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A new refinement of the Radon inequality

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Abstract. The aim of this paper is to give a new extension of the Radon inequality.
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1. Introduction

In this paper we study the following inequality due to J. Radon, first published in [12]. More precisely, for every real numbers $p > 0$, $x_k \geq 0$, $a_k > 0$, for $1 \leq k \leq n$, the following inequality holds true:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^{p+1}}{\left(\sum_{k=1}^{n} a_k\right)^p}, \quad p > 0. \tag{1}$$

For the proof and other comments, see [12, p. 1351], or the monograph [7, p. 31]. Inequality (1) is widely studied by many authors because of its intrinsec beauty and also because of its utility in practical applications, as in the case of obtaining more general inequalities involving manifolds [9, p. 692]. A particular case $p = 2$,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + ... + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + ... + x_n)^2}{a_1 + a_2 + ... + a_n},$$

is also well-known under the name of Bergström's inequality, e.g., [3, 4, 6]. It is equivalent to the Cauchy-Buniakovski-Schwarz inequality, while the Radon's inequality (1) is equivalent to the more general Hölder's inequality.

Generalizations of Bergström inequality and of the Cauchy-Buniakovski-Schwarz inequality are established by many authors, in particular by [1, 2, 5] and [11] and later in [8]. We give here more general versions of the results from [8] and these results are also generalizations of the Radon inequality and the Hölder inequality. In this sense, note that the Radon's inequality (1) follows from

$$\left[\sum_{k=1}^{n} \left(\frac{x_k}{a_k^{1/q}}\right)^{p+1}\right]^{1/(p+1)} \cdot \left[\sum_{k=1}^{n} \left(a_k^{1/q}\right)^q\right]^{1/q} \geq \sum_{k=1}^{n} x_k$$

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by raising to the $(p+1)$-th power, where $(p+1)^{-1}+q^{-1}=1$.

2. The results

It is established as the main result in [8] that:

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+...+\frac{x_n^2}{a_n}\geq\frac{(x_1+x_2+...+x_n)^2}{a_1+a_2+...+a_n}+\max_{1\leq i<j\leq n}\frac{(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)},\tag{2}$$

for $x_k\geq 0$, $a_k>0$, $1\leq k\leq n$, which is considered as an extension of the Bergström inequality. It was proved in [8], and before as a special case in [2, 5] and [11], that the sequence

$$d_n=\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+...+\frac{x_n^2}{a_n}-\frac{(x_1+x_2+...+x_n)^2}{a_1+a_2+...+a_n}\quad,\qquad n\geq 2,$$

is increasing and inequality (2) is $d_n\geq d_2$, where

$$d_2=\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}-\frac{(x_1+x_2)^2}{a_1+a_2}=\frac{(a_1x_2-a_2x_1)^2}{a_1a_2(a_1+a_2)}.$$

We complete here that by putting $x_k=\lambda_ka_k$, $1\leq k\leq n$, in inequality (2), a more convenient form is obtained, at least if we think that the involved maximum can be calculated more easily. More precisely,

$$a_1\lambda_1^2+a_1\lambda_2^2+...+a_n\lambda_n^2\geq\frac{(a_1\lambda_1+a_2\lambda_2+...+a_n\lambda_n)^2}{a_1+a_2+...+a_n}+\max_{1\leq i<j\leq n}\frac{a_ia_j(\lambda_i-\lambda_j)^2}{a_i+a_j}.$$

In a general case, first we establish the following

Theorem 1. *For every $p>0$, a, $b>0$, x, $y\geq 0$,*

$$\frac{x^{p+1}}{a^p}+\frac{y^{p+1}}{b^p}-\frac{(x+y)^{p+1}}{(a+b)^p}\geq\frac{p(x+y)^{p-1}(bx-ay)^2}{ab(a+b)^p}$$

holds.

Proof. The expression on the left-hand side of the inequality can be written as

$$\begin{aligned}&\frac{x^{p+1}}{a^p}+\frac{y^{p+1}}{b^p}-\frac{(x+y)^{p+1}}{(a+b)^p}\\&=\frac{x\,[bx(a+b)]^p+y\,[ay(a+b)]^p-(x+y)\,[ab(x+y)]^p}{a^pb^p(a+b)^p}\\&=\frac{x\left[(abx+b^2x)^p-(abx+aby)^p\right]-y\left[(abx+aby)^p-(a^2y+aby)^p\right]}{a^pb^p(a+b)^p}.\end{aligned}\tag{3}$$

Without loss of generality, we assume that $bx\geq ay$. As a simple consequence of the Lagrange theorem, for every $p>0$ and $0<u\leq v$, we have

$$pu^{p-1}(v-u) \le v^p - u^p \le pv^{p-1}(v-u).$$

Using this double inequality in the square brackets from (3), we obtain

$$\begin{aligned}
&\frac{x\left[(abx+b^2x)^p-(abx+aby)^p\right]-y\left[(abx+aby)^p-(a^2y+aby)^p\right]}{a^pb^p(a+b)^p}\\
&\ge \frac{xp(abx+aby)^{p-1}(b^2x-aby)-yp(abx+aby)^{p-1}(abx-a^2y)}{a^pb^p(a+b)^p}\\
&= \frac{p(x+y)^{p-1}(bx-ay)^2}{ab(a+b)^p}.
\end{aligned}$$

□

Now we are in the position to give the extension of inequality (2), which is stronger than the Radon's inequality.

Theorem 2. *For every $n \ge 2$, $p > 0$, $a_k > 0$, $x_k \ge 0$, $1 \le k \le n$, it holds:*

$$\begin{aligned}
\frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}+...+\frac{x_n^{p+1}}{a_n^p} \ge \frac{(x_1+x_2+...+x_n)^{p+1}}{(a_1+a_2+...+a_n)^p}\\
+p\cdot \max_{1\le i<j\le n}\frac{(x_i+x_j)^{p-1}(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)^p}.
\end{aligned} \tag{4}$$

Denoting $x_k = \lambda_k a_k$, $1 \le k \le n$, we have the equivalent form:

$$\begin{aligned}
a_1\lambda_1^{p+1}+a_2\lambda_2^{p+1}+...+a_n\lambda_n^{p+1} \ge \frac{(a_1\lambda_1+a_2\lambda_2+...+a_n\lambda_n)^{p+1}}{(a_1+a_2+...+a_n)^p}\\
+p\cdot \max_{1\le i<j\le n}\frac{a_ia_j(a_i\lambda_i+a_j\lambda_j)^{p-1}(\lambda_i-\lambda_j)^2}{(a_i+a_j)^p}.
\end{aligned} \tag{5}$$

Proof. Note first that inequality (2) is obtained as a particular case of our result (4) for $p = 1$. Let us define the sequence $(\delta_n)_{n\ge2}$ by the formula

$$\delta_n = \frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}+...+\frac{x_n^{p+1}}{a_n^p}-\frac{(x_1+x_2+...+x_n)^{p+1}}{(a_1+a_2+...+a_n)^p}.$$

We claim that the sequence $(\delta_n)_{n\ge2}$ is increasing. In this sense, we have:

$$\delta_{n+1}-\delta_n = \frac{x_{n+1}^{p+1}}{a_{n+1}^p}+\frac{(x_1+x_2+...+x_n)^{p+1}}{(a_1+a_2+...+a_n)^p}-\frac{(x_1+x_2+...+x_{n+1})^{p+1}}{(a_1+a_2+...+a_{n+1})^p} \ge 0,$$

because of the classical Radon inequality (1). Now, the monotonicity of the sequence $(\delta_n)_{n\ge2}$ implies that $\delta_n \ge \delta_2$, for every $n \ge 2$, and by Theorem 1,

$$\delta_2 = \frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}-\frac{(x_1+x_2)^{p+1}}{(a_1+a_2)^p} \ge \frac{p(x_1+x_2)^{p-1}(a_1x_2-a_2x_1)^2}{a_1a_2(a_1+a_2)^p}.$$

Now note that the obtained inequality

$$\delta_n \geq \frac{p(x_1+x_2)^{p-1}(a_1x_2-a_2x_1)^2}{a_1a_2(a_1+a_2)^p}$$

remains true even if we replace a_1, a_2, x_1, x_2 by any a_i, a_j, x_i, x_j, respectively, with $1 \leq i < j \leq n$, so the Theorem is proved. □

A different bound was proved in [1] for $p \geq 1$.

In the first part of this section we show how Theorem 2 can be used to establish generalizations of the inequalites from [8]. If we put $x_k = 1$ in (4), we obtain the following

Corollary 1. *For every* $n \geq 2$, $p > 0$, $a_k > 0$, $1 \leq k \leq n$, *it holds:*

$$\frac{1}{a_1^p}+\frac{1}{a_2^p}+...+\frac{1}{a_n^p}-\frac{n^{p+1}}{(a_1+a_2+...+a_n)^p} \geq p\cdot 2^{p-1}\cdot \max_{1\leq i<j\leq n}\frac{(a_i-a_j)^2}{a_ia_j(a_i+a_j)^p}.$$

By adding all the $n(n-1)/2$ inequalities of the form

$$\delta_n \geq \frac{p(x_i+x_j)^{p-1}(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)^p}, \qquad 1 \leq i < j \leq n,$$

we can state the next

Corollary 2. *For every* $n \geq 2$, $p > 0$, $x_k \geq 0$, $a_k > 0$, $1 \leq k \leq n$, *the following extension of the Radon's inequality holds:*

$$\begin{aligned}&\frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}+...+\frac{x_n^{p+1}}{a_n^p}-\frac{(x_1+x_2+...+x_n)^{p+1}}{(a_1+a_2+...+a_n)^p}\\ &\geq p\cdot \max_{1\leq i<j\leq n}\frac{(x_i+x_j)^{p-1}(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)^p}\\ &\geq \frac{2p}{n(n-1)}\sum_{1\leq i<j\leq n}\frac{(x_i+x_j)^{p-1}(a_ix_j-a_jx_i)^2}{a_ia_j(a_i+a_j)^p}.\end{aligned}$$

Another place to use our result is Nesbitt's inequality. For every $x_k > 0$, $1 \leq k \leq n$, with $x_1+x_2+...+x_n = s$, the following inequality holds:

$$\sum_{k=1}^n \frac{x_k}{s-x_k} \geq \frac{n}{n-1}. \tag{6}$$

It was strengthened in [8] in the following way:

$$\sum_{k=1}^n \frac{x_k}{s-x_k} \geq \frac{n}{n-1} + \max_{1\leq i<j\leq n}\frac{x_ix_j(x_i-x_j)^2}{(s-x_i)(s-x_j)\left[(x_i+x_j)s-(x_i^2+x_j^2)\right]}. \tag{7}$$

Next we give the following inequality which generalizes (6) and (7).

Corollary 3. *For every $n \geq 2$, $p > 0$, $x_k \geq 0$, $a_k > 0$, $1 \leq k \leq n$, with $s = x_1 + x_2 + ... + x_n$, the following extension of Nesbitt's inequality holds:*

$$\sum_{k=1}^{n} \frac{x_k}{(s-x_k)^p} \geq \frac{1}{s^{p-1}} \cdot \left(\frac{n}{n-1}\right)^p + p \cdot \max_{1 \leq i < j \leq n} \frac{x_i x_j (x_i + x_j)^{p-1} (x_i - x_j)^2}{(s-x_i)(s-x_j)\left[(x_i + x_j)s - (x_i^2 + x_j^2)\right]^p}.$$

Proof. We use Theorem 2. We have

$$\begin{aligned}\sum_{k=1}^{n} \frac{x_k}{(s-x_k)^p} &= \sum_{k=1}^{n} \frac{x_k^{p+1}}{(sx_k - x_k^2)^p} \\ &\geq \frac{(x_1 + x_2 + ... + x_n)^{p+1}}{[s^2 - (x_1^2 + x_2^2 + ... + x_n^2)]^p} \\ &\quad + p \cdot \max_{1 \leq i < j \leq n} \frac{x_i x_j (x_i + x_j)^{p-1} (x_i - x_j)^2}{(s-x_i)(s-x_j)\left[(x_i + x_j)s - (x_i^2 + x_j^2)\right]^p}.\end{aligned}$$

Then, the conclusion follows by the inequality $n(x_1^2 + x_2^2 + ... + x_n^2) \geq s^2$. □

Finally, remark that the reverse inequality of (1) is true in case $p \in \langle -1, 0 \rangle$,

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \leq \frac{\left(\sum_{k=1}^{n} x_k\right)^{p+1}}{\left(\sum_{k=1}^{n} a_k\right)^p}, \qquad p \in \langle -1, 0 \rangle.$$

(see for example [12]).

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