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Harmonic number sums in closed form

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Abstract. We extend some results of Euler related sums. Integral and closed form representation of sums with products of harmonic numbers and cubed binomial coefficients are developed in terms of Polygamma functions. The given representations are new.
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1. Introduction

The generalized harmonic numbers of order α are given by

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha}, \quad \text{for} \quad n = 1, 2, 3, \dots, \ \alpha = 1, 2, 3, \dots$$

and

$$H_n^{(1)} = H_n = \int_{t=0}^{1} \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log n \right) = -\psi(1) \approx 0.57721566490153286\dots$$

Recently Jung, Cho and Choi [10] evaluated Euler sums from integrals and obtained results like

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{2^n (n+1)^2} = \frac{1}{4}\zeta(3) - \frac{1}{3}(\ln 2)^3. \tag{1}$$

In this paper we expand (1) to obtain results of the form

$$\begin{aligned} \sum_{n\ge1} \frac{H_n^{(1)}}{n^5 (2n+1)^3} = & \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3) - 18\zeta(5) + 6\zeta(3)\zeta(2) - 15\zeta(4) \\ & -440\,\zeta(3) + 56\ln 2\,\zeta(3) + 240\ln 2\,\zeta(2) + 480(\ln 2)^2 \end{aligned} \tag{2}$$

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to add, in a small way, some results related to (1) and (3) and to extend the result of Cloitre, as reported in [19]. A remarkable recursion known to Euler [8] is

$$2\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^q}=(q+2)\,\zeta\,(q+1)-\sum_{r=1}^{q-2}\zeta\,(r+1)\,\zeta\,(q-r) \tag{3}$$

and there is also a recurrence formula

$$(2n+1)\,\zeta\,(2n)=2\sum_{r=1}^{n-1}\zeta\,(2r)\,\zeta\,(2n-2r)$$

which shows that in particular, for $n=2,\ 5\zeta\,(4)=2\,(\zeta\,(2))^2$ and more generally that $\zeta\,(4n)$ is a rational multiple of $(\zeta\,(2n))^2$. The Riemann zeta function

$$\zeta\,(z)=\sum_{r=1}^{\infty}\frac{1}{r^z},\ \ \operatorname{Re}(z)>1.$$

Here we are also interested in evaluating closed form and integral representations of Euler type sums containing both harmonic numbers, $H_n^{(\alpha)}$ and powers of binomial coefficients, two types of special numbers of enumerative combinatorics. There are many works investigating sums of both harmonic numbers and binomial coefficients, see for example [1, 2, 3, 4, 6, 9, 13, 14, 15, 16, 17], and references therein. Chu and Zheng [7] also obtained many other identities involving harmonic numbers and central binomial coefficients.

2. Identities

Theorem 1. *Let $a,b,c,d\geq 0$ be real positive numbers, $|t|<1$ and $j,k,l,m\geq 0$, then*

$$\begin{aligned}\sum_{n\geq 1}\frac{t^n\sum_{r=1}^{an}\frac{1}{r+j}}{n^5\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}} &= -abcd\int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1\frac{(1-x)^j\,(1-y)^k\,(1-z)^l\,(1-w)^m}{xyzw}\\ &\quad\times\ln(1-x)\ln\left(1-tx^ay^bz^cw^d\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\end{aligned} \tag{4}$$

Proof. Consider

$$\begin{aligned}&\sum_{n\geq 1}\frac{t^n}{n^5\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}\\ &\quad=abcd\sum_{n\geq 1}\frac{t^nn^4\Gamma\,(an)\,\Gamma\,(bn)\,\Gamma\,(cn)\,\Gamma\,(dn)\,\Gamma\,(j+1)\,\Gamma\,(k+1)\,\Gamma\,(l+1)\,\Gamma\,(m+1)}{n^5\Gamma\,(an+j+1)\,\Gamma\,(bn+k+1)\,\Gamma\,(cn+l+1)\,\Gamma\,(dn+m+1)}\end{aligned}$$

$$= abcd \sum_{n\geq 1} \frac{t^n \Gamma(bn)\Gamma(cn)\Gamma(dn)\Gamma(k+1)\Gamma(l+1)\Gamma(m+1)}{n\Gamma(bn+k+1)\Gamma(cn+l+1)\Gamma(dn+m+1)} B(an, j+1)$$

$$= abcd \sum_{n\geq 1} \frac{t^n}{n} B(bn, k+1) B(cn, l+1) B(dn, m+1) \int_0^1 x^{an-1}(1-x)^j \,\mathrm{d}x,$$

where the Beta function

$$B(s,z) = \int_0^1 w^{s-1}(1-w)^{z-1}\,\mathrm{d}w = \frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}$$

for $\mathrm{Re}(s) > 0$, $\mathrm{Re}(z) > 0$ and the Gamma function

$$\Gamma(z) = \int_0^\infty w^{z-1}e^{-w}\,\mathrm{d}w \quad \text{for } \mathrm{Re}(z) > 0.$$

Now differentiate with respect to the parameter j, then

$$\sum_{n\geq 1} \frac{t^n \sum_{r=1}^{an} \frac{1}{r+j}}{n^5 \binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}$$
$$= -abcd \int_0^1 \frac{(1-x)^j}{x} \ln(1-x)$$
$$\times \sum_{n\geq 1} \frac{x^{an}t}{n} \left[\int_0^1 y^{bn-1}(1-y)^k \int_0^1 z^{cn-1}(1-z)^l \int_0^1 w^{dn-1}(1-w)^m \,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}y\right]\mathrm{d}x$$
$$= -abcd \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(1-x)^j \ln(1-x)\cdot(1-y)^k(1-z)^l(1-w)^m}{xyzw}$$
$$\times \sum_{n\geq 1} \frac{\left(tx^ay^bz^cw^d\right)^n}{n} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w$$

by a subsequent allowable change of sum and integral, hence

$$\sum_{n\geq 1} \frac{t^n \sum_{r=1}^{an} \frac{1}{r+j}}{n^5 \binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}$$
$$= -abcd \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(1-x)^j(1-y)^k(1-z)^l(1-w)^m}{xyzw}$$
$$\times \ln(1-x)\ln\left(1-tx^ay^bz^cw^d\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w.$$

□

We now investigate three new corollaries that are a consequence of the Main Theorem 1.

Corollary 1. *For $b=c=d=a>0$, $t=1$, $j=0=l=m$ and $k\geq 1$ an integer, we have*

$$\begin{aligned}
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{n+k}{k}}\\
&=-a^4\int_0^1\int_0^1\int_0^1\int_0^1\frac{(1-y)^k\ln(1-x)\ln(1-(xyzw)^a)}{xyzw}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\\
&=\frac{7}{4}\zeta(6)-\frac{\zeta^2(3)}{2}-3aH_k^{(1)}\zeta(5)+aH_k^{(1)}\zeta(3)\zeta(2)+\frac{5a^2}{8}\left\{\left(H_k^{(1)}\right)^2+H_k^{(2)}\right\}\zeta(4)\\
&\quad-\frac{a^3}{3}\left\{\left(H_k^{(1)}\right)^3+3H_k^{(1)}H_k^{(2)}+2H_k^{(3)}\right\}\zeta(3)\\
&\quad+\frac{a^4}{24}\left\{\left(H_k^{(1)}\right)^4+6\left(H_k^{(1)}\right)^2H_k^{(2)}+3\left(H_k^{(2)}\right)^2+8H_k^{(1)}H_k^{(3)}+6H_k^{(4)}\right\}\zeta(2)\\
&\quad+\frac{a^4}{2}\sum_{r=1}^{k}\frac{(-1)^{r+1}}{r^4}\binom{k}{r}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2+H_{\frac{r}{a}-1}^{(2)}\right\}. \qquad (5)
\end{aligned}$$

Proof. Expand

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}}=\sum_{n\geq 1}\frac{k!H_n^{(1)}}{n^5\prod_{r=1}^{k}(an+r)}=\sum_{n\geq 1}\frac{k!H_n^{(1)}}{n^5}\sum_{r=1}^{k}\frac{A_r}{an+r},$$

where

$$A_s=\lim_{n\to\left(-\frac{s}{a}\right)}\left\{\frac{an+s}{\prod_{s=1}^{k}(an+s)}\right\}=(-1)^{s+1}\left(\frac{s}{k!}\binom{k}{s}\right),$$

hence

$$\begin{aligned}
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5}\sum_{r=1}^{k}\frac{(-1)^{r+1}r\binom{k}{r}}{an+r}\\
&=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5(an+r)}\\
&=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{rn^5}-\frac{a}{r^2n^4}+\frac{a^2}{r^3n^3}-\frac{a^3}{r^4n^2}+\frac{a^4}{r^4n(an+r)}\right]\\
&=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\left[\frac{7\zeta(6)}{4r}-\frac{\zeta^2(3)}{2r}-\frac{3a\zeta(5)}{r^2}+\frac{a\zeta(3)\zeta(2)}{r^2}\right.\\
&\quad\left.+\frac{5a^2\zeta(4)}{4r^3}-\frac{2a^3\zeta(3)}{r^4}+\frac{a^4\zeta(2)}{r^5}+\frac{a^4}{2r^5}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2+H_{\frac{r}{a}-1}^{(2)}\right\}\right]. \qquad (6)
\end{aligned}$$

Simplifying (6), we obtain (5). □

Remark 1. *The degenerate case, $k = 0$ gives the known result*

$$\sum_{n\geq 1} \frac{H_n^{(1)}}{n^5} = \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3).$$

Similarly,

$$\begin{aligned}\sum_{n\geq 1} \frac{H_n^{(1)}}{n^5 \binom{4n+3}{3}} &= \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3) - 22\zeta(5) + \frac{22}{3}\zeta(3)\zeta(2) + \frac{425}{9}\zeta(4) - \frac{9200}{27}\zeta(3) \\ &\quad -\frac{15616}{27}\zeta(2) + \frac{30976}{27}\pi\ln 2 + \frac{30368}{9}(\ln 2)^2 - \frac{247808}{81}G, \\ \sum_{n\geq 1} \frac{H_n^{(1)}}{n^5 \binom{\frac{n}{4}+2}{2}} &= \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3) - \frac{9}{8}\zeta(5) + \frac{3}{8}\zeta(3)\zeta(2) \\ &\quad +\frac{315}{2305}\zeta(4) - \frac{15}{256}\zeta(3) + \frac{31}{4096}\zeta(2) + \frac{12601699}{722534400},\end{aligned}$$

where G is Catalan's constant, defined by

$$G = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \approx 0.915965\ldots$$

Corollary 2. *For $d = c = b = a > 0$, $t = 1$, $j = 0 = m$, $l = k \geq 1$ integers, we have*

$$\sum_{n\geq 1} \frac{H_n^{(1)}}{n^5 \binom{an+k}{k}^2} \tag{7}$$

$$\begin{aligned}&= -a^4 \int_0^1\int_0^1\int_0^1\int_0^1 \frac{[(1-y)(1-z)]^k}{(1-xyzw)} \ln(1-x)\ln\left(1-(xyzw)^a\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \\ &= \frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3) + \sum_{r=1}^{k}\binom{k}{r}^2 \left[-\frac{6a}{r}(1 - rX(k,r))\zeta(5)\right. \\ &\quad + \frac{2a}{r}(1 - rX(k,r))\zeta(3)\zeta(2) + \frac{5a^2}{2r^2}(3 - 2rX(k,r))\zeta(4) \\ &\quad - \frac{a^3}{r^3}(9 - 4rX(k,r))\zeta(3) + \frac{a^3}{r^4}\left(5a - rH_{\frac{r}{a}-1}^{(1)} - 2raX(k,r)\right)\zeta(2) \\ &\quad + \frac{1}{r^4}\left[\frac{5a^4}{2}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2 + H_{\frac{r}{a}-1}^{(2)}\right\} + ra^3\left(H_{\frac{r}{a}-1}^{(1)}H_{\frac{r}{a}-1}^{(2)} + H_{\frac{r}{a}-1}^{(3)}\right)\right. \\ &\quad \left.\left. - a^4 r\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2 + H_{\frac{r}{a}-1}^{(2)}\right\}X(k,r)\right]\right],\end{aligned} \tag{8}$$

where

$$X(k,r) := H_{k-r}^{(1)} - H_{r-1}^{(1)}. \tag{9}$$

Proof. Expand

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^2}=\sum_{n\geq 1}\frac{H_n^{(1)}(k!)^2}{n^5\left((an+1)_{k+1}\right)^2}$$
$$=\sum_{n\geq 1}\frac{H_n^{(1)}(k!)^2}{n^5}\sum_{r=1}^{k}\left[\frac{A_r}{an+r}+\frac{B_r}{(an+r)^2}\right],$$

where

$$B_s=\lim_{n\to\left(-\frac{s}{a}\right)}\left\{\frac{(an+s)^2}{\prod_{s=1}^{k}(an+s)^2}\right\}=\left(\frac{s}{k!}\binom{k}{s}\right)^2$$

and

$$A_s=\lim_{n\to\left(-\frac{s}{a}\right)}\frac{\mathrm{d}}{\mathrm{d}n}\left\{\frac{(an+s)^2}{\prod_{s=1}^{k}(an+s)^2}\right\}=-2\left(\frac{s}{k!}\binom{k}{s}\right)^2X(k,s),$$

where $X(k,r)$ is given by (9). Now, by interchanging the order of summation, we have

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^2}=\sum_{r=1}^{k}(k!)^2A_r\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5(an+r)}+\sum_{r=1}^{k}(k!)^2B_r\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5(an+r)^2},\quad(10)$$

where

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5(an+r)^2}$$
$$=\sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{r^2n^5}-\frac{2a}{r^3n^4}+\frac{3a^2}{r^4n^3}-\frac{4a^3}{r^5n^2}-\frac{a^5}{r^5(an+r)^2}+\frac{5a^4}{r^5n(an+r)}\right]$$
$$=\frac{7\zeta(6)}{4r^2}-\frac{\zeta^2(3)}{2r^2}-\frac{6a\zeta(5)}{r^3}+\frac{2a\zeta(3)\zeta(2)}{r^3}$$
$$+\frac{15a^2\zeta(4)}{4r^4}-\frac{9a^3\zeta(3)}{r^5}+\frac{5a^4\zeta(2)}{r^6}-\frac{a^3\zeta(2)H_{\frac{r}{a}-1}^{(1)}}{r^5}$$
$$+\frac{5a^4}{2r^6}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2+H_{\frac{r}{a}-1}^{(2)}\right\}+\frac{a^3}{r^5}\left\{H_{\frac{r}{a}-1}^{(1)}H_{\frac{r}{a}-1}^{(2)}+H_{\frac{r}{a}-1}^{(3)}\right\}\quad(11)$$

and from (11) and the inner part of (6) into (10), we obtain

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^2}$$
$$=\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[\frac{7\zeta(6)}{4r^2}-\frac{\zeta^2(3)}{2r^2}-\frac{6a\zeta(5)}{r^3}+\frac{2a\zeta(3)\zeta(2)}{r^3}\right.$$

$$+\frac{15a^2\zeta(4)}{4r^4}-\frac{9a^3\zeta(3)}{r^5}+\frac{5a^4\zeta(2)}{r^6}-\frac{a^3\zeta(2)H^{(1)}_{\frac{r}{a}-1}}{r^5}$$
$$+\frac{5a^4}{2r^6}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^2+H^{(2)}_{\frac{r}{a}-1}\right\}+\frac{a^3}{r^5}\left\{H^{(1)}_{\frac{r}{a}-1}H^{(2)}_{\frac{r}{a}-1}+H^{(3)}_{\frac{r}{a}-1}\right\}\Bigg]$$
$$-2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[\frac{7\zeta(6)}{4r}-\frac{3a\zeta(5)}{r^2}+\frac{5a^2\zeta(4)}{4r^3}-\frac{\zeta^2(3)}{2r}\right.$$
$$\left.+\frac{a\zeta(3)\zeta(2)}{r}-\frac{2a^3\zeta(3)}{r^4}+\frac{a^4\zeta(2)}{r^5}+\frac{a^4}{2r^5}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^2+H^{(2)}_{\frac{r}{a}-1}\right\}\right]X(k,r).$$

Now we collect the zeta functions, and using the property that

$$\sum_{r=1}^{k}\binom{k}{r}^2[1-2rX(k,r)]=1$$

then (8) is confirmed. □

Remark 2. *Some examples are*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{4n+2}{2}^2}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)-36\zeta(5)+12\zeta(3)\zeta(2)+115\zeta(4)-8376\zeta(3)$$
$$-128\pi^3-2304\zeta(2)+4656\zeta(2)\ln 2+4608\pi\ln 2$$
$$+14112(\ln 2)^2-12288G+1024\pi G+6144G\ln 2,$$
$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{\frac{n}{3}+3}{3}^2}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)-\frac{11}{3}\zeta(5)+\frac{11}{9}\zeta(3)\zeta(2)$$
$$-\frac{970}{216}\zeta(4)-\frac{305}{216}\zeta(3)-\frac{2496877}{7348320}\zeta(2)+\frac{1108486911367}{648121824000}.$$

Corollary 3. *For $d=c=b=a>0$, $t=1$, $j=0$, $m=l=k\geq 1$ an integer, then*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^3}$$
$$=-a^4\int_0^1\int_0^1\int_0^1\int_0^1\frac{[(1-w)(1-z)(1-y)]^k}{xyzw}$$
$$\times\ln(1-x)\ln\left(1-(xyzw)^a\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \tag{12}$$
$$=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^2(3)+\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3\left[-a\left(\frac{9}{r}+18X(k,r)+3rY(k,r)\right)\zeta(5)\right.$$
$$+a\left(\frac{3}{r}-6X(k,r)+rY(k,r)\right)\zeta(3)\zeta(2)$$
$$+a^2\left(\frac{29}{4r^2}-\frac{45X(k,r)}{4r}+\frac{5}{4}Y(k,r)\right)\zeta(4)$$

$$+a^2\left(-\frac{25a}{r^3}-\frac{H^{(1)}_{\frac{r}{a}-1}}{r^2}+\frac{27aX(k,r)}{r^2}+\frac{2a}{r}Y(k,r)\right)\zeta(3)$$
$$+a^2\left(\frac{15a^2}{r^4}-\frac{5aH^{(1)}_{\frac{r}{a}-1}}{r^3}-\frac{H^{(2)}_{\frac{r}{a}-1}}{r^2}-3\left\{\frac{5a^2}{r^3}-\frac{aH^{(1)}_{\frac{r}{a}-1}}{r^2}\right\}X(k,r)\right.$$
$$\left.+\frac{a^2}{r^2}Y(k,r)\right)\zeta(2)+\frac{15a^4}{2r^4}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^2+H^{(2)}_{\frac{r}{a}-1}\right\}+\frac{5a^3}{r^3}\left\{H^{(1)}_{\frac{r}{a}-1}H^{(2)}_{\frac{r}{a}-1}+H^{(3)}_{\frac{r}{a}-1}\right\}$$
$$+\frac{a^2}{2r^2}\left\{\left(H^{(2)}_{\frac{r}{a}-1}\right)^2+2H^{(1)}_{\frac{r}{a}-1}H^{(3)}_{\frac{r}{a}-1}+3H^{(4)}_{\frac{r}{a}-1}\right\}$$
$$-3\left(\frac{5a^4}{2r^3}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^2+H^{(2)}_{\frac{r}{a}-1}\right\}+\frac{a^3}{r^2}\left\{H^{(1)}_{\frac{r}{a}-1}H^{(2)}_{\frac{r}{a}-1}+H^{(3)}_{\frac{r}{a}-1}\right\}\right)X(k,r)$$
$$\left.+\frac{a^4}{2r^2}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^2+H^{(2)}_{\frac{r}{a}-1}\right\}Y(k,r)\right], \tag{13}$$

where $X(k,r)$ is given by (9) and

$$Y(k,r):=\frac{3}{2}\left\{3X^2(k,r)+H^{(2)}_{k-r}+H^{(2)}_{r-1}\right\}. \tag{14}$$

Proof. Expand

$$\sum_{n\geq1}\frac{H^{(1)}_n}{n^5\binom{an+k}{k}^3}=\sum_{n\geq1}\frac{H^{(1)}_n(k!)^3}{n^5\prod_{r=1}^{k}(an+r)^3}$$
$$=\sum_{n\geq1}\frac{H^{(1)}_n(k!)^3}{n^5}\sum_{r=1}^{k}\left[\frac{A_r}{an+r}+\frac{B_r}{(an+r)^2}+\frac{C_r}{(an+r)^3}\right],$$

where

$$C_s=\lim_{n\to\left(-\frac{s}{a}\right)}\left\{\frac{(an+s)^3}{\prod_{s=1}^{k}(an+s)^3}\right\}=(-1)^{s+1}\left(\frac{s}{k!}\binom{k}{s}\right)^3,$$
$$B_s=\lim_{n\to\left(-\frac{s}{a}\right)}\frac{\mathrm{d}}{\mathrm{d}n}\left\{\frac{(an+s)^3}{\prod_{s=1}^{k}(an+s)^3}\right\}=3(-1)^s\left(\frac{s}{k!}\binom{k}{s}\right)^3X(k,s)$$

and

$$A_s=\frac{1}{2}\lim_{n\to\left(-\frac{s}{a}\right)}\frac{\mathrm{d}^2}{\mathrm{d}n^2}\left\{\frac{(an+s)^3}{\prod_{s=1}^{k}(an+s)^3}\right\}$$
$$=\frac{3}{2}(-1)^{s+1}\left(\frac{s}{k!}\binom{k}{s}\right)^3\left[3X^2(k,s)+H^{(2)}_{k-s}+H^{(2)}_{s-1}\right],$$

where $X(k,r)$ is given by (9).

Now we can write

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^3}=\sum_{r=1}^{k}A_r\,(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\,(an+r)}+\sum_{r=1}^{k}B_r\,(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\,(an+r)^2}+\sum_{r=1}^{k}C_r\,(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\,(an+r)^3},\tag{15}$$

where

$$\begin{aligned}\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\,(an+r)^3}&=\sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{r^3n^5}-\frac{3a}{r^4n^4}+\frac{6a^2}{r^5n^3}-\frac{10a^3}{r^6n^2}\right.\\&\left.-\frac{a^5}{r^5\,(an+r)^3}-\frac{5a^5}{r^6\,(an+r)^2}+\frac{15a^4}{r^6n\,(an+r)}\right]\\&=\frac{7\zeta(6)}{4r^3}-\frac{\zeta^2(3)}{2r^3}-\frac{9a\zeta(5)}{r^4}+\frac{3a\zeta(3)\,\zeta(2)}{r^4}+\frac{29a^2\zeta(4)}{4r^5}\\&-\frac{25a^3\zeta(3)}{r^6}-\frac{a^2\zeta(3)\,H_{\frac{r}{a}-1}^{(1)}}{r^5}+\frac{15a^4\zeta(2)}{r^7}\\&-\frac{5a^3\zeta(2)\,H_{\frac{r}{a}-1}^{(1)}}{r^6}-\frac{a^2\zeta(2)\,H_{\frac{r}{a}-1}^{(2)}}{r^5}+\frac{15a^4}{2r^7}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2+H_{\frac{r}{a}-1}^{(2)}\right\}\\&+\frac{5a^3}{r^6}\left\{H_{\frac{r}{a}-1}^{(1)}H_{\frac{r}{a}-1}^{(2)}+H_{\frac{r}{a}-1}^{(3)}\right\}\\&+\frac{a^2}{2r^5}\left\{\left(H_{\frac{r}{a}-1}^{(2)}\right)^2+2H_{\frac{r}{a}-1}^{(1)}H_{\frac{r}{a}-1}^{(3)}+3H_{\frac{r}{a}-1}^{(4)}\right\}\end{aligned}\tag{16}$$

Substituting (16), (11) and the inner part of (6) into (15) we get

$$\begin{aligned}\sum_{n\geq 1}\frac{H_n^{(1)}}{n^5\binom{an+k}{k}^3}&=\sum_{r=1}^{k}\frac{3}{2}(-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3\left[3X^2(k,r)+H_{k-r}^{(2)}+H_{r-1}^{(2)}\right]\\&\times\left[\frac{7\zeta(6)}{4r}-\frac{3a\zeta(5)}{r^2}+\frac{5a^2\zeta(4)}{4r^3}-\frac{\zeta^2(3)}{2r}\right.\\&\left.+\frac{a\zeta(3)\,\zeta(2)}{r}-\frac{2a^3\zeta(3)}{r^4}+\frac{a^4\zeta(2)}{r^5}+\frac{a^4}{2r^5}\left\{\left(H_{\frac{r}{a}-1}^{(1)}\right)^2+H_{\frac{r}{a}-1}^{(2)}\right\}\right]\\&+\sum_{r=1}^{k}3(-1)^r\left(r\binom{k}{r}\right)^3X(k,r)\left[\frac{7\zeta(6)}{4r^2}-\frac{\zeta^2(3)}{2r^2}-\frac{6a\zeta(5)}{r^3}\right.\\&+\frac{2a\zeta(3)\,\zeta(2)}{r^3}-\frac{15a^2\zeta(4)}{4r^4}-\frac{9a^3\zeta(3)}{r^5}+\frac{5a^4\zeta(2)}{r^6}\end{aligned}$$

$$
-\frac{a^{3}\zeta(2)H^{(1)}_{\frac{r}{a}-1}}{r^{5}}+\frac{5a^{4}}{2r^{6}}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^{2}+H^{(2)}_{\frac{r}{a}-1}\right\}+\frac{a^{3}}{r^{5}}\left\{H^{(1)}_{\frac{r}{a}-1}H^{(2)}_{\frac{r}{a}-1}+H^{(3)}_{\frac{r}{a}-1}\right\}\Bigg]
$$

$$
+\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[\frac{7\zeta(6)}{4r^{3}}-\frac{\zeta^{2}(3)}{2r^{3}}-\frac{9a\zeta(5)}{r^{4}}+\frac{3a\zeta(3)\zeta(2)}{r^{4}}\right.
$$

$$
+\frac{29a^{2}\zeta(4)}{4r^{5}}-\frac{25a^{3}\zeta(3)}{r^{6}}-\frac{a^{2}\zeta(3)H^{(1)}_{\frac{r}{a}-1}}{r^{5}}+\frac{15a^{4}\zeta(2)}{r^{7}}-\frac{5a^{3}\zeta(2)H^{(1)}_{\frac{r}{a}-1}}{r^{6}}
$$

$$
-\frac{a^{2}\zeta(2)H^{(2)}_{\frac{r}{a}-1}}{r^{5}}+\frac{a^{2}}{2r^{5}}\left\{\left(H^{(2)}_{\frac{r}{a}-1}\right)^{2}+2H^{(1)}_{\frac{r}{a}-1}H^{(3)}_{r-1}+3H^{(4)}_{\frac{r}{a}-1}\right\}
$$

$$
\left.+\frac{15a^{4}}{2r^{7}}\left\{\left(H^{(1)}_{\frac{r}{a}-1}\right)^{2}+H^{(2)}_{\frac{r}{a}-1}\right\}+\frac{5a^{3}}{r^{6}}\left\{H^{(1)}_{\frac{r}{a}-1}H^{(2)}_{\frac{r}{a}-1}+H^{(3)}_{\frac{r}{a}-1}\right\}\right],
$$

and after much algebraic manipulation with the aid of say, *Mathematica* [20], simplification and collection of the zeta functions, we arrive at (13) where we have used the fact that

$$
\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[1-3rX(k,r)+r^{2}Y(k,r)\right]=1,
$$

and $Y(k,r)$ is given by (14). □

Remark 3. *Some examples are*

$$
\sum_{n\geq 1}\frac{H^{(1)}_{n}}{n^{5}\binom{2n+4}{4}^{3}}=\frac{7}{4}\zeta(6)-\frac{1}{2}\zeta^{2}(3)-\frac{75}{2}\zeta(5)+\frac{25}{2}\zeta(3)\zeta(2)-\frac{145805}{48}\zeta(4)
$$

$$
-\frac{68053}{216}\zeta(3)+\frac{35840\ln 2}{9}\zeta(3)-\frac{2767747}{144}\zeta(2)-\frac{4096\ln 2}{9}\zeta(2)
$$

$$
+\frac{311296}{27}(\ln 2)^{2}-\frac{157696}{27}\ln 2+\frac{391657}{48}.
$$

For $a=2,\ k=1$ we obtain (2).

Remark 4. *In Corollaries 1, 2 and 3 we encounter harmonic numbers at possible rational values of the argument, of the form $H^{(\alpha)}_{\frac{r}{a}-1}$ where $r=1,2,3,\ldots,k$ and $k\in\mathbb{N}$. To evaluate $H^{(\alpha)}_{\frac{r}{a}-1}$ we have available a relation in terms of the Polygamma function $\psi^{(\alpha)}(z)$, for rational arguments z,*

$$
H^{(\alpha+1)}_{\frac{r}{a}-1}=\zeta(\alpha+1)+\frac{(-1)^{\alpha}}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{a}\right),
$$

where $\zeta(z)$ is the Riemann Zeta function. We also define

$$
H^{(1)}_{\frac{r}{a}-1}=\gamma+\psi\left(\frac{r}{a}\right),\ \text{and}\ H^{(\alpha)}_{0}=0.
$$

The evaluation of the Polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [12], (see also [11]), or Choi and Cvijović [5] in terms of the Polylogarithmic or other special functions.

In concluding we remark that the general results obtained by the evaluation of identities (5), (8) and (12) are an extension and generalization of the identities obtained by Sofo [18].

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