Math. Commun. **16**(2011), 383–398.

On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ II

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Received October 19, 2009; accepted November 29, 2010

Abstract. In the present paper, which is a natural continuation of the work done in [13], we determine the α-, β- and γ-duals of the sequence spaces ℓ_p^λ and ℓ_∞^λ of non-absolute type, where $1 \le p < \infty$. Further, we characterize some related matrix classes and deduce the characterizations of some other classes by means of a given basic lemma.
AMS subject classifications: 40C05, 40H05, 46A45

Key words: sequence spaces, BK-space, α-, β- and γ-duals, matrix mappings

1. Introduction

By w, we shall denote the space of all real or complex valued sequences, and any vector subspace of w is called a *sequence space*. For simplicity in notation, if $x \in w$, then we may write $x = (x_k)$ instead of $x = (x_k)_{k=0}^\infty$.

A sequence space is called an *FK-space* if it is a complete metrizable locally convex space (F-space) with the property that convergence implies coordinatewise convergence (K-space). A normable FK-space is called a *BK-space* (see [8, p.338] and [18, p.55]).

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively; which are BK-spaces with the same sup-norm defined by

$$\|x\|_{\ell_\infty} = \sup_k |x_k|.$$

Here, and in the sequel, the supremum $\sup_k$ is taken over all $k \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Also, by ℓ_p $(1 \le p < \infty)$, we denote the space of all sequences associated with p-absolutely convergent series; which is a BK-space with the usual ℓ_p-norm given by

$$\|x\|_{\ell_p} = \Big(\sum_k |x_k|^p\Big)^{1/p} \quad \text{for } 1 \le p < \infty,$$

where, here and in what follows, the summation without limits runs from 0 to ∞. Further, we shall write bs and cs for the spaces of all sequences associated with

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bounded and convergent series, respectively; which are BK-spaces with their natural norm [8, Example 7.3.2].

Let X and Y be sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a *matrix mapping* from X into Y if for every sequence $x = (x_k) \in X$, the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, the A-*transform* of x, exists and is in Y; where

$$A_n(x) = \sum_k a_{nk} x_k \tag{1}$$

for all $n \in \mathbb{N}$. By $(X : Y)$ we denote the class of all infinite matrices that map X into Y. Thus $A \in (X : Y)$ if and only if the series on the right-hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$.

The theory of FK-spaces is the most powerful tool in the characterizations of matrix mappings between sequence spaces, and the most important result was that matrix mappings between FK-spaces are continuous [18, Theorem 4.2.8]. We refer the reader to [16] for the characterizations of matrix mappings between the classical sequence spaces.

For an arbitrary sequence space X, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \big\{x \in w : \ Ax \in X\big\}, \tag{2}$$

which is a sequence space.

An infinite matrix $A = (a_{nk})$ is called a *triangle* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. The study of matrix domains of triangles has a special importance due to the various properties which they have. For example, if A is a triangle and X is a BK-space, then X_A is also a BK-space with the norm given by $\|x\|_{X_A} = \|Ax\|_X$ for all $x \in X_A$ (see [8, Theorem 8.1.4]).

The approach constructing a new sequence space by means of the matrix domain of a particular triangle has recently been employed by several authors in many research papers. For instance, they introduced the sequence spaces $(\ell_p)_{E^r} = e_p^r$ and $(\ell_\infty)_{E^r} = e_\infty^r$ in [3], $(\ell_p)_{A^r} = a_p^r$ and $(\ell_\infty)_{A^r} = a_\infty^r$ in [6], $(\ell_p)_\Delta = bv_p$ and $(\ell_\infty)_\Delta = bv_\infty$ in [7], $(c_0)_{R^t} = r_0^t$, $c_{R^t} = r_c^t$ and $(\ell_\infty)_{R^t} = r_\infty^t$ in [11], $(\ell_p)_{C_1} = X_p$ and $(\ell_\infty)_{C_1} = X_\infty$ in [14], and $(\ell_\infty)_{N_q}$ and c_{N_q} in [17]; where E^r, R^t, C_1 and N_q denote the matrices of Euler, Riesz, Cesàro and Nörlund means, respectively, the matrix A^r is defined in [6], Δ denotes the band matrix defining the difference operator and $1 \leq p < \infty$. In [13], following [3, 6, 7, 11, 14] and [17], the sequence spaces ℓ_p^λ and ℓ_∞^λ of non-absolute type have been introduced, some related results and inclusion relations have been given and the Schauder basis for the space ℓ_p^λ has been constructed, where $1 \leq p < \infty$. In the present paper, we determine the α-, β- and γ-duals of the spaces ℓ_p^λ and ℓ_∞^λ. Further, we characterize some related matrix classes and derive the characterizations of some other classes by means of a given basic lemma.

2. The sequence spaces ℓ_p^λ and ℓ_∞^λ

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. Then, by using the convention that any term with a negative subscript is equal to zero, we define the infinite matrix $\Lambda = (\lambda_{nk})$, for all $n, k \in \mathbb{N}$, by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Recently, the sequence spaces ℓ_p^λ and ℓ_∞^λ of non-absolute type have been introduced in [13] as the spaces of all sequences whose Λ-transforms are in the spaces ℓ_p and ℓ_∞, respectively; where $1 \le p < \infty$, that is

$$\ell_p^\lambda = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}$$

and

$$\ell_\infty^\lambda = \left\{ x = (x_k) \in w : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}.$$

With the notation of (2), we can redefine the space ℓ_p^λ $(1 \le p \le \infty)$ as the matrix domain of the triangle Λ in the space ℓ_p, that is $\ell_p^\lambda = (\ell_p)_\Lambda$ for $1 \le p \le \infty$. Then, it is obvious that ℓ_p^λ $(1 \le p \le \infty)$ is a BK-space with the norm $\|x\|_{\ell_p^\lambda} = \|\Lambda(x)\|_{\ell_p}$, where $\Lambda(x)$ denotes the Λ-transform of $x \in \ell_p^\lambda$.

Also, it has been shown that the linear operator defined from ℓ_p^λ to ℓ_p by $x \mapsto \Lambda(x)$ is bijective and norm preserving, which yields the fact that the spaces ℓ_p^λ and ℓ_p are norm isomorphic for $1 \le p \le \infty$.

Further, we may note that the spaces ℓ_p^λ and ℓ_∞^λ are reduced, in the special case $\lambda_k = k + 1$, to the Cesàro sequence spaces X_p and X_∞, which are defined in [14] as the spaces of all sequences whose C_1-transforms are in the spaces ℓ_p and ℓ_∞, respectively; where $1 \le p < \infty$.

Moreover, let us recall that the sequence spaces $\mathrm{ces}[p, q]$ and $\mathrm{ces}[\infty, q]$ are defined in [10] (see also [9, Example 7.4]) as follows:

$$\mathrm{ces}[p, q] = \left\{ x = (x_k) \in w : \sum_n \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k| \right)^p < \infty \right\}$$

and

$$\mathrm{ces}[\infty, q] = \left\{ x = (x_k) \in w : \sup_n \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k| \right) < \infty \right\},$$

where $0 < p < \infty$ and $q = (q_k)_{k=0}^\infty$ is a sequence of positive reals with $Q_n = \sum_{k=0}^n q_k$ for all $n \in \mathbb{N}$. Then, by taking $q_k = \lambda_k - \lambda_{k-1}$ for all $k \in \mathbb{N}$, it can easily be seen that the inclusions $\mathrm{ces}[p, q] \subset \ell_p^\lambda$ and $\mathrm{ces}[\infty, q] \subset \ell_\infty^\lambda$ strictly hold, where $1 \le p < \infty$.

Furthermore, the sequence spaces $c(a, p, q)$ and $c(a, p, \infty)$ have been introduced in [9] as follows:

$$c(a, p, q) = \left\{ x = (x_k) \in w : \sum_n \left[a_n \left(\sum_{k=0}^{n} |x_k|^p \right)^{1/p} \right]^q < \infty \right\}$$

and

$$c(a,p,\infty)=\left\{x=(x_k)\in w:\ \sup_n\left[a_n\left(\sum_{k=0}^{n}|x_k|^p\right)^{1/p}\right]<\infty\right\},$$

where $a=(a_n)_{n=0}^{\infty}$ is a sequence of non-negative reals and $0<p,q<\infty$.

On the other hand, let $1<p<\infty$ and $n\in\mathbb{N}$. Then, it follows by applying the Hölder's inequality that

$$\begin{aligned}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k\right| &\le \sum_{k=0}^{n}\Big(\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}\Big)|x_k| \\ &\le \left[\sum_{k=0}^{n}\Big(\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}\Big)|x_k|^p\right]^{1/p}\left[\sum_{k=0}^{n}\Big(\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}\Big)\right]^{(p-1)/p} \\ &= \left[\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|^p\right]^{1/p}\end{aligned}$$

which is also true for $p=1$. Therefore, by taking[‡] $a_n=[\max_{0\le k\le n}(\lambda_k-\lambda_{k-1})/\lambda_n]^{1/p}$ for all $n\in\mathbb{N}$, we obtain that

$$\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k\right|\le a_n\left(\sum_{k=0}^{n}|x_k|^p\right)^{1/p}\quad\text{for }1\le p<\infty$$

which implies both inclusions $c(a,p,q)\subset\ell_q^{\lambda}$ and $c(a,p,\infty)\subset\ell_{\infty}^{\lambda}$, where $1\le p,q<\infty$.

Finally, for any sequence $x=(x_k)\in w$, we define the *associated sequence* $y=(y_k)$, which will frequently be used, as the Λ-transform of x, that is

$$y_k=\sum_{j=0}^{k}\Big(\frac{\lambda_j-\lambda_{j-1}}{\lambda_k}\Big)x_j \tag{3}$$

and hence

$$x_k=\sum_{j=k-1}^{k}(-1)^{k-j}\frac{\lambda_j}{\lambda_k-\lambda_{k-1}}\,y_j \tag{4}$$

for all $k\in\mathbb{N}$.

Remark 1. *We shall assume, throughout the remaining part of this paper, that the sequences x and y are connected by relation* (3), *that is $y=\Lambda(x)$ and hence $x\in\ell_p^{\lambda}$ if and only if $y\in\ell_p$, where $1\le p\le\infty$. Also, we shall assume that q is the conjugate number of p for $1\le p\le\infty$, that is $q=\infty$ for $p=1$, $q=p/(p-1)$ for $1<p<\infty$, and $q=1$ for $p=\infty$. Further, we shall write $\mathcal{F}$ for the collection of all nonempty and finite subsets of* $\mathbb{N}$.

[‡]In the special case $\Delta\lambda=(\lambda_k-\lambda_{k-1})_{k=0}^{\infty}\in\ell_{\infty}$, we may replace the term $\max_{0\le k\le n}(\lambda_k-\lambda_{k-1})$ by $\sup_k(\lambda_k-\lambda_{k-1})$.

3. α-, β- and γ-duals of the spaces ℓ_p^λ and ℓ_∞^λ

For arbitrary sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y)=\Big\{a=(a_k)\in w:\ ax=(a_kx_k)\in Y \text{ for all } x=(x_k)\in X\Big\} \tag{5}$$

is called the *multiplier space* of X and Y.

One can easily observe for a sequence space Z with $Y\subset Z\subset X$ that the inclusions $M(X,Y)\subset M(X,Z)$ and $M(X,Y)\subset M(Z,Y)$ hold, respectively.

With the notation of (5), α-, β- and γ-*duals* of a sequence space X, which are respectively denoted by X^α, X^β and X^γ, are defined by

$$X^\alpha=M(X,\ell_1),\quad X^\beta=M(X,cs)\quad\text{and}\quad X^\gamma=M(X,bs).$$

It is obvious that $X^\alpha\subset X^\beta\subset X^\gamma$. Also, it can easily be seen that the inclusions $X^\alpha\subset Y^\alpha$, $X^\beta\subset Y^\beta$ and $X^\gamma\subset Y^\gamma$ hold whenever $Y\subset X$. We refer the reader to [8, pp.341–369] and [18, pp.105–111] for further study concerning α-, β- and γ-duals of some sequence spaces.

Now, we may begin with quoting the following lemmas (see [16, pp.2–9]) which are needed for proving Theorems 1–3, below.

Lemma 1. *$A\in(\ell_p:\ell_1)$ if and only if*

(i) For $1<p\le\infty$,

$$\sup_{F\in\mathcal{F}}\sum_k\Big|\sum_{n\in F}a_{nk}\Big|^q<\infty. \tag{6}$$

(ii) For $p=1$,

$$\sup_k\sum_n|a_{nk}|<\infty. \tag{7}$$

Lemma 2. *$A\in(\ell_p:c)$ if and only if*

(i) For $1<p<\infty$,

$$\lim_n a_{nk} \text{ exists for every } k\in\mathbb{N}, \tag{8}$$

$$\sup_n\sum_k|a_{nk}|^q<\infty. \tag{9}$$

(ii) For $p=1$, (8) holds and

$$\sup_{n,k}|a_{nk}|<\infty. \tag{10}$$

(iii) For $p=\infty$, (8) holds and

$$\sup_n\sum_k|a_{nk}|<\infty, \tag{11}$$

$$\lim_n\sum_k\big|a_{nk}-\lim_n a_{nk}\big|=0. \tag{12}$$

Lemma 3. *$A \in (\ell_p : \ell_\infty)$ if and only if*

(i) For $1 < p \leq \infty$, (9) holds.

(ii) For $p = 1$, (10) holds.

Now, we prove the following results determining the α-, β- and γ-duals of the spaces ℓ_p^λ for $1 \leq p \leq \infty$. In proving Theorems 1 and 2, we apply the technique used in [7] and [1] for the spaces of single and double sequences, respectively. This technique has also been used in [2]–[6].

Theorem 1. *Define the sets d_q^λ and d_∞^λ as follows:*

$$d_q^\lambda = \Big\{a = (a_k) \in w : \sum_k \Big|\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\Big|^q < \infty\Big\}$$

and

$$d_\infty^\lambda = \Big\{a = (a_k) \in w : \sup_k \Big|\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\Big| < \infty\Big\}.$$

Then $(\ell_1^\lambda)^\alpha = d_\infty^\lambda$ and $(\ell_p^\lambda)^\alpha = d_q^\lambda$, where $1 < p \leq \infty$.

Proof. Let $a = (a_k) \in w$ and $1 < p \leq \infty$. Then, by using (3) and (4), we immediately derive for every $n \in \mathbb{N}$ that

$$a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}} a_n y_k = B_n(y), \tag{13}$$

where the matrix $B = (b_{nk}^\lambda)$ is defined for all $n, k \in \mathbb{N}$ by

$$b_{nk}^\lambda = \begin{cases} (-1)^{n-k} \dfrac{\lambda_k}{\lambda_n - \lambda_{n-1}} a_n & \text{if } n-1 \leq k \leq n, \\ 0 & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

Thus, we observe by (13) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell_p^\lambda$ if and only if $By \in \ell_1$ whenever $y = (y_k) \in \ell_p$. This means that $a = (a_k) \in (\ell_p^\lambda)^\alpha$ if and only if $B \in (\ell_p : \ell_1)$. We therefore obtain by Lemma 1 with B instead of A that $a \in (\ell_p^\lambda)^\alpha$ if and only if

$$\sup_{F \in \mathcal{F}} \sum_k \Big|\sum_{n \in F} b_{nk}^\lambda\Big|^q < \infty. \tag{14}$$

On the other hand, we have for any $F \in \mathcal{F}$ that

$$\sum_{n \in F} b_{nk}^\lambda = \begin{cases} 0 & \text{if } k \notin F \text{ and } k+1 \notin F, \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k & \text{if } k \in F \text{ and } k+1 \notin F, \\ \dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k} a_{k+1} & \text{if } k \notin F \text{ and } k+1 \in F, \\ \Big(\dfrac{a_k}{\lambda_k - \lambda_{k-1}} - \dfrac{a_{k+1}}{\lambda_{k+1} - \lambda_k}\Big)\lambda_k & \text{if } k \in F \text{ and } k+1 \in F. \end{cases}$$

Hence, we deduce that (14) holds if and only if

$$\sum_k \Big|\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_k\Big|^q < \infty$$

which leads us to the consequence that $(\ell_p^\lambda)^\alpha = d_q^\lambda$, where $1 < p \le \infty$.

Similarly, we obtain from (13) that $a = (a_k) \in (\ell_1^\lambda)^\alpha$ if and only if $B \in (\ell_1 : \ell_1)$ which can equivalently be written as

$$\sup_k \sum_n \big|b_{nk}^\lambda\big| < \infty \tag{15}$$

by (7) of Lemma 1. Further, we have for every $k \in \mathbb{N}$ that

$$\sum_n \big|b_{nk}^\lambda\big| = \sum_{n=k}^{k+1}\Big|\frac{\lambda_k}{\lambda_n-\lambda_{n-1}}a_n\Big|.$$

Thus, we conclude that (15) holds if and only if

$$\sup_k \Big|\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_k\Big| < \infty$$

which shows that $(\ell_1^\lambda)^\alpha = d_\infty^\lambda$ and this completes the proof. □

Remark 2. *We may note that if* $\liminf \lambda_{k+1}/\lambda_k > 1$*, then there is a constant* $b > 1$ *such that* $1 \le \lambda_k/(\lambda_k - \lambda_{k-1}) \le b$ *for all* $k \in \mathbb{N}$*. This yields that* $d_q^\lambda = \ell_q$ *and* $d_\infty^\lambda = \ell_\infty$*, i.e.,* $(\ell_p^\lambda)^\alpha = \ell_q$ *for* $1 \le p \le \infty$ *which is compatible with the fact that* $\ell_p^\lambda = \ell_p$ *in this particular case (see [13, Corollary 4.19]).*

Theorem 2. *Define the sets* e_q^λ *and* e_0^λ *by*

$$e_q^\lambda = \Big\{a = (a_k) \in w : \sum_k \Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k-\lambda_{k-1}}\Big)\lambda_k\Big|^q < \infty\Big\}$$

and

$$e_0^\lambda = \Big\{a = (a_k) \in w : \lim_k \frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_k = 0\Big\},$$

where

$$\bar{\Delta}\Big(\frac{a_k}{\lambda_k-\lambda_{k-1}}\Big) = \frac{a_k}{\lambda_k-\lambda_{k-1}} - \frac{a_{k+1}}{\lambda_{k+1}-\lambda_k} \quad \textit{for all } k \in \mathbb{N}.$$

Then $(\ell_1^\lambda)^\beta = d_\infty^\lambda$*,* $(\ell_p^\lambda)^\beta = d_\infty^\lambda \cap e_q^\lambda$ *and* $(\ell_\infty^\lambda)^\beta = e_0^\lambda \cap e_1^\lambda$*, where* $1 < p < \infty$*.*

Proof. Let us consider the equation

$$\begin{aligned}\sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \Big[\sum_{j=k-1}^k (-1)^{k-j}\frac{\lambda_j}{\lambda_k-\lambda_{k-1}}y_j\Big]a_k \\ &= \sum_{k=0}^{n-1}\bar{\Delta}\Big(\frac{a_k}{\lambda_k-\lambda_{k-1}}\Big)\lambda_k y_k + \frac{\lambda_n}{\lambda_n-\lambda_{n-1}}a_n y_n = T_n(y),\end{aligned} \tag{16}$$

where $n \in \mathbb{N}$ and $T = (t_{nk}^{\lambda})$ is the matrix defined for $n, k \in \mathbb{N}$ by

$$t_{nk}^{\lambda} = \begin{cases} \bar{\Delta}\Big(\dfrac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k & \text{if } k < n, \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

Then, it is clear that the columns of the matrix T are in the space c, since

$$\lim_n t_{nk}^{\lambda} = \bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k \tag{17}$$

for all $k \in \mathbb{N}$. Thus, we deduce from (16) with Lemma 2 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell_p^{\lambda}$ if and only if $Ty \in c$ whenever $y = (y_k) \in \ell_p$. This yields that $a = (a_k) \in (\ell_p^{\lambda})^{\beta}$ if and only if $T \in (\ell_p : c)$, where $1 \le p \le \infty$.

Let us firstly begin with the case $1 < p < \infty$. Then, we derive from (9) that

$$\sum_k \Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big|^q < \infty$$

and

$$\sup_n \Big|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\Big| < \infty. \tag{18}$$

This leads us to the consequence that $(\ell_p^{\lambda})^{\beta} = d_{\infty}^{\lambda} \cap e_q^{\lambda}$.

Similarly, for $p = 1$, we deduce from (10) that (18) holds and

$$\sup_k \Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big| < \infty. \tag{19}$$

But it is obvious that condition (19) is redundant, since it is obtained from (18). Hence, we conclude that $(\ell_1^{\lambda})^{\beta} = d_{\infty}^{\lambda}$.

Finally, if $p = \infty$, then we deduce from (11) that (18) holds and

$$\sum_k \Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big| < \infty. \tag{20}$$

On the other hand, for every $n \in \mathbb{N}$, we have by (17) that

$$\begin{aligned} \sum_k \big|t_{nk}^{\lambda} - \lim_n t_{nk}^{\lambda}\big| &= \sum_{k=n}^{\infty} \Big|t_{nk}^{\lambda} - \bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big| \\ &= \Big|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n - \bar{\Delta}\Big(\frac{a_n}{\lambda_n - \lambda_{n-1}}\Big)\lambda_n\Big| + \sum_{k=n+1}^{\infty} \Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big|. \end{aligned}$$

This yields, by passing to the limits as $n \to \infty$ and using (20), that

$$\lim_n \sum_k \big|t_{nk}^{\lambda} - \lim_n t_{nk}^{\lambda}\big| = \lim_n \Big|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\Big|.$$

Therefore, we obtain by (12) that

$$\lim_n \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n = 0.$$

Thus, the weaker condition (18) is redundant. Consequently, we deduce that $(\ell_\infty^\lambda)^\beta = e_0^\lambda \cap e_1^\lambda$. This concludes the proof. □

Finally, we end this section with the following theorem which determines the γ-dual of the space ℓ_p^λ, where $1 \leq p \leq \infty$.

Theorem 3. *Let* $1 < p \leq \infty$. *Then* $(\ell_1^\lambda)^\gamma = d_\infty^\lambda$ *and* $(\ell_p^\lambda)^\gamma = d_\infty^\lambda \cap e_q^\lambda$.

Proof. This can be proved similarly to the proof of Theorem 2 with Lemma 3 instead of Lemma 2. □

4. Certain matrix mappings on the spaces ℓ_p^λ and ℓ_∞^λ

In the present section, we essentially characterize the matrix classes $(\ell_p^\lambda : \ell_\infty)$, $(\ell_p^\lambda : c)$, $(\ell_p^\lambda : c_0)$, $(\ell_p^\lambda : \ell_1)$, $(\ell_1^\lambda : \ell_p)$ and $(\ell_\infty^\lambda : \ell_p)$, where $1 \leq p \leq \infty$. Further, we deduce the characterizations of some other classes by means of a given basic lemma.

For any infinite matrix $A = (a_{nk})$, we shall write for brevity that

$$\tilde{a}_{nk} = \bar{\Delta}\Big(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k = \Big(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} - \frac{a_{n,k+1}}{\lambda_{k+1} - \lambda_k}\Big)\lambda_k \quad (n, k \in \mathbb{N}).$$

The following lemmas (see [16, pp.4–9]) will be needed in the proofs of our main results on matrix transformations.

Lemma 4. $A \in (\ell_p : c_0)$ *if and only if*

(i) For $p = 1$,

$$\lim_n a_{nk} = 0 \text{ for all } k \in \mathbb{N}, \qquad (21)$$
$$\sup_{n,k} |a_{nk}| < \infty.$$

(ii) For $1 < p < \infty$, (21) *holds and*

$$\sup_n \sum_k |a_{nk}|^q < \infty.$$

(iii) For $p = \infty$,

$$\lim_n \sum_k |a_{nk}| = 0.$$

Lemma 5. *Let $1 \leq p < \infty$. Then $A \in (\ell_1 : \ell_p)$ if and only if*

$$\sup_k \sum_n |a_{nk}|^p < \infty.$$

Lemma 6. *Let $1 < p < \infty$. Then $A \in (\ell_\infty : \ell_p)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \Big| \sum_{k \in K} a_{nk} \Big|^p < \infty.$$

Now, we prove the following results characterizing the matrix mappings on the spaces ℓ_p^λ for $1 \leq p \leq \infty$. Because the cases $p = 1$ and $p = \infty$ can be proved by analogy, we shall omit the proof of these cases and only consider the case $1 < p < \infty$ in the proofs of Theorems 4–7 below. Also, these results will be proved by applying the same technique used in [6, 7, 12].

Theorem 4.

(i) $A \in (\ell_1^\lambda : \ell_\infty)$ if and only if

$$\Big(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \Big)_{k=0}^\infty \in \ell_\infty \ \textit{for every}\ n \in \mathbb{N}, \tag{22}$$

$$\sup_{n,\,k} |\tilde{a}_{nk}| < \infty. \tag{23}$$

(ii) Let $1 < p < \infty$. Then $A \in (\ell_p^\lambda : \ell_\infty)$ if and only if (22) *holds and*

$$\sup_n \sum_k |\tilde{a}_{nk}|^q < \infty. \tag{24}$$

(iii) $A \in (\ell_\infty^\lambda : \ell_\infty)$ if and only if

$$\lim_k \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} = 0 \ \textit{for all}\ n \in \mathbb{N}, \tag{25}$$

$$\sup_n \sum_k |\tilde{a}_{nk}| < \infty. \tag{26}$$

Proof. Suppose that conditions (22) and (24) hold and take any $x = (x_k) \in \ell_p^\lambda$, where $1 < p < \infty$. Then, we have by Theorem 2 that $(a_{nk})_{k=0}^\infty \in (\ell_p^\lambda)^\beta$ for all $n \in \mathbb{N}$ and this implies the existence of the A-transform of x, i.e., Ax exists. Further, it is clear that the associated sequence $y = (y_k)$ is in the space ℓ_p and hence $y \in c_0$.

Let us now consider the following equality derived by using relations (3) and (4) from the m^{th} partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} y_m \qquad (n, m \in \mathbb{N}), \tag{27}$$

where the summation running from 0 to $m-1$ is equal to zero when $m=0$. Then, by using (22) and (24), from (27) as $m\to\infty$ we obtain that

$$\sum_k a_{nk}x_k=\sum_k \tilde{a}_{nk}y_k \ \text{ for all } \ n\in\mathbb{N}. \tag{28}$$

Further, since the matrix $\tilde{A}=(\tilde{a}_{nk})$ is in the class $(\ell_p:\ell_\infty)$ by (24) and Lemma 3; we have $\tilde{A}y\in\ell_\infty$. Therefore, we deduce from (1) and (28) that $Ax\in\ell_\infty$ and hence $A\in(\ell_p^\lambda:\ell_\infty)$.

Conversely, suppose that $A\in(\ell_p^\lambda:\ell_\infty)$, where $1<p<\infty$. Then $(a_{nk})_{k=0}^\infty\in(\ell_p^\lambda)^\beta$ for all $n\in\mathbb{N}$ and this, with Theorem 2, implies both (22) and

$$\sum_k |\tilde{a}_{nk}|^q<\infty \ \text{ for each } \ n\in\mathbb{N}$$

which together imply that relation (28) holds for all sequences $x\in\ell_p^\lambda$ and $y\in\ell_p$ which are connected by relation (3).

Let us now consider the continuous linear functionals f_n $(n\in\mathbb{N})$ defined on ℓ_p^λ by the sequences $A_n=(a_{nk})_{k=0}^\infty$ as follows:

$$f_n(x)=\sum_k a_{nk}x_k.$$

Then, since ℓ_p^λ and ℓ_p are norm isomorphic; it should follow with (28) that

$$\|f_n\|=\|\tilde{A}_n\|_{\ell_q}=\Big(\sum_k|\tilde{a}_{nk}|^q\Big)^{1/q}$$

for all $n\in\mathbb{N}$, where $\tilde{A}_n=(\tilde{a}_{nk})_{k=0}^\infty\in\ell_q$ for every $n\in\mathbb{N}$ as we have shown above. This just shows that the functionals defined by the rows of A on ℓ_p^λ are pointwise bounded. Thus, we deduce by the Banach-Steinhaus Theorem that these functionals are uniformly bounded. Hence, there exists a constant $M>0$ such that $\|f_n\|\le M$ for all $n\in\mathbb{N}$ which yields the necessity of (24). This completes the proof of part (ii).

Similarly, parts (i) and (iii) can be proved by means of Theorem 2 and Lemma 3, and so we leave the details to the reader. □

Theorem 5.

(i) $A\in(\ell_1^\lambda:c)$ *if and only if* (22) *and* (23) *hold and*

$$\lim_n \tilde{a}_{nk}=\alpha_k \ \textit{ for every } \ k\in\mathbb{N}. \tag{29}$$

(ii) Let $1<p<\infty$*. Then* $A\in(\ell_p^\lambda:c)$ *if and only if* (22), (24) *and* (29) *hold.*

(iii) $A\in(\ell_\infty^\lambda:c)$ *if and only if* (25), (26) *and* (29) *hold and*

$$\lim_n\sum_k|\tilde{a}_{nk}-\alpha_k|=0.$$

Proof. Suppose that A satisfies conditions (22), (24) and (29), and take any $x \in \ell_p^\lambda$, where $1 < p < \infty$. Then Ax exists. Also, by using (29), we have for every $k \in \mathbb{N}$ that $|\tilde{a}_{nk}|^q \to |\alpha_k|^q$ as $n \to \infty$. Thus, we deduce from (24) that the inequality

$$\sum_{j=0}^{k} |\alpha_j|^q \leq \sup_n \sum_j |\tilde{a}_{nj}|^q = M < \infty$$

holds for every $k \in \mathbb{N}$ which yields that $(\alpha_k) \in \ell_q$. Further, since $x \in \ell_p^\lambda$; we have $y \in \ell_p$. Consequently, we obtain by applying the Hölder's inequality that $(\alpha_k y_k) \in \ell_1$.

Now, for any given $\epsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ such that

$$\Big[\sum_{k=k_0+1}^{\infty} |y_k|^p\Big]^{1/p} < \frac{\epsilon}{4M^{1/q}}.$$

Then, it follows by (29) that there is some $n_0 \in \mathbb{N}$ such that

$$\Big|\sum_{k=0}^{k_0} (\tilde{a}_{nk} - \alpha_k) y_k\Big| < \frac{\epsilon}{2}$$

for every $n \geq n_0$. Therefore, by using (28), we derive that

$$\begin{aligned}
\Big|\sum_k a_{nk} x_k - \sum_k \alpha_k y_k\Big| &= \Big|\sum_k (\tilde{a}_{nk} - \alpha_k) y_k\Big| \\
&\leq \Big|\sum_{k=0}^{k_0} (\tilde{a}_{nk} - \alpha_k) y_k\Big| + \Big|\sum_{k=k_0+1}^{\infty} (\tilde{a}_{nk} - \alpha_k) y_k\Big| \\
&< \frac{\epsilon}{2} + \Big[\sum_{k=k_0+1}^{\infty} (|\tilde{a}_{nk}| + |\alpha_k|)^q\Big]^{1/q} \Big[\sum_{k=k_0+1}^{\infty} |y_k|^p\Big]^{1/p} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4M^{1/q}} \Big[\Big(\sum_{k=k_0+1}^{\infty} |\tilde{a}_{nk}|^q\Big)^{1/q} + \Big(\sum_{k=k_0+1}^{\infty} |\alpha_k|^q\Big)^{1/q}\Big] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4M^{1/q}} 2M^{1/q} = \epsilon
\end{aligned}$$

for all sufficiently large $n \geq n_0$. This leads us to the consequence that $A_n(x) \to \sum_k \alpha_k y_k$ as $n \to \infty$, which means that $Ax \in c$ and hence $A \in (\ell_p^\lambda : c)$.

Conversely, suppose that $A \in (\ell_p^\lambda : c)$, where $1 < p < \infty$. Then $A \in (\ell_p^\lambda : \ell_\infty)$. This leads us with Theorem 4 to the necessity of conditions (22) and (24) which together imply that (28) holds for all sequences $x \in \ell_p^\lambda$ and $y \in \ell_p$ which are connected by the relation $y = \Lambda(x)$.

Now, let $y \in \ell_p$ be given and let x be the sequence defined by (4). Then $y = \Lambda(x)$ and hence $x \in \ell_p^\lambda$. Further, since $Ax \in c$ by the hypothesis; we obtain by (28) that $\tilde{A}y \in c$ which shows that $\tilde{A} \in (\ell_p : c)$, where $\tilde{A} = (\tilde{a}_{nk})$. Hence, the necessity of (29) is immediate by (8) of Lemma 2. This concludes the proof of part (ii).

Since parts (i) and (iii) can be proved similarly, we omit their proofs. □

Theorem 6.

(i) $A \in (\ell_1^\lambda : c_0)$ *if and only if* (22) *and* (23) *hold and*

$$\lim_n \tilde{a}_{nk} = 0 \text{ for all } k \in \mathbb{N}. \tag{30}$$

(ii) Let $1 < p < \infty$. *Then* $A \in (\ell_p^\lambda : c_0)$ *if and only if* (22), (24) *and* (30) *hold.*

(iii) $A \in (\ell_\infty^\lambda : c_0)$ *if and only if* (25) *holds and*

$$\lim_n \sum_k |\tilde{a}_{nk}| = 0. \tag{31}$$

Proof. This theorem can be proved by the same technique used in the proof of Theorem 5 with Lemma 4 instead of Lemma 2, and by using the fact that (31) implies both (26) and (30). Thus, we leave the proof to the reader. □

Theorem 7.

(i) $A \in (\ell_1^\lambda : \ell_1)$ *if and only if* (22) *holds and*

$$\sup_k \sum_n |\tilde{a}_{nk}| < \infty.$$

(ii) Let $1 < p < \infty$. *Then* $A \in (\ell_p^\lambda : \ell_1)$ *if and only if* (22) *holds and*

$$\sup_{F \in \mathcal{F}} \sum_k \Big| \sum_{n \in F} \tilde{a}_{nk} \Big|^q < \infty. \tag{32}$$

(iii) $A \in (\ell_\infty^\lambda : \ell_1)$ *if and only if* (25) *holds and*

$$\sup_{F \in \mathcal{F}} \sum_k \Big| \sum_{n \in F} \tilde{a}_{nk} \Big| < \infty.$$

Proof. Suppose that conditions (22) and (32) hold and take any $x \in \ell_p^\lambda$, where $1 < p < \infty$. Then $y \in \ell_p$. Also, it is obvious by (32) that (24) holds. Therefore, we have by Theorem 2 that $(a_{nk})_{k=0}^\infty \in (\ell_p^\lambda)^\beta$ for all $n \in \mathbb{N}$ and hence Ax exists. Further, it follows by combining (32) and Lemma 1 that the matrix $\tilde{A} = (\tilde{a}_{nk})$ is in the class $(\ell_p : \ell_1)$ and hence $\tilde{A}y \in \ell_1$. Moreover, we deduce by (22) and (24) that the relation (28) holds which yields that $Ax \in \ell_1$ and hence $A \in (\ell_p^\lambda : \ell_1)$.

Conversely, suppose that $A \in (\ell_p^\lambda : \ell_1)$, where $1 < p < \infty$. Then $A \in (\ell_p^\lambda : \ell_\infty)$. Thus, Theorem 4 implies both (24) and the necessity of (22), which together imply that (28) holds for all $x \in \ell_p^\lambda$ and $y \in \ell_p$ such that $y = \Lambda(x)$. Therefore, the necessity of (32) can be deduced similarly as the necessity of (29) in the proof of Theorem 5 with Lemma 1 instead of Lemma 2. This completes the proof of part (ii).

Similarly, one can prove the other two parts by means of Theorems 2, 4 and Lemma 1. □

Theorem 8. *Let $1 \leq p < \infty$. Then $A \in (\ell_1^\lambda : \ell_p)$ if and only if* (22) *holds and*

$$\sup_k \sum_n |\tilde{a}_{nk}|^p < \infty. \tag{33}$$

Proof. Suppose that A satisfies conditions (22) and (33), and take any $x \in \ell_1^\lambda$. Then $y \in \ell_1$. Further, we have by Theorem 2 that $(a_{nk})_{k=0}^\infty \in (\ell_1^\lambda)^\beta$ for all $n \in \mathbb{N}$ and hence Ax exists. Moreover, by (33) we obtain that

$$\sup_k |\tilde{a}_{nk}| \leq \sup_k \Big(\sum_n |\tilde{a}_{nk}|^p \Big)^{1/p} < \infty \text{ for each } n \in \mathbb{N}.$$

Therefore, the series $\sum_k \tilde{a}_{nk} y_k$ converges absolutely for each fixed $n \in \mathbb{N}$. Thus, if we pass to the limits in (27) as $m \to \infty$, then it follows by (22) that (28) holds. Hence, by applying the Minkowski's inequality and using (28) and (33), we derive that

$$\Big(\sum_n |A_n(x)|^p \Big)^{1/p} = \Big(\sum_n \Big| \sum_k \tilde{a}_{nk} y_k \Big|^p \Big)^{1/p} \leq \sum_k \Big[|y_k| \Big(\sum_n |\tilde{a}_{nk}|^p \Big)^{1/p} \Big] < \infty$$

which yields that $Ax \in \ell_p$ and so $A \in (\ell_1^\lambda : \ell_p)$.

Conversely, suppose that $A \in (\ell_1^\lambda : \ell_p)$, where $1 \leq p < \infty$. Then $A \in (\ell_1^\lambda : \ell_\infty)$. Thus, Theorem 4 implies both (23) and the necessity of (22). Therefore, it follows by combining (22) and (23) that relation (28) holds for all sequences $x \in \ell_1^\lambda$ and $y \in \ell_1$ such that $y = \Lambda(x)$. This leads us with the hypothesis to the consequence that $\tilde{A} = (\tilde{a}_{nk}) \in (\ell_1 : \ell_p)$. Hence, the necessity of (33) is immediate by Lemma 5 and this concludes the proof. □

Theorem 9. *Let $1 < p < \infty$. Then $A \in (\ell_\infty^\lambda : \ell_p)$ if and only if* (25) *holds and*

$$\sum_k |\tilde{a}_{nk}| \text{ converges for every } n \in \mathbb{N},$$
$$\sup_{K \in \mathcal{F}} \sum_n \Big| \sum_{k \in K} \tilde{a}_{nk} \Big|^p < \infty.$$

Proof. It can be proved similarly to the proof of Theorem 8 with Lemma 6 instead of Lemma 5. Thus, we omit the proof. □

Now, we may present the following basic lemma [7, Lemma 5.3] (see also [12, p.713]) which is useful for deriving the characterizations of some other matrix classes via Theorems 4–9.

Lemma 7. *Let X and Y be sequence spaces, A an infinite matrix and B a triangle. Then $A \in (X : Y_B)$ if and only if $C = BA \in (X : Y)$.*

As an immediate consequence of Lemma 7, we conclude our work by the following corollary in which $\lambda' = (\lambda'_k)$ is a strictly increasing sequence of positive reals tending to infinity, $\Lambda' = (\Lambda'_{nk})$ is the triangle defined in Section 2 with λ' instead of λ, and $c_0^{\lambda'}$, $c^{\lambda'}$, $\ell_p^{\lambda'}$ and $\ell_\infty^{\lambda'}$ are the matrix domains of Λ' in the spaces c_0, c, ℓ_p and ℓ_∞, respectively; where $1 \leq p < \infty$.

Corollary 1. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{\lambda'_n} \sum_{j=0}^{n} (\lambda'_j - \lambda'_{j-1}) a_{jk}$$

for all $n, k \in \mathbb{N}$. Then, the necessary and sufficient conditions such that A belongs to any of the classes $(\ell_p^\lambda : \ell_\infty^{\lambda'})$, $(\ell_p^\lambda : c^{\lambda'})$, $(\ell_p^\lambda : c_0^{\lambda'})$, $(\ell_p^\lambda : \ell_1^{\lambda'})$, $(\ell_1^\lambda : \ell_p^{\lambda'})$ or $(\ell_\infty^\lambda : \ell_p^{\lambda'})$ are obtained from the respective ones in Theorems 4–9 by replacing the entries of the matrix A by those of C, where $1 \leq p \leq \infty$.

Remark 3. *It is obvious that Lemma 7 has several consequences, some of them give the characterization of matrix mappings from the space ℓ_p^λ $(1 \leq p \leq \infty)$ into a suitable space of those studied in [2, 3, 4, 5, 6, 7, 11, 12, 14, 15] and [17], and this can be achieved similarly to Corollary 1.*

Acknowledgement

Research of the first author was supported by the Department of Science and Technology, New Delhi, under grant No.SR/S4/MS:505/07, and research of the second author was supported by Al Bayda University, Yemen.

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