)})^*}$ and this is the codomain of the other function appearing on the right-hand side of (19).

We omit the somewhat tedious verification that the mappings $K_{\boldsymbol{\beta}}$ are well defined on all of $X \times I \times \Delta^n$. Moreover, they form a coherent mapping $\boldsymbol{K}\colon X \times I \to \boldsymbol{Y}^*$. Finally, the basis $0 \times \Delta^n$ of $I \times \Delta^n$ is triangulated by the intersections $T^{n+1}_{\rho^n(0)\ldots\rho^n(n)} \cap (0 \times \Delta^n) = 0 \times \Delta^n_{\rho^n}$ and $\zeta^{n+1}(0,t) = d_{n+1}\eta^n(t)$, for $(0,t) \in 0 \times \Delta^n_{\rho^n}$. Therefore, formulae (19) and (18) show that

$$\begin{aligned} K_{\beta_0\ldots\beta_n}(x,0,t) &= q_{\beta_0^*,(\beta_{\rho^n(0)})^*} g_{\{(\beta_{\rho^n(0)})^*\}\ldots\{(\beta_{\rho^n(0)})^*,\ldots,(\beta_{\rho^n(n)})^*\}\beta_n}(x,\zeta^{n+1}(0,t)) \\ &= q_{\beta_0^*,(\beta_{\rho^n(0)})^*} g_{\{(\beta_{\rho^n(0)})^*\}\ldots\{(\beta_{\rho^n(0)})^*,\ldots,(\beta_{\rho^n(n)})^*\}}(x,\eta^n(t)) \\ &= g^{\bullet *}_{\boldsymbol{\beta}}(x,t). \end{aligned} \tag{20}$$

Similarly, the triangulation T^{n+1}, restricted to the basis $1 \times \Delta^n$ of $I \times \Delta^n$ consists of a single n-simplex $1 \times \Delta^n = T^{n+1}_{\rho^n(0)} \cap (1 \times \Delta^n)$ and its faces and $\zeta^{n+1}(1,t) = d_0 t$, for $(1,t) \in I \times \Delta^n_{\rho^n}$. Therefore, formula (19) shows that

$$\begin{aligned} K_{\beta_0\ldots\beta_n}(x,1,t) &= q_{\beta_0^*,(\beta_{\rho^n(0)})^*} g_{\{(\beta_{\rho^n(0)})^*\}\beta_0\ldots\beta_n}(x,\zeta^{n+1}(1,t)) \\ &= q_{\beta_0^*,(\beta_{\rho^n(0)})^*} q^*_{\{(\beta_{\rho^n(0)})^*\}\beta_0} g_{\beta_0\ldots\beta_n}(x,t) \\ &= g_{\beta_0\ldots\beta_n}(x,t), \end{aligned} \tag{21}$$

because $q_{\beta_0^*,(\beta_{\rho^n(0)})^*} q^*_{\{(\beta_{\rho^n(0)})^*\}\beta_0} = q_{\beta_0^*,(\beta_{\rho^n(0)})^*} q_{(\beta_{\rho^n(0)})^*\beta_0^*} = q_{\beta_0^*\beta_0^*} = \mathrm{id}$.

We will now show that $f^{*\bullet} \simeq \boldsymbol{f}$, for every coherent mapping $\boldsymbol{f}\colon X \to \boldsymbol{Y}$ and thus, $\Psi_{\boldsymbol{Y}}\Phi_{\boldsymbol{Y}}[\boldsymbol{f}] = [\boldsymbol{f}]$. Indeed, $\boldsymbol{f}^{*\bullet}$ consists of mappings $f^{*\bullet}_{\boldsymbol{\mu}} = f^{*\bullet}_{\mu_0\ldots\mu_n}\colon X \times \Delta^n \to Y_{\mu_0}$, where by (14) and (10), for $t \in \Delta^n_{\rho^n}$, one has

$$f^{*\bullet}_{\boldsymbol{\mu}}(x,t) = q_{\mu_0\beta_0^*} f^*_{\beta_0\ldots\beta_n}(x,\eta^n(t)) = q_{\mu_0\beta_0^*} f_{\beta_0^*\ldots\beta_n^*}(x,\eta^n(t)), \tag{22}$$

where $\beta_k = \{\mu_{\rho^n(0)},\ldots,\mu_{\rho^n(k)}\}$, $0 \le k \le n$, and thus,

$$f^{*\bullet}_{\boldsymbol{\mu}}(x,t) = q_{\mu_0\mu_{\rho^n(0)}} f_{\{\mu_{\rho^n(0)}\}^*\ldots\{\mu_{\rho^n(0)},\ldots,\mu_{\rho^n(n)}\}^*}(x,\eta^n(t)). \tag{23}$$

We now define a homotopy $\boldsymbol{H}\colon X \times I \to \boldsymbol{Y}$, which connects $\boldsymbol{f}^{*\bullet}$ to $\boldsymbol{f}$. It consists of mappings $H_{\boldsymbol{\mu}}\colon X \times I \times \Delta^n \to Y_{\mu_0}$. For $(s,t) \in T^{n+1}_{\rho^n(0)\ldots\rho^n(k)} \subseteq I \times \Delta^n$, we put

$$H_{\boldsymbol{\mu}}(x,s,t) = q_{\mu_0\mu_{\rho^k(0)}} f_{\{\mu_{\rho^k(0)}\}^*\ldots\{\mu_{\rho^k(0)},\ldots,\mu_{\rho^k(k)}\}^*\mu_k\ldots\mu_n}(x,\zeta^{n+1}(s,t)). \tag{24}$$

Since , $\{\rho^k(0),\dots,\rho^k(k)\}=\{0,\dots,k\}$, it follows that $\{\mu_{\rho^k(0)},...,\mu_{\rho^k(k)}\}=\{\mu_0,\dots,\mu_k\}$ and thus, $\{\mu_{\rho^k(0)},...,\mu_{\rho^k(k)}\}^*=\{\mu_0,\dots,\mu_k\}^*=\mu_k$. Therefore, the index of f in (24) is a (degenerate) multiindex of length $n+1$. Moreover, $\mu_0\le\mu_{\rho^k(0)}$, because $0\le\rho^k(0)$. All this shows that the right-hand side of (24) is well defined.

One can verify that the mappings $H_{\boldsymbol{\mu}}$ are well defined on all of $X\times I\times\Delta^n$. Moreover, they form a coherent mapping $\boldsymbol{H}\colon X\times I\to\boldsymbol{Y}$. Finally, the basis $0\times\Delta^n$ of $I\times\Delta^n$ is triangulated by the intersections $T^{n+1}_{\rho^n(0)\dots\rho^n(n)}\cap(0\times\Delta^n)=0\times\Delta^n_{\rho^n}$ and $\zeta^{n+1}(0,t)=d_{n+1}\eta^n(t)$, for $(0,t)\in 0\times\Delta^n_{\rho^n}$. Therefore, formulae (24) and (23) show that

$$\begin{aligned} H_{\boldsymbol{\mu}}(x,0,t) &= q_{\mu_0\mu_{\rho^n(0)}}f_{\{\mu_{\rho^n(0)}\}^*\dots\{\mu_{\rho^n(0)},\dots,\mu_{\rho^n(n)}\}^*\mu_n}(x,\zeta^{n+1}(0,t))\\ &= q_{\mu_0\mu_{\rho^n(0)}}f_{\{\mu_{\rho^n(0)}\}^*\dots\{\mu_{\rho^n(0)},\dots,\mu_{\rho^n(n)}\}^*}(x,\eta^n(t))\\ &= f^{*\bullet}_{\boldsymbol{\mu}}(x,t). \end{aligned}\tag{25}$$

Similarly, for $t\in\Delta^n$, one has $(1,t)\in T^{n+1}_{\rho^0(0)}$ and $\zeta^{n+1}(t,1)=d_0t$. Since $\{\mu_{\rho^0(0)}\}^*=\mu_{\rho^0(0)}$ and $\rho^0(0)=0$, formula (24) shows that

$$\begin{aligned} H_{\boldsymbol{\mu}}(x,1,t) &= f_{\mu_0\mu_0\dots\mu_n}(x,\zeta^{n+1}(1,t))\\ &= f_{\mu_0\dots\mu_n}(x,t)\\ &= f_{\boldsymbol{\mu}}(x,t). \end{aligned}\tag{26}$$

□

3.6. The following technical lemma plays an important role in the proof of Theorem 2, given in Section 5.

Lemma 3. *Let X,Y,Z be spaces, let $F\colon Z\to X$ and $H\colon Z\to Y$ be strong shape morphisms and let $\pi\colon Y\to X$ be a mapping. Furthermore, let $\boldsymbol{p}\colon X\to\boldsymbol{X}$ be a cofinite HPol-resolution of X, let $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ be an HPol-resolution (which need not be cofinite) and let $\boldsymbol{\pi}\colon\boldsymbol{Y}\to\boldsymbol{X}$ be a mapping of systems such that $\boldsymbol{\pi q}=\boldsymbol{p}\pi$. If $[\boldsymbol{f}]\colon Z\to\boldsymbol{X}$ and $[\boldsymbol{h}]\colon Z\to\boldsymbol{Y}$ are homotopy classes of coherent mappings associated with F and H, respectively, then $\overline{S}[\pi]H=F$ if and only if $[C(\boldsymbol{\pi})][\boldsymbol{h}]=[\boldsymbol{f}]$.*

Note that the classes $[\boldsymbol{f}]\colon Z\to\boldsymbol{X}$, $[\boldsymbol{h}]\colon Z\to\boldsymbol{Y}$, $[C(\boldsymbol{\pi})]\colon\boldsymbol{Y}\to\boldsymbol{X}$ and $[C(\boldsymbol{\pi})\boldsymbol{h}]\colon Z\to\boldsymbol{X}$ are well defined and $[C(\boldsymbol{\pi})][\boldsymbol{h}]=[C(\boldsymbol{\pi})\boldsymbol{h}]$ (see Lemma 1 (i)).

Proof. We first consider the case when $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ is cofinite. By 2.7, the strong shape morphism $\overline{S}[\pi]\colon Y\to X$ is associated with the class of coherent mappings $[C(\boldsymbol{\pi})]\colon\boldsymbol{Y}\to\boldsymbol{X}$. Since H is associated with $[\boldsymbol{h}]$, 2.6 shows that $\overline{S}[\pi]H$ is associated with $[\boldsymbol{\pi}][\boldsymbol{h}]$, i.e.,

$$\Gamma_{\boldsymbol{p}}(\overline{S}[\pi]H)=[C(\boldsymbol{\pi})][\boldsymbol{h}].\tag{27}$$

Since

$$\Gamma_{\boldsymbol{p}}(F)=[\boldsymbol{f}],\tag{28}$$

we see that $\overline{S}[\pi]H=F$ implies $[C(\boldsymbol{\pi})][\boldsymbol{h}]=[\boldsymbol{f}]$. Conversely, if $[C(\boldsymbol{\pi})][\boldsymbol{h}]=[\boldsymbol{f}]$, then $\Gamma_{\boldsymbol{p}}(\overline{S}[\pi]H)=\Gamma_{\boldsymbol{p}}(F)$. It follows that $\overline{S}[\pi]H=F$, because $\Gamma_{\boldsymbol{p}}$ is a bijection.

We will now assume that $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ is not cofinite. Consider the cofinite HPol-resolution $\boldsymbol{q}^*\colon Y\to\boldsymbol{Y}^*$, induced by $\boldsymbol{q}\colon Y\to\boldsymbol{Y}$ (see Lemma 2), the cofinite HPol-system $\boldsymbol{X}$ and the homotopy class of coherent mappings $[C(\boldsymbol{p}\pi)]\colon Y\to\boldsymbol{X}$. Since

$\boldsymbol{p}$ and $\boldsymbol{q}^*$ are cofinite HPol-resolutions, 2.7 applies and yields a class of coherent mappings $[\boldsymbol{\pi}^+]\colon \boldsymbol{Y}^* \to \boldsymbol{X}$ such that $[C(\boldsymbol{p}\pi)] = [C(\boldsymbol{p})][C(\pi)] = [\boldsymbol{\pi}^+][C(\boldsymbol{q}^*)]$. By (9), $[C(\boldsymbol{u}_{\boldsymbol{Y}})][C(\boldsymbol{q})] = [C(\boldsymbol{q}^*)]$ and thus, $([\boldsymbol{\pi}^+][C(\boldsymbol{u}_{\boldsymbol{Y}})])[C(\boldsymbol{q})] = [\boldsymbol{\pi}^+]([C(\boldsymbol{u}_{\boldsymbol{Y}})][C(\boldsymbol{q})]) = [\boldsymbol{\pi}^+][C(\boldsymbol{q}^*)] = [C(\boldsymbol{p}\pi)]$. Since $\boldsymbol{p}\pi = \boldsymbol{\pi}\boldsymbol{q}$, we see that $([\boldsymbol{\pi}^+][C(\boldsymbol{u}_{\boldsymbol{Y}})])[C(\boldsymbol{q})] = [C(\boldsymbol{\pi}\boldsymbol{q})] = [C(\boldsymbol{\pi})][C(\boldsymbol{q})]$. Since $\boldsymbol{q}$ is a resolution and $\boldsymbol{X}$ is a cofinite HPol-system, the uniqueness part of 2.8 implies that

$$[\boldsymbol{\pi}^+][C(\boldsymbol{u}_{\boldsymbol{Y}})] = [C(\boldsymbol{\pi})]. \tag{29}$$

Let $[\boldsymbol{h}^+]\colon Z \to \boldsymbol{Y}^*$ be the class of coherent mappings, which is associated with the strong shape morphism $H\colon Z \to Y$, i.e., let $\Gamma_{\boldsymbol{q}^*}(H) = [\boldsymbol{h}^+]$. By 2.7, the strong shape morphism $\overline{S}[\pi]\colon Y \to X$ is associated with the class of coherent mappings $[\boldsymbol{\pi}^+]\colon \boldsymbol{Y}^* \to \boldsymbol{X}$, i.e., $\Gamma_{\boldsymbol{p}\boldsymbol{q}^*}(\overline{S}[\pi]) = [\boldsymbol{\pi}^+]$. It follows, by 2.6, that $\overline{S}[\pi]H$ is associated with the class $[\boldsymbol{\pi}^+][\boldsymbol{h}^+]$, i.e., $\Gamma_{\boldsymbol{p}}(\overline{S}[\pi]H) = [\boldsymbol{\pi}^+][\boldsymbol{h}^+]$. Since $[\boldsymbol{h}]\colon Z \to \boldsymbol{Y}$ is associated with H, i.e., $\Gamma_{\boldsymbol{q}}(H) = [\boldsymbol{h}]$, and by (1), $\Phi_{\boldsymbol{Y}}\Gamma_{\boldsymbol{q}}(H) = \Gamma_{\boldsymbol{q}^*}(H)$, we see that $\Phi_{\boldsymbol{Y}}[\boldsymbol{h}] = \Gamma_{\boldsymbol{q}^*}(H) = [\boldsymbol{h}^+]$. However, by (12), $\Phi_{\boldsymbol{Y}}[\boldsymbol{h}] = [C(\boldsymbol{u}_{\boldsymbol{Y}})][\boldsymbol{h}]$ and thus, $[\boldsymbol{h}^+] = [C(\boldsymbol{u}_{\boldsymbol{Y}})][\boldsymbol{h}]$. Now note that, by (29), $[\boldsymbol{\pi}^+][\boldsymbol{h}^+] = [\boldsymbol{\pi}^+]([C(\boldsymbol{u}_{\boldsymbol{Y}})][\boldsymbol{h}]) = ([\boldsymbol{\pi}^+][C(\boldsymbol{u}_{\boldsymbol{Y}}])[\boldsymbol{h}] = [C(\boldsymbol{\pi})][\boldsymbol{h}]$. Consequently, (27) holds again. On the other hand, $F\colon Z \to X$ is associated with $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$, i.e., (28) also holds. Comparing (27) with (28), we conclude as in the case of cofinite $\boldsymbol{q}$, that $\overline{S}[\pi]H = F$ if and only if $[C(\boldsymbol{\pi})][\boldsymbol{h}] = [\boldsymbol{f}]$. □

4. The standard resolution of $X \times P$

4.1. Following the author's paper [9], we now describe the standard resolution $\boldsymbol{q}\colon Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ of the product $Y = X \times P$ of a compact Hausdorff space X and a polyhedron P (CW-topology). It consists of an inverse system $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ (sometimes also called the standard resolution of $X \times P$) and of a mapping of systems $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$, which consists of mappings $q_\mu\colon X \times P \to Y_\mu$, $\mu \in M$, into spaces Y_μ. It is determined by a triangulation K of P and by the limit $\boldsymbol{p}\colon X \to \boldsymbol{X}$ of a cofinite inverse system of compact polyhedra $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$.

Order the simplicial complex K by putting $\sigma \le \sigma'$, whenever the simplex σ is a face of the simplex $\sigma' \in K$. Let M be the set of all increasing functions $\mu\colon K \to \Lambda$, i.e., functions such that $\sigma \le \sigma'$ implies $\mu(\sigma) \le \mu(\sigma')$. Endow M with the natural ordering, i.e., put $\mu \le \mu'$ provided $\mu(\sigma) \le \mu'(\sigma)$, for every $\sigma \in K$. It is easy to see that $(M, \le)$ is a directed ordered set, but in general, M fails to be cofinite. In order to define the spaces Y_μ, one first associates with every $\sigma \in K$ and $\mu \in M$ the product space $X_{\mu(\sigma)} \times \sigma$. Then one considers the coproduct (disjoint sum)

$$\tilde{Y}_\mu = \coprod_{\sigma \in K} (X_{\mu(\sigma)} \times \sigma). \tag{30}$$

By definition, Y_μ is the quotient space

$$Y_\mu = \tilde{Y}_\mu / \sim_\mu, \tag{31}$$

where $\sim_\mu$ denotes the equivalence relation determined by considering points $(x, t) \in X_{\mu(\sigma)} \times \sigma \subseteq \tilde{Y}_\mu$ and $(x', t') \in X_{\mu(\sigma')} \times \sigma' \subseteq \tilde{Y}_\mu$ equivalent, provided $\sigma \le \sigma'$, $x =$

$p_{\mu(\sigma)\mu(\sigma')}(x')$ and $t' = i_{\sigma\sigma'}(t)$, where $i_{\sigma\sigma'}\colon \sigma \to \sigma'$ is the inclusion mapping (we usually simplify the notation by putting $i_{\sigma\sigma'}(t) = t$). The corresponding quotient mapping is denoted by $\phi_\mu\colon \tilde{Y}_\mu \to Y_\mu$.

In order to define the mappings $q_{\mu\mu'}\colon Y_{\mu'} \to Y_\mu$, one first defines mappings $\tilde{q}_{\mu\mu'}\colon \tilde{Y}_{\mu'} \to \tilde{Y}_\mu$, by putting

$$\tilde{q}_{\mu\mu'}(x,t) = (p_{\mu(\sigma)\mu'(\sigma)}(x), t), \tag{32}$$

for $(x,t) \in X_{\mu(\sigma)} \times \sigma \subseteq \tilde{Y}_\mu$. It is readily seen that there exist unique mappings $q_{\mu\mu'}\colon Y_{\mu'} \to Y_\mu$ such that

$$q_{\mu\mu'}\phi_{\mu'} = \phi_\mu \tilde{q}_{\mu\mu'}. \tag{33}$$

Moreover, $q_{\mu\mu'}q_{\mu'\mu''} = q_{\mu\mu''}$, for $\mu \le \mu' \le \mu''$, so that $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu}, M)$ is an inverse system.

The mapping $\boldsymbol{q}\colon X \times P \to \boldsymbol{Y}$ consists of mappings $q_\mu\colon X \times P \to Y_\mu$, $\mu \in M$, defined as follows. With every $\sigma \in K$ and $\mu \in M$ one associates the mapping $p_{\mu(\sigma)} \times 1_\sigma\colon X \times \sigma \to X_{\mu(\sigma)} \times \sigma$, where $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$, are the projections forming $\boldsymbol{p}\colon X \to \boldsymbol{X}$. Put

$$\tilde{Y} = \coprod_{\sigma \in K} (X \times \sigma) = X \times \coprod_{\sigma \in K} \sigma \tag{34}$$

and define mappings $\tilde{q}_\mu\colon \tilde{Y} \to \tilde{Y}_\mu$, by putting

$$\tilde{q}_\mu(x,t) = (p_{\mu(\sigma)}(x), t), \tag{35}$$

for $(x,t) \in X \times \sigma \subseteq \tilde{Y}$. We also consider the quotient mapping $\phi = 1_X \times u\colon \tilde{Y} \to X \times P$, where $u\colon \coprod_{\sigma \in K} \sigma \to P$ is the quotient mapping, defined by the requirement that the restrictions $u|\sigma\colon \sigma \to P$ are inclusion mappings $\sigma \hookrightarrow P$. It is readily seen that there exist unique mappings $q_\mu\colon X \times P \to Y_\mu$ such that

$$\phi_\mu \tilde{q}_\mu = q_\mu \phi. \tag{36}$$

Moreover, $q_\mu = q_{\mu\mu'}q_{\mu'}$, for $\mu \le \mu'$.

We also consider two mappings of systems $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$ and $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$, defined as follows. With every $\lambda \in \Lambda$ one associates the constant function $\sigma \mapsto \lambda$, $\sigma \in K$, denoted by $\overline{\lambda}$. Clearly, $\overline{\lambda} \in M$. By (30), $\tilde{Y}_{\overline{\lambda}} = X_\lambda \times (\coprod_{\sigma \in K} \sigma)$. Moreover, if $(x,t) \in X_{\overline{\lambda}(\sigma)} \times \sigma = X_\lambda \times \sigma \subseteq \tilde{Y}_{\overline{\lambda}}$, $(x',t') \in X_{\overline{\lambda}(\sigma')} \times \sigma' = X_\lambda \times \sigma' \subseteq \tilde{Y}_{\overline{\lambda}}$ and $(x,t) \sim_{\overline{\lambda}} (x',t')$, then $x = x'$ and $u(t) = u(t')$. To verify this assertion, it suffices to consider the case when $\sigma \le \sigma'$. In that case, $x = p_{\overline{\lambda}(\sigma)\overline{\lambda}(\sigma')}(x') = p_{\lambda\lambda}(x') = x'$ and $t' = i_{\sigma\sigma'}(t)$, hence also $u(t) = u(t')$. All this shows that $Y_{\overline{\lambda}} = X_\lambda \times P$ and the quotient mapping $\phi_{\overline{\lambda}}\colon \tilde{Y}_{\overline{\lambda}} \to Y_{\overline{\lambda}}$ is the mapping $1_{X_\lambda} \times u\colon X_\lambda \times (\coprod_{\sigma \in K} \sigma) \to X_\lambda \times P$.

By definition, the mapping $\boldsymbol{\pi}_X$ is given by the increasing function $\lambda \mapsto \overline{\lambda}$ and by the first projections $\pi_\lambda\colon Y_{\overline{\lambda}} = X_\lambda \times P \to X_\lambda$. Since $q_{\overline{\lambda}\overline{\lambda'}} = p_{\lambda\lambda'} \times 1_P$, one has $\pi_\lambda q_{\overline{\lambda}\overline{\lambda'}} = p_{\lambda\lambda'}\pi_{\lambda'}$ and thus, $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$ is a mapping. Since P is a polyhedron, the mapping $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$ is determined (up to equivalence), by any index $\lambda \in \Lambda$ and by the second projection $\pi_P\colon Y_{\overline{\lambda}} = X_\lambda \times P \to P$. It is readily seen that

$$\boldsymbol{\pi}_X \boldsymbol{q} = \boldsymbol{p}\pi_X, \quad \boldsymbol{\pi}_P \boldsymbol{q} = \pi_P, \tag{37}$$

where $\pi_X : X \times P \to X$ and $\pi_P : X \times P \to P$ are the first and the second projections.

4.2. In [9], it was proved that the spaces Y_μ are (Hausdorff) paracompact spaces, belonging to the class HPol. Consequently, the standard resolution $\boldsymbol{q}: X \times P \to \boldsymbol{Y}$ is a non-cofinite HPol-resolution. Recently, the author showed that the spaces Y_μ are (Hausdorff) stratifiable k-spaces (see [11], Lemmas 4 and 5). Recall that stratifiable spaces were introduced in 1961 by J. Ceder as a generalization of metrizable spaces. Ceder proved that polyhedra (even CW-complexes), which are in general non-metrizable, belong to the class of stratifiable spaces. Moreover, he proved that stratifiable spaces are (Hausdorff) paracompact and perfectly normal spaces.

5. Proof of Theorem 2

5.1. $(\text{ECH})_Z \Rightarrow (\text{ESS})_Z$. Let $F: Z \to X$ be a strong shape morphism and let $[g]: Z \to P$ be a homotopy class of mappings. We must find a strong shape morphism $H: Z \to X \times P$ such that $\overline{S}[\pi_X]H = F$ and $\overline{S}[\pi_P]H = \overline{S}[g]$. Since $\boldsymbol{p}: X \to \boldsymbol{X}$ is a cofinite HPol-resolution of X, with the strong shape morphism $F: Z \to X$ is associated a homotopy class of coherent mappings $[\boldsymbol{f}]: Z \to \boldsymbol{X}$. Now condition $(\text{ECH})_Z$ yields a homotopy class of coherent mapping $[\boldsymbol{h}]: Z \to \boldsymbol{Y}$ such that $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}] = [\boldsymbol{f}]$ and $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}] = [C(g)]$. Since $\boldsymbol{q}$ is an HPol-resolution, there is a strong shape morphism $H: Z \to X \times P$, which is associated with $[\boldsymbol{h}]$.

Recall that $\boldsymbol{\pi}_X \boldsymbol{q} = \boldsymbol{p}\pi_X$ (see (37)) and apply Lemma 3 to $X, Y = X \times P, Z, F, H, \pi = \pi_X, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\pi} = \boldsymbol{\pi}_X, \boldsymbol{f}$ and $\boldsymbol{h}$. Since $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}] = [\boldsymbol{f}]$, it follows that indeed, $\overline{S}[\pi_X]H = F$. Also recall that $\boldsymbol{\pi}_P \boldsymbol{q} = \pi_P$ (see (37)) and apply Lemma 3 to $X = P, Y = X \times P, Z, F = \overline{S}[g], H, \pi = \pi_P, \boldsymbol{p} = 1_P, \boldsymbol{q}, \boldsymbol{\pi} = \boldsymbol{\pi}_P, \boldsymbol{f} = C(g)$ and $\boldsymbol{h}$. Since $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}] = [C(g)]$, it follows that also $\overline{S}[\pi_P]H = \overline{S}[g]$.

5.2. $(\text{ECH})_Z \Leftarrow (\text{ESS})_Z$. Given a homotopy class of coherent mappings $[\boldsymbol{f}]: Z \to \boldsymbol{X}$, we choose a strong shape morphism $F: Z \to X \times P$, which is associated with $[\boldsymbol{f}]$. Now condition $(\text{ESS})_Z$ yields a strong shape morphism $H: Z \to X \times P$ such that $\overline{S}[\pi_X]H = F$ and $\overline{S}[\pi_P]H = \overline{S}[g]$. Using again Lemma 3, one concludes that $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}] = [\boldsymbol{f}]$ and $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}] = [C(g)]$.

5.3. $(\text{UCH})_Z \Rightarrow (\text{USS})_Z$. Let $H_i: Z \to X \times P$, $i = 1, 2$, be two strong shape morphisms such that $\overline{S}[\pi_X]H_1 = \overline{S}[\pi_X]H_2$ and $\overline{S}[\pi_P]H_1 = \overline{S}[\pi_P]H_2$, $i = 1, 2$. We must prove that $H_1 = H_2$. Denote by $F: Z \to X$ the strong shape morphism $F = \overline{S}[\pi_X]H_i$ and note that it does not depend on i. Denote by $[\boldsymbol{f}]: Z \to \boldsymbol{X}$ the homotopy classes of coherent mappings associated with $\boldsymbol{F}$. Since the codomain of $\overline{S}[\pi_P]H_i$ is the polyhedron P, there is a mapping $g: Z \to P$ such that $\overline{S}[\pi_P]H_i = \overline{S}[g]$. Note that $[g]$ too does not depend on i. Since $\boldsymbol{q}: X \times P \to \boldsymbol{Y}$ is an HPol-resolution of $X \times P$, with the strong shape morphisms H_i, one can associate homotopy classes of coherent mappings $[\boldsymbol{h}_i]: Z \to \boldsymbol{Y}$, $i = 1, 2$. Note that $\boldsymbol{\pi}_X \boldsymbol{q} = \boldsymbol{p}\pi_X$ and apply Lemma 3 to $X, Y = X \times P, Z, F, H_i, \pi = \pi_X, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\pi} = \boldsymbol{\pi}_X, \boldsymbol{f}$ and $\boldsymbol{h}_i$. Since $F = \overline{S}[\pi_X]H_i$, it follows that $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}_i] = [\boldsymbol{f}]$, $i = 1, 2$, and thus, $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}_1] = [C(\boldsymbol{\pi}_X)][\boldsymbol{h}_2]$. A similar argument, using $\boldsymbol{\pi}_P \boldsymbol{q} = \pi_P$ and Lemma 3, where $X = P, Y = X \times P, Z, F = \overline{S}[g], H_i, \pi = \pi_P, \boldsymbol{p} = 1_P : P \to \{P\}, \boldsymbol{q}, \boldsymbol{\pi} = \boldsymbol{\pi}_P, \boldsymbol{f} = C(g)$ and $\boldsymbol{h}_i$ shows that $[\boldsymbol{\pi}_P][\boldsymbol{h}_1] = [C(g)] = [\boldsymbol{\pi}_P][\boldsymbol{h}_2]$. Now (UCH) implies that $[\boldsymbol{h}_1] = [\boldsymbol{h}_2]$ and thus, $H_1 = H_2$.

5.4. $(UCH)_Z \Leftarrow (USS)_Z$. This implication is proved by repeating the above argument with interchanged roles of strong shape morphisms and homotopy classes of coherent mappings.

6. Results concerning ordinary shape

Results developed in previous sections for strong shape and coherent homotopy have their analogues in (ordinary) shape and pro-homotopy. Using the definitions from [8], the proofs follow the same pattern and will therefore be omitted.

The analogue of the category CH is the category pro-H. Its objects are cofinite inverse systems of spaces $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. To define morphisms, we first consider homotopy mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ between arbitrary inverse systems. They consist of an increasing function $f\colon M \to \Lambda$ (the index function) and of a collection of mappings $f_\mu\colon X_{f(\mu)} \to Y_\mu$ such that

$$f_\mu p_{f(\mu)f(\mu')} \simeq q_{\mu\mu'} f_{\mu'}, \quad \mu \le \mu'. \tag{38}$$

If $\boldsymbol{X}$ is a rudimentary system, i.e., it consists of a single space X, then $\boldsymbol{f}\colon X \to \boldsymbol{Y}$ consists of a collection of mappings $f_\mu\colon X \to Y_\mu$ such that

$$f_\mu \simeq q_{\mu\mu'} f_{\mu'}, \quad \mu \le \mu'. \tag{39}$$

Two homotopy mappings $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$, given by increasing index functions f, f' and mappings f_μ, f'_μ, $\mu \in M$, are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, if there exists an increasing function $F\colon M \to \Lambda$, $F \ge f, f'$, such that

$$f_\mu p_{f(\mu_n)F(\mu_n)} \simeq f'_\mu p_{f'(\mu_n)F(\mu_n)}. \tag{40}$$

If $\boldsymbol{Y}$ is a cofinite system, homotopy of homotopy mappings is an equivalence relation. Therefore, the homotopy classes $[\boldsymbol{f}]\colon \boldsymbol{X} \to \boldsymbol{Y}$ of homotopy mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ are well defined. By definition, they are the morphisms of the category pro-H.

If $\boldsymbol{Y}$ is an arbitrary system, but $\boldsymbol{X}$ is a single space X, then the homotopy of homotopy mappings $\boldsymbol{f}\colon X \to \boldsymbol{Y}$ is also an equivalence relation and therefore, the homotopy classes $[\boldsymbol{f}]\colon X \to \boldsymbol{Y}$ and the set $H(X, \boldsymbol{Y})$ of all such classes are well defined. By the definition of shape morphisms, if $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ is a cofinite HPol-resolution of Y, there is a bijection $\Gamma_{\boldsymbol{q}}$ between the set $\mathrm{Sh}(X, Y)$ of shape morphisms $F\colon X \to Y$ and the set $\mathrm{H}(X, \boldsymbol{Y})$. As in the case of Theorem 1, one can extend the definition of $\Gamma_{\boldsymbol{q}}$ to the case when $\boldsymbol{q}$ is not cofinite.

The analogue of Theorem 2 assumes the following form.

Theorem 4. *Let $\boldsymbol{X}$ be a cofinite inverse system of compact polyhedra with limit $\boldsymbol{p}\colon X \to \boldsymbol{X}$ and let K be a simplicial complex with carrier $P = |K|$. Let $\boldsymbol{q}\colon X \times P \to \boldsymbol{Y}$ be the standard resolution of $X \times P$ associated with $\boldsymbol{p}$ and K and let $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$ be mappings of systems, induced by the canonical projections π_X, π_P. For every topological space Z, the statements (ES)$_Z$ for X,P and (EH)$_Z$ for $\boldsymbol{X}$, K and the statements (US)$_Z$ for X,P and (UH)$_Z$ for $\boldsymbol{X}$, K are equivalent, respectively.*

Hereby, (EH)$_Z$ and (UH)$_Z$ read as follows.

(EH)$_Z$ For every homotopy class of homotopy mappings $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$ and every homotopy class of mappings $[g]\colon Z \to P$, there exists a homotopy class of homotopy mappings $[\boldsymbol{h}]\colon Z \to \boldsymbol{Y}$ such that $[\boldsymbol{\pi}_X][\boldsymbol{h}] = [\boldsymbol{f}]$ and $[\boldsymbol{\pi}_P][\boldsymbol{h}] = [g]$.

(UH)$_Z$ If $[\boldsymbol{h}_i]\colon Z \to \boldsymbol{Y}$, $i = 1, 2$, are two homotopy classes of homotopy mappings such that $[\boldsymbol{\pi}_X][\boldsymbol{h}_1] = [\boldsymbol{\pi}_X][\boldsymbol{h}_2]$ and $[\boldsymbol{\pi}_P][\boldsymbol{h}_1] = [\boldsymbol{\pi}_P][\boldsymbol{h}_2]$, then $[\boldsymbol{h}_1] = [\boldsymbol{h}_2]$.

Remark 3. *There is an alternative definition of the category Sh, which does not require monotonicity of the index functions (see [13]). It yields the same notion of shape. Here we preferred to use the definition of [8], because it is closer to the definition of strong shape.*

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