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New matching theorems and their applications

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Abstract. In this paper, we establish two matching theorems involving three maps. As applications, KKM type theorems, intersections theorems, analytic alternatives and minimax inequalities are obtained.

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1. Introduction

In [9], by using his own generalization of the classical Knaster-Kuratowski-Mazurkievicz theorem (simply, KKM theorem), Ky Fan obtains a matching theorem [9, Theorem 3] for open coverings of convex sets. From this, he also obtains another matching theorem [9, Theorem 4] for closed coverings of convex sets. Kim [15, Theorem 1] and Shih and Tan [26, Theorem 2] establish the open version of the KKM principle and prove that this is equivalent to Fan's matching theorem for closed covering. In [21, Theorem 5], Park obtains a matching theorem involving two maps, one of them being acyclic. Under many aspects Park's result generalizes Fan's matching theorem for closed coverings.

In this paper, we establish two matching theorems involving three maps. Applications of our matching theorems are given in Sections 3 (KKM type theorems), 4 (intersection theorems) and 5 (analytic alternatives, minimax inequalities and an existence theorem for the solutions of certain variational inequalities). Our results seem to be new, although they are closely related to some known results (see, for instance, [5, 7, 10, 14, 16, 17, 20, 21, 24, 28, 29]).

2. Preliminaries

A set-valued mapping (simply, a map) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y, that is a function with the values $T(x) \subseteq Y$ for each $x \in X$. As usual, the set $\{(x, y) \in X \times Y | y \in T(x)\}$ is called the *graph* of T. For any $A \subseteq X$, let $T(A) = \cup\{T(x) : x \in A\}$. To a map $T : X \multimap Y$ are associated two maps $T^- : Y \multimap X$, the *(lower) inverse* of T, defined by $T^-(y) = \{x \in X : y \in T(x)\}$ and the map $T^* : Y \multimap X$, the *dual* of T, defined by $T^*(y) = \{x \in X : y \notin T(x)\}$. Given

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two maps $T : X \multimap Y$ and $S : Y \multimap Z$, the composite $ST : X \multimap Z$ is defined by $(ST)(x) = S(T(x)) = \cup\{S(y) : y \in T(x)\}$.

If Y is a subset of a topological vector space, by $\overline{Y}$, co Y and $\overline{\text{co}}\, Y$, we denote the closure, convex hull and closed convex hull of Y, respectively. Given a map $T : X \multimap Y$, the maps co $T : X \multimap$ co Y, $\overline{\text{co}}\ T : X \multimap \overline{\text{co}}\ Y$ are defined by $(\text{co}\, T)(x) = \text{co}\ (T(x))$, $(\overline{\text{co}}\, T)(x) = \overline{\text{co}}\ (T(x))$ for all $x \in X$.

For topological spaces X and Y a map $T : X \multimap Y$ is said to be *upper semicontinuous* (abbreviated as u.s.c.) (respectively, *lower semicontinuous* (abbreviated as l.s.c.)) if for every closed subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; *continuous* if it is u.s.c. and l.s.c.; *compact* if $T(X)$ is contained in a compact subset of Y; *closed* if its graph is closed in $X \times Y$.

The following lemma collects two known facts [2].

Lemma 1. *Let X and Y be two topological spaces and $T : X \multimap Y$ a map. Then:*

(i) if T is closed and Y is compact, then T is u.s.c. with compact values;

(ii) if T is u.s.c. with compact values, then $T(K)$ is compact whenever $K \subseteq X$ is compact.

Suppose that X and Y are topological vector spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $T : X \multimap Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. According to Park ([23]), a class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathcal{C}$ of single-valued continuous functions;

(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and

(iii) for any polytope Δ, each $T \in \mathfrak{A}_c(X, X)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the admissible maps of Gòrniewicz, the permissible maps of Dzedzej and many others (for details see [22] and [25]).

Throughout this paper a real Hausdorff topological vector space is abbreviated as *t.v.s.* and a real locally convex Hausdorff topological vector space as *l.c.s.*

3. Matching theorems

The following lemma is Theorem 4 in [23].

Lemma 2. *Let X be a nonempty convex subset of a l.c.s. and $T \in \mathfrak{A}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.*

We also need the following

Lemma 3. *Let X be a topological space and Y a nonempty convex set in a l.c.s. E. Let $T : X \multimap Y$ be a nonempty valued u.s.c. map such that $\overline{co}\, T(x)$ is compact for each $x \in X$. Then the map $\overline{co}\, T$ is u.s.c.*

Proof. Let $\mathcal{V}$ be a basis of open convex symmetric neighborhoods of the origin of E. Let $x_0 \in X$ be arbitrarily fixed and let G be an open subset of Y such that $\overline{\text{co}}\, T(x_0) \subseteq G$. We prove that for some $V \in \mathcal{V}$

$$\overline{\text{co}}\, T(x_0) + V \subseteq G. \tag{1}$$

In the contrary case, for each $V \in \mathcal{V}$ there exists a point $y_V \in \overline{\text{co}}\, T(x_0)$ such that $(y_V + V) \cap (E \setminus G) \neq \emptyset$. Since V is symmetric, we infer that

$$y_V \in (E \setminus G) + V. \tag{2}$$

Since (y_V) is a net in the compact $\overline{\text{co}}\, T(x_0)$, we may suppose that (y_V) converges to a point $y_0 \in \overline{\text{co}}\, T(x_0)$. From (2) we get $y_0 \in \overline{E \setminus G} = E \setminus G$, hence $y_0 \in \overline{\text{co}}\, T(x_0) \cap (E \setminus G)$; this contradicts $\overline{\text{co}}\, T(x_0) \subseteq G$.

Let $V \in \mathcal{V}$ for which (1) holds and $U \in \mathcal{V}$ such that $\overline{U} \subseteq V$. Since T is u.s.c. and $T(x_0)$ is contained in the open set $\text{co}\, T(x_0) + U$, there exists a neighborhood W of x_0 such that $T(x) \subseteq \text{co}\, T(x_0) + U$, for all $x \in W$. The set $\text{co}\, T(x_0) + U$ is convex, hence for all $x \in W$ we have $\text{co}\, T(x) \subseteq \text{co}\, T(x_0) + U$, whence

$$\overline{\text{co}}\, T(x) \subseteq \overline{\text{co} T(x_0) + U} = \overline{\text{co}}\, T(x_0) + \overline{U} \subseteq \overline{\text{co}}\, T(x_0) + V \subseteq G.$$

Thus, the map $\overline{\text{co}}\, T$ is u.s.c. □

A particular case of Lemma 3 (when E is a Banach space) appears in [1].

A nonempty topological space is called *acyclic* if all its reduced Čech homology groups over the rationals are trivial. For nonempty sets in topological vector spaces, convex ⇒ star-shaped ⇒ contractible ⇒ acyclic ⇒ connected and not conversely. If X and Y are topological spaces, $T : X \multimap Y$ is called an *acyclic map* whenever T is u.s.c. with compact acyclic values. Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $T : X \multimap Y$. As we have already mentioned, in the second section of the paper, $\mathbb{V}$ is an example of map class $\mathfrak{A}$.

Theorem 1. *Let X be a nonempty convex set in a l.c.s. and Y a nonempty convex set in a t.v.s. Suppose that either X or Y is compact. Let $S \in \mathbb{V}_c(X, Y)$ and $T : X \multimap Y$, $R : Y \multimap X$ two maps such that:*

(i) $R(y) \subseteq T^-(y)$ for each $y \in Y$;

(ii) R is u.s.c. with nonempty values;

(iii) co $T^-(y)$ is compact for each $y \in Y$.

Then there exists a nonempty finite set $A \subset X$ such that $S(\text{co}\, A) \cap \bigcap_{x \in A} T(x) \neq \emptyset$.

Proof. For each $y \in Y$, by (i) and (iii), $\overline{\text{co}}\, R(y)$ is a compact subset of $\text{co}\, T^-(y)$. By Lemma 3 the map $\overline{\text{co}}\, R : Y \multimap X$ is u.s.c., hence $\overline{\text{co}}\, R$ is a Kakutani map. Then $(\overline{\text{co}}\, R)S \in \mathbb{V}_c(X, X)$. Lemma 2 is applicable to the map $(\overline{\text{co}}\, R)S$ as soon as we prove that this map is compact. Clearly, this happens if X is compact. When Y is compact, since the maps S and $\overline{\text{co}}\, R$ are u.s.c. with compact values, by Lemma 1 (ii), we infer successively that $S(X)$ and $\overline{\mathrm{co}}\, R(S(X))$ are compact, hence $(\overline{\mathrm{co}}\, R)S$ is a compact map.

By Lemma 2, there is a point $x_0 \in X$ such that $x_0 \in \overline{\mathrm{co}}\, R(S(x_0)) \subseteq (\mathrm{co}\, T^-)(S(x_0))$. This implies that there exist $y_0 \in S(x_0)$ and a finite set $A \subseteq T^-(y_0)$ such that $x_0 \in \mathrm{co}\, A$. But $A \subseteq T^-(y_0)$ is equivalent to $y_0 \in \bigcap_{x\in A} T(x)$. On the other hand, $y_0 \in S(x_0) \subseteq S(\mathrm{co}\, A)$, hence $y_0 \in S(\mathrm{co}\, A) \cap \bigcap_{x\in A} T(x)$. □

Theorem 2. *Let X be a nonempty metrizable convex set in a l.c.s. and Y a nonempty convex set in a t.v.s. Suppose that either X is compact and Y is paracompact or Y is compact. Let $S \in \mathbb{V}_c(X,Y)$ and $T : X \multimap Y$, $R : Y \multimap X$ satisfying conditions (i) and (iii) in Theorem 1 and*

(ii') R is l.s.c. with nonempty values.

Then there exists a nonempty finite set $A \subset X$ such that $S(co\, A) \cap \bigcap_{x\in A} T(x) \neq \emptyset$.

Proof. As in the previous proof, for each $y \in Y$, $\overline{\mathrm{co}}\, R(y)$ is a compact subset of $\mathrm{co}\, T^-(y)$. Since R is l.s.c., by Proposition 2.3 and 2.6 in [18], the map $\overline{\mathrm{co}}\, R$ is also l.s.c. and obviously, every $\overline{\mathrm{co}}\, R(y)$ is nonempty and complete. By Theorem 1.1 in [19], there exists an u.s.c map $R_1 : Y \multimap X$ with nonempty values such that $R_1(y) \subseteq \overline{\mathrm{co}}\, R(y) \subseteq \mathrm{co}\, T^-(y)$ for all $y \in Y$. Following the same argument as in the proof of Theorem 1 we find a fixed point x_0 for $(\overline{\mathrm{co}}\, R_1)S$ which is a fixed point for $(\mathrm{co}\, T^-)S$, too. From $x_0 \in (\mathrm{co}\, T^-(S(x_0))$ it follows that there exist $y_0 \in S(x_0)$ and a finite set $A \subseteq T^-(y_0)$ such that $y_0 \in S(\mathrm{co}\, A) \cap \bigcap_{x\in A} T(x)$. □

Let X be a convex set in a vector space and Y a nonempty set. If $S, F : X \multimap Y$ are two maps such that $S(\mathrm{co}\, A) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a *generalized* KKM map w.r.t. S [6].

From each of Theorems 1 and 2 we derive a KKM type theorem.

Theorem 3. *Let X be a nonempty convex set in a l.c.s. and Y a nonempty convex set in a t.v.s. Suppose that either X or Y is compact. Let $F : X \multimap Y$ be a map such that F^* is u.s.c. and co $F^*(y)$ is compact for all $y \in Y$. If there exists a map $S \in \mathbb{V}_c(X,Y)$ such that F is a generalized KKM map w.r.t. S, then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Proof. Define $T : X \multimap Y$ and $R : Y \multimap X$ by

$$T(x) = Y \setminus F(x) \text{ and } R(y) = F^*(y)$$

Suppose that $\bigcap_{x\in X} F(x) = \emptyset$. Then for each $y \in Y$ there exists $x \in X$ such that $y \notin F(x)$, that is $x \in R(y)$. Thus, R has nonempty values. It is easy to verify that T and R satisfy conditions (i), (ii) and (iii) of Theorem 1. By Theorem 1, there exists a nonempty finite set $A \subseteq X$ such that $S(\mathrm{co}\, A) \cap \bigcap_{x\in A} T(x) \neq \emptyset$, that is $S(\mathrm{co}\, A) \nsubseteq \bigcup_{x\in A} F(x) = F(A)$. This contradicts the fact that F is a generalized KKM map w.r.t. S. □

In a similar manner, using Theorem 2 instead of Theorem 1 as an argument we can prove

Theorem 4. *Let X be a nonempty metrizable convex set in a l.c.s. and Y a nonempty convex set in a t.v.s. Suppose that either X is compact and Y is paracompact or Y is compact. Let $F : X \multimap Y$ be a map such that F^* is l.s.c. and co $F^*(y)$ is compact for all $y \in Y$. If there exists a map $S \in \mathbb{V}_c(X, Y)$ such that F is a generalized KKM map w.r.t. S, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

4. Intersection theorems

Theorem 5. *Let X be a compact convex set in a l.c.s. and Y a compact convex set in a t.v.s. Let B, C, D, E be four subsets of $X \times Y$ such that:*

(i) C and E are closed in $X \times Y$;

(ii) $C \subseteq B$ and $E \subseteq D$;

(iii) for each $y \in Y$, $\{x \in X : (x, y) \notin B\}$ is convex;

(iv) for each $x \in X$, $\{y \in Y : (x, y) \in C\}$ is acyclic;

(v) for each $y \in Y$, co$\{x \in X : (x, y) \in D\}$ is compact;

(vi) for each $x \in X$, $\{y \in Y : (x, y) \in E\}$ is nonempty.

Then $B \cap D \neq \emptyset$.

Proof. Define the maps $S : X \multimap Y$, $T : X \multimap Y$, $R : Y \multimap X$ by

$$S(x) = \{y \in Y : (x, y) \in C\}, \;\; T(x) = \{y \in Y : (x, y) \in D\},$$

$$R(y) = \{x \in X : (x, y) \in E\}.$$

Since Y is compact and the graph of S is closed in $X \times Y$, S is u.s.c. and compact valued, by Lemma 1.(i). By (iv), S has acyclic values, hence $S \in \mathbb{V}(X, Y)$. Similarly, one can prove that R is u.s.c. For each $y \in Y$, $R(y) \subseteq T^-(y)$, since $E \subseteq D$. Furthermore, R has nonempty values (by (vi)) and for each $y \in Y$ co $T^-(y)$ is compact (by (v)). Then, Theorem 1 implies that there exist a finite set $A \subseteq X$, $x_0 \in$ co A and $y_0 \in Y$ such that $y_0 \in S(x_0) \cap \bigcap_{x \in A} T(x)$. Therefore $(x_0, y_0) \in C$ and $(x, y_0) \in D$ for all $x \in A$.

We show that for some $x_1 \in A$, $(x_1, y_0) \in B$ and thus $(x_1, y_0) \in B \cap D$. If $(x, y_0) \notin B$ for all $x \in A$, by (iii) we get $(x_0, y_0) \notin B$, and, since $C \subseteq B$, $(x_0, y_0) \notin C$; a contradiction. □

Theorem 6. *Let X be a nonempty metrizable convex set in a l.c.s. and Y a nonempty compact convex set in a t.v.s. Let B, C, D, E be four subsets of $X \times Y$ satisfying conditions (ii), (iii), (iv), (v) and (vi) of Theorem 5 and*

(vii) C is closed in $X \times Y$;

(viii) for each open subset U of X the set $\bigcup_{x \in U} \{y \in Y : (x, y) \in E\}$ is open in Y.

Then $B \cap D \neq \emptyset$.

Proof. Let the maps S, T, R be defined as in the proof of Theorem 5. We show that R is l.s.c. Let U be an open subset of X. Then

$$R^-(U) = \{y \in Y : R(y) \cap U \neq \emptyset\} = \bigcup_{x \in U} \{y \in Y : (x, y) \in E\}.$$

By (viii), $R^-(U)$ is open, hence R is l.s.c. Now the proof is similar to the previous proof using Theorem 2 instead of Theorem 1 as an argument. $\square$

Remark 1. *Condition (viii) of Theorem 6 is fulfilled if E is open in $X \times Y$. Indeed, in this case, for any $x \in X$ the function $q_x : Y \to X \times Y$, $q_x(y) = (x, y)$ is continuous and the set $\cup_{x \in U}\{y \in Y : (x, y) \in E\} = \bigcup_{x \in U} q_x^{-1}(E)$ is a union of open sets, hence it is open.*

5. Analytic alternatives, minimax inequalities, variational inequalities

Let X be a convex subset of a vector space, Y a set and $f, g : X \times Y \to \mathbb{R}$ two real functions. We say that f is *g-quasiconcave in x* [6] if for any nonempty finite subset A of X we have

$$f(u, y) \geq \min_{x \in A} g(x, y) \quad \text{for any } u \in \text{co } A \text{ and all } y \in Y.$$

It is clear that if $g(x, y) \leq f(x, y)$ for each $(x, y) \in X \times Y$ and for each $y \in Y$ one of the functions $x \to f(x, y)$, $x \to g(x, y)$ is quasiconcave, then f is g-quasiconcave in x.

Theorem 7. *Let X be a compact convex set in a l.c.s., Y a compact convex set in a t.v.s., $f, g, h : X \times Y \to \mathbb{R}$ three real functions and α, β, γ three real numbers such that $\alpha < \beta \leq \gamma$. Suppose that:*

(i) the sets $\{(x, y) \in X \times Y : f(x, y) \leq \alpha\}$ and $\{(x, y) \in X \times Y : h(x, y) \geq \gamma\}$ are closed in $X \times Y$;

(ii) $h(x, y) \leq g(x, y)$ for each $(x, y) \in X \times Y$;

(iii) for each $x \in X$ the set $\{y \in Y : f(x, y) \leq \alpha\}$ is acyclic or empty;

(iv) f is g-quasiconcave in x;

(v) for each $y \in Y$, the set co $\{x \in X : g(x, y) \geq \beta\}$ is compact or empty.

Then, at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $f(x_0, y) > \alpha$ for all $y \in Y$;

(b) There exists $y_0 \in Y$ such that $h(x, y_0) < \gamma$ for all $x \in X$.

Proof. Let $S : X \multimap Y$, $T : X \multimap Y$, $R : Y \multimap X$ be defined by

$$\begin{aligned} S(x) &= \{y \in Y : f(x,y) \le \alpha\}, \\ T(x) &= \{y \in Y : g(x,y) \ge \beta\}, \\ R(y) &= \{x \in X : h(x,y) \ge \gamma\}. \end{aligned}$$

Suppose that both assertions (a) and (b) are not true. This means that S and R have nonempty values. Since the graph of S is closed and Y is compact, S is u.s.c. with compact values. By (iii), $S \in \mathbb{V}(X,Y)$. Similarly, one obtains that R is u.s.c. For all $y \in Y$, $R(y) \subseteq T^-(y)$ (by (ii) and $\beta \le \gamma$) and $\operatorname{co} T^-(y)$ is compact (by (v)).

Applying Theorem 1 we find a finite set $A \subseteq X$, a point $x_0 \in \operatorname{co} A$ and a point $y_0 \in Y$ such that $y_0 \in S(x_0) \cap \bigcap_{x \in A} T(x)$. Thus

$$f(x_0,y_0) \le \alpha \text{ and } \min_{x \in A} g(x,y_0) \ge \beta. \tag{3}$$

By (3) and (iv) we obtain

$$\beta \le \min_{x \in A} g(x,y_0) \le f(x_0,y_0) \le \alpha,$$

which contradicts the hypothesis $\alpha < \beta$. □

From Theorem 7 we derive the following minimax inequality:

Corollary 1. *Let X be a compact convex set in a l.c.s., Y a compact convex set in a t.v.s. and $f,g,h : X \times Y \to \mathbb{R}$ three real functions satisfying:*

(i) f is l.s.c. and h is u.s.c. on $X \times Y$;

(ii) $h(x,y) \le g(x,y)$ for each $(x,y) \in X \times Y$;

(iii) for each $\alpha > \sup_{x \in X} \min_{y \in Y} f(x,y)$ and any $x \in X$, $\{y \in Y : f(x,y) \le \alpha\}$ is acyclic;

(iv) f is g-quasiconcave in x;

(v) for each $\beta < \inf_{y \in Y} \max_{x \in X} h(x,y)$ and any $y \in Y$, co $\{x \in X : g(x,y) \ge \beta\}$ is compact.

Then $\inf_{y \in Y} \max_{x \in X} h(x,y) \le \sup_{x \in X} \min_{y \in Y} f(x,y)$.

Proof. First, let us observe that if f is l.s.c. on $X \times Y$, then for each $x \in X$, $f(x,\cdot)$ is also an l.s.c. function of y on Y and therefore its minimum $\min_{y \in Y} f(x,y)$ on the compact set Y exists. Similarly, $\sup_{x \in X} h(x,y)$ can be replaced by $\max_{x \in X} h(x,y)$. Note also that condition (i) implies that f and h satisfy condition (i) of Theorem 7.

Suppose the conclusion were false and choose three real numbers α, β, γ such that

$$\sup_{x \in X} \min_{y \in Y} f(x,y) < \alpha < \beta \le \gamma < \inf_{y \in Y} \max_{x \in X} h(x,y).$$

We prove that neither assertion (a) nor assertion (b) of the conclusion of Theorem 7 can take place.

If (a) happens, then

$$\sup_{x\in X}\min_{y\in Y} f(x,y) \geq \min_{y\in Y} f(x_0,y) > \alpha; \text{ a contradiction.}$$

If (b) happens, then

$$\inf_{y\in Y}\max_{x\in X} h(x,y) \leq \max_{x\in X} h(x,y_0) < \gamma; \text{ a contradiction again.}$$

□

Further on, versions of Theorem 7 and Corollary 1 will be obtained using Theorem 2 instead of Theorem 1 as an argument.

For X, Y topological spaces a function $h : X \times Y \to \mathbb{R}$ is said to be *marginally l.s.c.* in y [3] if for every open subset U of X the function $y \to \sup_{x\in U} h(x,y)$ is l.s.c. on Y. Obviously, every function l.s.c. in y is marginally l.s.c. in y. The example given in [3, p. 249] shows that the converse is not true.

Theorem 8. *Let X be a nonempty metrizable convex set in a l.c.s., Y a nonempty compact convex set in a t.v.s., $f, g, h : X \times Y \to \mathbb{R}$ three real functions and α, β, γ three real numbers such that $\alpha < \beta \leq \gamma$. Suppose that f, g, h satisfy conditions (ii), (iii), (iv), (v) of Theorem 7 and:*

(vi) the set $\{(x,y) \in X \times Y : f(x,y) \leq \alpha\}$ is closed in $X \times Y$;

(vii) h is marginally l.s.c. in y.

Then, at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $f(x_0,y) > \alpha$ for all $y \in Y$.

(b) There exists $y_0 \in Y$ such that $h(x,y_0) < \gamma$ for all $x \in X$.

Proof. Suppose that both assertions (a) and (b) are not true and define the maps $S, T : X \multimap Y$, $R : Y \multimap X$ by

$$S(x) = \{y \in Y : f(x,y) \leq \alpha\}, \ \ T(x) = \{y \in Y : g(x,y) \geq \beta\},$$

$$R(y) = \{x \in X : h(x,y) > \gamma\}.$$

Let U be an open subset of X. Since

$$\{y \in Y : R(y) \cap U \neq \emptyset\} = \{y \in Y : \sup_{x\in U} h(x,y) > \gamma\},$$

by (vii), it follows that R is l.s.c. Other requirements of Theorem 2 are easily checked and, as in the proof of Theorem 7, Theorem 2 yields a contradiction. □

From Theorem 8 we immediately obtain the following

Corollary 2. *Let X be a nonempty metrizable convex set in a l.c.s., Y a nonempty compact convex set in a t.v.s. and $f, g, h : X \times Y \to \mathbb{R}$ three real functions. Suppose that f is l.s.c. on $X \times Y$ and that f, g, h satisfy conditions (ii)÷(v) in Corollary 1 and condition (vii) in Theorem 8. Then* $\inf_{y\in Y} \sup_{x\in X} h(x, y) \le \sup_{x\in X} \min_{y\in Y} f(x, y)$.

The origin of Corollaries 1 and 2 goes back to the famous Sion's minimax theorem [27].

We need now to recall Berge's maximum theorem [4].

Lemma 4. *Let X and Y be topological spaces, $f : X \times Y \to \mathbb{R}$ a continuous function and $F : X \multimap Y$ a continuous map with nonempty compact values. Then the map $G : X \multimap Y$ defined by $G(x) = \{y \in F(x) : f(x, y) = \max_{v\in F(x)} f(x, v)\}$ is u.s.c.*

Note also that if f and F are as in Lemma 4, then the map $S : X \multimap Y$ defined by $S(x) = \{y \in F(x) : f(x, y) = \min_{v\in F(x)} f(x, v)\}$ is u.s.c., too.

Theorem 9. *Let X be a nonempty compact convex set in a l.c.s. and Y a nonempty convex set in a t.v.s. Let $f, g : X \times Y \to \mathbb{R}$ be two continuous functions and $F : X \multimap Y$ a continuous map with nonempty compact values. Suppose that:*

(i) for each $x \in X$ the set $\{y \in F(x) : f(x, y) = \min_{v\in F(x)} f(x, v)\}$ is acyclic;

(ii) for each $y \in Y$ the set co$\{x \in X : g(x, y) = \max_{u\in X} g(u, y)\}$ is nonempty compact;

(iii) f is g-quasiconcave in x on $F(X)$.

Then

(a) there exists $(x_0, y_0) \in graphF$ such that

$$\max_{x\in X} g(x, y_0) \le \min_{y\in F(x_0)} f(x_0, y);$$

(b) the following minimax inequality holds:

$$\inf_{y\in Y} \max_{x\in X} g(x, y) \le \max_{x\in X} \min_{y\in F(x)} f(x, y).$$

Proof. Consider the maps $S, T : X \multimap Y$ defined by

$$S(x) = \{y \in F(x) : f(x, y) = \min_{v\in F(x)} f(x, v)\},$$
$$T(x) = \{y \in Y : g(x, y) = \max_{u\in X} g(u, y)\}.$$

By Lemma 4 the maps S and T^- are u.s.c. Since f is continuous and F is nonempty compact valued, for each $x \in X$, $S(x)$ is a nonempty compact subset of $F(x)$. By (i) the values of S are acyclic, hence $S \in \mathbb{V}(X, Y)$. Note also that for each $y \in Y$, co $T^-(y)$ is nonempty compact.

Applying Theorem 1 in the particular case $R = T^-$ we get a finite set $\{x_1, \ldots, x_n\} \subseteq X$ and two points $x_0 \in \text{co}\,\{x_1, \ldots, x_n\}$ and $y_0 \in Y$ such that $y_0 \in S(x_0) \cap \bigcap_{i=1}^n T(x_i)$.

Since $y_0 \in S(x_0)$, for each $x \in X$ we have

$$f(x_0, y_0) \leq f(x_0, y). \tag{4}$$

Since $y_0 \in \cap_{i=1}^n T(x_i)$, for each $x \in X$ we have

$$g(x_0, y_0) \leq \max_{u \in X} g(u, y_0) = g(x_1, y_0) = \cdots = g(x_n, y_0) = \min_{1 \leq i \leq n} g(x_i, y_0). \tag{5}$$

Then, for each $y \in F(x_0)$ and $x \in X$, by (4), (5) and (iii) we have

$$g(x, y_0) \leq \min_{1 \leq i \leq n} g(x_i, y_0) \leq f(x_0, y_0) \leq f(x_0, y).$$

Assertion (b) follows immediately from (a) since

$$\inf_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} g(x, y_0) \leq \min_{y \in F(x_0)} f(x_0, y) \leq \max_{x \in X} \min_{y \in F(x)} f(x, y).$$

□

The next two corollaries are particular cases of Theorem 9.

Corollary 3. *Let X be a compact convex set in a l.c.s., Y a compact convex set in a t.v.s., and $f, g : X \times Y \to \mathbb{R}$ two continuous functions such that:*

(i) for each $x \in X$ the set $\{y \in Y : f(x, y) = \min_{v \in Y} f(x, v)\}$ is acyclic;

(ii) for each $y \in Y$ the set co $\{x \in X : g(x, y) = \max_{u \in X} g(u, y)\}$ is nonempty compact;

(iii) f is g-quasiconcave in x.

Then

$$\min_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} \min_{y \in Y} f(x, y).$$

Proof. Take $F(x) = Y$ for all $x \in X$ and apply Theorem 9. □

The origin of the following corollary goes back to Ky Fan's minimax inequality [8].

Corollary 4. *Let X be a compact convex set in a l.c.s. and $f, g : X \times X \to \mathbb{R}$ two continuous functions such that:*

(i) for each $x \in X$ the set co $\{x \in X : g(x, y) = \max_{u \in X} g(u, y)\}$ is nonempty and compact;

(ii) f is g-quasiconcave in x.

Then

$$\min_{y \in X} \max_{x \in X} g(x, y) \leq \max_{x \in X} f(x, x).$$

Proof. Apply Theorem 9 with $X = Y$, $F(x) = \{x\}$. $\square$

Corollary 4 can be applied to the existence of solutions of certain variational inequalities:

Corollary 5. *Let X be a compact conex set in a l.c.s. and $p, q : X \times X \to \mathbb{R}$, $h : X \to \mathbb{R}$ continuous functions satisfying:*

(i) $\sum_{i=1}^{n} \lambda_i p(x_i, y) \leq q(\sum_{i=1}^{n} \lambda_i x_i, y)$ *for all* $\{x_1, \ldots, x_n\} \subset X$, $y \in Y$, $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$;

(ii) $q(x, x) \leq 0$ *for each* $x \in X$;

(iii) h *is quasiconvex;*

(iv) for each $y \in X$ *the set co* $\{x \in X : p(x, y) - h(x) = \max_{u \in X}[p(u, y) - h(u)]\}$ *is nonempty and compact.*

Then there exists $y_0 \in X$ such that

$$p(x, y_0) + h(y_0) \leq h(x) \text{ for all } x \in X.$$

Proof. Define the continuous functions $f, g : X \times X \to \mathbb{R}$ by

$$f(x, y) = q(x, y) + h(y) - h(x),$$
$$g(x, y) = p(x, y) + h(y) - h(x).$$

By (iv) it follows that condition (i) in Corollary 4 is fulfilled. We prove now that f is g-quasiconcave in x. Suppose that there exist a finite set $\{x_1, \ldots, x_n\} \subseteq X$, $x_0 \in \text{co}\, \{x_1, \ldots, x_n\}$ and $y_0 \in Y$ such that $f(x_0, y_0) < \min_{1 \leq i \leq n} g(x_i, y_0)$, that is

$$q(x_0, y_0) - h(x_0) < p(x_i, y_0) - h(x_i) \text{ for } 1 \leq i \leq n. \tag{6}$$

Suppose that $x_0 = \sum_{i=1}^{n} \lambda_i x_i$, with $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$. Multiplying (6) by λ_i and summing over i we find

$$q(x_0, y_0) - h(x_0) < \sum_{i=1}^{n} \lambda_i p(x_i, y_0) - \sum_{i=1}^{n} \lambda_i h(x_i),$$

and since the function h is quasiconvex we obtain

$$q(\sum_{i=1}^{n} \lambda_i x_i, y_0) - h(x_0) < \sum_{i=1}^{n} \lambda_i p(x_i, y_0) - h(x_0),$$

which contradicts (ii).

Then f and g satisfy the requirements of Corollary 4. Furthermore $f(x, x) = q(x, x) \leq 0$.

Therefore, the conclusion follows by Corollary 4. $\square$

Remark 2. *It is easily seen that condition (i) holds if $p(x, y) \leq q(x, y)$ for each $(x, y) \in X \times X$ and for every $y \in X$ one of the function f and g is concave in x.*

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