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Characteristic classes of vector bundles over $CP(j) \times HP(k)$ and involutions fixing $CP(2m+1) \times HP(k)$*

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Abstract. In this paper, we determine the total Stiefel-Whitney classes of vector bundles over the product of the complex projective space $CP(j)$ with the quaternionic projective space $HP(k)$. Moreover, we show that every involution fixing $CP(2m+1) \times HP(k)$ bounds.

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1. Introduction

In 1962, Steenrod raised to Conner the following questions:

Given a smooth closed manifold F (not necessarily connected), does there exist a non-trivial smooth involution T on a smooth closed manifold M with F as its fixed point set? Can we determine all involutions $(M,\ T)$ up to bordism for the manifold F?

When F is the disjoint union of some spaces, there have been many results, see [3, 4, 7, 8, 11, 12]. But there are few results for the case that F is the product of some spaces, see [6, 10, 13]. We shall particularly be concerned with the case in which $F = CP(2m+1) \times HP(k)$, where by $CP(2m+1)$ and $HP(k)$ we denote a $(2m+1)$-dimensional complex projective space and a k-dimensional quaternionic projective space, respectively.

From [1], we know that the bordism class of an involution $(M,\ T)$ with F as its fixed point set is determined by the bordism class of the normal bundle over F. To calculate characteristic numbers of the normal bundle over $F = CP(2m+1) \times HP(k)$, we need to know the possible form of the total Stiefel-Whitney classes of vector bundles over it. We have the following theorem:

Theorem 1. *The total Stiefel-Whitney class of a vector bundle ξ over $CP(j) \times HP(k)$ has the form*

$$w(\xi) = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^{\varepsilon},$$

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where $\alpha \in H^2(CP(j);Z_2)$, $\beta \in H^4(HP(k);Z_2)$ *are nonzero classes,* a, b, d *are non-negative integers, and* $\varepsilon = 0$ *or 1. When* $\varepsilon = 1$*, we must have*

$$\begin{cases} i = 2^t(2p+1), & t \geq 1, \\ j = 2^t(2p+1) + x, & 0 \leq x < 2^t, \\ 4k = 2^s - 2^{t+1}(2p+1) + y, & 0 \leq y < 2^{t+1}. \end{cases}$$

By using this result, we prove

Theorem 2. *Every involution fixing* $CP(2m+1) \times HP(k)$ *bounds.*

The paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we discuss the existence of involutions fixing $CP(2m+1) \times HP(k)$ and prove Theorem 2.

2. Characteristic classes of vector bundles

Let

$$H^*(CP(j) \times HP(k); Z) = Z[\alpha]/\alpha^{j+1} \bigotimes Z[\beta]/\beta^{k+1},$$

where $\alpha \in H^2(CP(j);Z)$, $\beta \in H^4(HP(k);Z)$ are generators. For convenience, we also denote generators of $H^2(CP(j);Z_2)$, $H^4(HP(k);Z_2)$ by α, β.

Let $P_1 : CP(j) \times HP(k) \longrightarrow CP(j)$, $P_2 : CP(j) \times HP(k) \longrightarrow HP(k)$ be the projection maps. We have a complex line bundle $P_1^*(L_\alpha)$ over $CP(j) \times HP(k)$, which is the pullback of the canonical complex line bundle L_α over $CP(j)$ with the total Chern class $c(P_1^*(L_\alpha)) = 1 + \alpha$, and a 2-dimensional complex bundle $P_2^*(L_\beta)$ over $CP(j) \times HP(k)$, which is the pullback of the canonical quaternionic line bundle L_β over $HP(k)$ with total Chern class $c(P_2^*(L_\beta)) = 1 + \beta$.

Lemma 1. *The total Chern class of the bundle* $P_1^*(L_\alpha) \otimes P_2^*(L_\beta)$ *over* $CP(j) \times HP(k)$ *is* $c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta)) = 1 + 2\alpha + \alpha^2 + \beta$.

Proof. We define a map $i_1 : CP(j) \longrightarrow CP(j) \times HP(k)$ by $i_1(x) = (x, pt_1)$, $x \in CP(j)$ and a map $i_2 : HP(k) \longrightarrow CP(j) \times HP(k)$ by $i_2(x) = (pt_2, x)$, $x \in HP(k)$, where $pt_1 \in HP(k)$, $pt_2 \in CP(j)$ are fixed points. Thus

$$P_1 i_1 : CP(j) \longrightarrow CP(j) \text{ is the identity on } CP(j),$$
$$P_2 i_2 : HP(k) \longrightarrow HP(k) \text{ is the identity on } HP(k).$$

So we have

$$(P_1 i_1)^*(L_\alpha) = i_1^* P_1^*(L_\alpha) = L_\alpha \tag{1}$$

and

$$(P_2 i_2)^*(L_\beta) = i_2^* P_2^*(L_\beta) = L_\beta. \tag{2}$$

From (1), we have

$$\begin{aligned} i_1^*(c(P_1^*(L_\alpha) \bigotimes P_2^*(L_\beta))) &= c(i_1^* P_1^*(L_\alpha) \bigotimes i_1^* P_2^*(L_\beta)) = c(L_\alpha \bigotimes C^2) \\ &= c(L_\alpha \bigotimes (C \bigoplus C)) = c(L_\alpha \bigoplus L_\alpha) = 1 + 2\alpha + \alpha^2, \quad (3) \end{aligned}$$

where C^i is an i-dimensional trivial complex bundle over $CP(j)$. Similarly, from (2) we have $i_2^*(c(P_1^*(L_\alpha) \bigotimes P_2^*(L_\beta))) = c(L_\beta) = 1 + \beta$.

Let $c(P_1^*(L_\alpha) \bigotimes P_2^*(L_\beta)) = 1 + \varepsilon_0\alpha + \varepsilon_1\alpha^2 + \varepsilon_2\beta$. Then

$$i_1^*(c(P_1^*(L_\alpha) \bigotimes P_2^*(L_\beta))) = i_1^*(1 + \varepsilon_0\alpha + \varepsilon_1\alpha^2 + \varepsilon_2\beta) = 1 + \varepsilon_0\alpha + \varepsilon_1\alpha^2.$$

From (3), $\varepsilon_0 = 2$ and $\varepsilon_1 = 1$. Similarly, we have $\varepsilon_2 = 1$. □

Lemma 2. *There is a 4-dimensional real vector bundle η over $CP(j) \times HP(k)$ such that the total Stiefel-Whitney class $w(\eta) = 1 + \alpha^2 + \beta$.*

Proof. Consider the 2-dimensional complex bundle $P_1^*(L_\alpha) \otimes P_2^*(L_\beta)$ as a real bundle. Let η be the real bundle. It follows from Lemma 2.1 that

$$w(\eta) = c(P_1^*(L_\alpha) \bigotimes P_2^*(L_\beta)) \bmod 2 = 1 + \alpha^2 + \beta.$$

□

Lemma 3. *Let the total Stiefel-Whitney class of a vector bundle ξ be $w(\xi) = 1 + w_{2^s} +$ higher terms. Then $w_{2^s+l} = 0$ and $Sq^l w_{2^s} = 0$ for $0 < l < 2^{s-1}$, where Sq^l is the Steenrod operation.*

Proof. If $0 < l < 2^{s-1}$, then $w_{2^{s-1}+l} = 0$. Using the Wu formula

$$Sq^i w_j = \sum_{t=0}^{i} \binom{j-i-1+t}{t} w_{i-t} w_{j+t} \qquad \text{for } i < j,$$

we have that for $0 < l < 2^{s-1}$,

$$\begin{aligned} 0 = Sq^{2^{s-1}} w_{2^{s-1}+l} &= \sum_{t=0}^{2^{s-1}} \binom{l-1+t}{t} w_{2^{s-1}-t} w_{2^{s-1}+l+t} \\ &= \binom{2^{s-1}+l-1}{2^{s-1}} w_0 w_{2^s+l} = w_{2^s+l}. \end{aligned}$$

Then $Sq^l w_{2^s} = \binom{2^s-1}{l} w_0 w_{2^s+l} = 0$. □

Proof of Theorem 1. Let $P_1^*(L_\alpha)$, $P_2^*(L_\beta)$ as above. Consider $P_1^*(L_\alpha)$ and $P_2^*(L_\beta)$ as real bundles. We have $w(P_1^*(L_\alpha)) = 1 + \alpha$ and $w(P_2^*(L_\beta)) = 1 + \beta$. We write $a\xi$ for $\underbrace{\xi \oplus \cdots \oplus \xi}_{a}$ and $\zeta = \xi - \eta$ for $\zeta \oplus \eta = \xi$.

If $w(\xi) = 1 + a_1\alpha +$ higher terms, then we have $w(\xi - a_1 P_1^*(L_\alpha)) = 1 + a_2\alpha^2 + b_1\beta +$ higher terms. Since $w(2P_1^*(L_\alpha)) = 1 + \alpha^2$, $w(\xi - a_1 P_1^*(L_\alpha) - 2a_2 P_1^*(L_\alpha) - b_1 P_2^*(L_\beta)) = 1 + w_8 +$ higher terms. We have

$$w(4P_1^*(L_\alpha)) = 1 + \alpha^4, w(2P_2^*(L_\beta)) = 1 + \beta^2, w(\eta) = 1 + \alpha^2 + \beta$$

and

$$w(2P_1^*(L_\alpha)+P_2^*(L_\beta)-\eta)=\frac{(1+\alpha^2)(1+\beta)}{1+\alpha^2+\beta}=1+\alpha^2\beta+\text{ higher terms.}$$

By subtracting multiples of these bundles, we may obtain a sum θ of vector bundles such that $w(\xi-\theta)=1+w_{16}+$higher terms. Proceeding inductively, we may suppose that there is a sum θ' of vector bundles such that $w(\xi-\theta')=1+w_{2^s}+$higher terms.

Since

$$w(2^{s-1}P_1^*(L_\alpha))=1+\alpha^{2^{s-1}},\quad w(2^{s-2}P_2^*(L_\beta))=1+\beta^{2^{s-2}}$$

and

$$w(2^{s-3}(2P_1^*(L_\alpha)+P_2^*(L_\beta)-\eta))=1+\alpha^{2^{s-2}}\beta^{2^{s-3}}+\text{higher terms,}$$

we may also suppose that $w_{2^s}(\xi-\theta')$ is a sum of monomials $\alpha^i\beta^{\frac{2^s-2i}{4}}$ with $i\neq 0$, $2^{s-2},2^{s-1}$. Among all such monomials we may suppose that the values of i are all divisible by $2^t(2\leq 2^t<2^{s-2})$ with at least one odd multiple of 2^t occurring. If a monomial $\alpha^h\beta^{\frac{2^s-2h}{4}}$ with $h=2^t(2p+1)$ occurs, then we have

$$\begin{aligned}Sq^{2^{t+1}}(\alpha^h\beta^{\frac{2^s-2h}{4}})&=\binom{h}{2^t}\alpha^{h+2^t}\beta^{\frac{2^s-2h}{4}}+\binom{\frac{2^s-2h}{4}}{2^{t-1}}\alpha^h\beta^{\frac{2^s-2h+2^{t+1}}{4}}\\&=\alpha^{h+2^t}\beta^{\frac{2^s-2h}{4}}+\alpha^h\beta^{\frac{2^s-2h+2^{t+1}}{4}}\end{aligned}$$

If h is an even multiple of 2^t, then $Sq^{2^{t+1}}(\alpha^h\beta^{\frac{2^s-2h}{4}})=0$. Thus, we have

$$0=Sq^{2^{t+1}}w_{2^s}=\sum_h Sq^{2^{t+1}}(\alpha^h\beta^{\frac{2^s-2h}{4}})=\sum_{h=2^t(odd)}(\alpha^{h+2^t}\beta^{\frac{2^s-2h}{4}}+\alpha^h\beta^{\frac{2^s-2h+2^{t+1}}{4}}).$$

However if h, h' are odd multiples of 2^t and $h\neq h'$, then

$$\begin{aligned}\alpha^{h+2^t}\beta^{\frac{2^s-2h}{4}}&\neq\alpha^{h'+2^t}\beta^{\frac{2^s-2h'}{4}},\quad \alpha^{h+2^t}\beta^{\frac{2^s-2h}{4}}\neq\alpha^{h'}\beta^{\frac{2^s-2h'+2^{t+1}}{4}},\\ \alpha^h\beta^{\frac{2^s-2h+2^{t+1}}{4}}&\neq\alpha^{h'+2^t}\beta^{\frac{2^s-2h'}{4}},\quad \alpha^h\beta^{\frac{2^s-2h+2^{t+1}}{4}}\neq\alpha^{h'}\beta^{\frac{2^s-2h'+2^{t+1}}{4}},\end{aligned}$$

i.e. cancellation does not occur among $Sq^{2^{t+1}}(\alpha^h\beta^{\frac{2^s-2h}{4}})$ and $Sq^{2^{t+1}}(\alpha^{h'}\beta^{\frac{2^s-2h'}{4}})$. So, if w_{2^s} is nonzero, there must be a monomial $\alpha^i\beta^{\frac{2^s-2i}{4}}$ with $i=2^t(2p+1)$ for which $\alpha^{i+2^t}\beta^{\frac{2^s-2i}{4}}$ and $\alpha^i\beta^{\frac{2^s-2i+2^{t+1}}{4}}$ are zero. For $\alpha^i\beta^{\frac{2^s-2i}{4}}$ to be nonzero, we have $i\leq j$ and $\frac{2^s-2i}{4}\leq k$. We must have $j<i+2^t$ and $k<\frac{2^s-2i+2^{t+1}}{4}$ so that $\alpha^{i+2^t}\beta^{\frac{2^s-2i}{4}}$ and $\alpha^i\beta^{\frac{2^s-2i+2^{t+1}}{4}}$ are zero. Since every other monomial in w_{2^s} is of the form $\alpha^h\beta^{\frac{2^s-2h}{4}}$ with h divisible by 2^t and $h\neq i$, then either $h>i$ or $\frac{2^s-2h}{4}>\frac{2^s-2i}{4}$, and so the monomials are zero. Thus $w_{2^s}=\alpha^i\beta^{\frac{2^s-2i}{4}}$ and

$$\begin{cases} i=2^t(2p+1), & t\geq 1,\\ j=2^t(2p+1)+x, & 0\leq x<2^t,\\ 4k=2^s-2^{t+1}(2p+1)+y, & 0\leq y<2^{t+1}.\end{cases}$$

From Lemma 3, we have $w_{2^s+l} = 0$ for $0 < l < 2^{s-1}$. For $l \geq 2^{s-1}$, suppose that w_{2^s+l} contains a monomial $\alpha^u\beta^v$ with $2u + 4v = 2^s + l \geq 2^s + 2^{s-1}$. If $u \geq i + 2^t$, then $u > j$. If $u < i + 2^t$, then

$$v \geq \frac{2^s + 2^{s-1} - 2u}{4} > \frac{2^s + 2^{s-1} - 2i - 2^{t+1}}{4} \geq \frac{2^s - 2i + 2^{t+1}}{4} > k.$$

For both cases we have $\alpha^u\beta^v = 0$. So $w_{2^s+l} = 0$ for $l > 0$.

The proof is completed. □

Corollary 1. *If ν is a non-bounding vector bundle over $CP(2m+1) \times HP(k)$ with the total Stiefel-Whitney class $w(\nu) = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$, then a is odd.*

Proof. ν has a nonzero characteristic number because it is non-bounding. A nonzero characteristic number must contain the monomial $\alpha^{2m+1}\beta^k$. Since the total Stiefel-Whitney class of $CP(2m+1) \times HP(k)$ is of the form $w = (1+\alpha)^{2m+2}(1+\beta)^{k+1}$ which contains only even powers of α, the class $w(\nu)$ must involve an odd power of α. By Theorem 1, we know that i is even. Thus the odd power of α can only be given by $(1+\alpha)^a$ and a is odd. □

3. Existence of involutions and their classification

Since $F = CP(2m+1) \times HP(k)$ bounds, there exists a manifold $V^{4m+2+4k+1}$ such that $CP(2m+1) \times HP(k) = \partial V$. Let $\xi^r \to V$ be the r-dimensional trivial bundle over V. If ν^r is the boundary of $\xi^r \to V$, then the disc bundle $D\xi^r$ has the boundary $D\nu^r \bigcup S\xi^r$. Multiplying the fibers of ξ^r by -1 induces an involution on $D\xi^r$. The restriction on $S\xi^r$ of the involution is free and on $D\nu^r$ is to multiply the fibers by -1, so it fixes the zero section, which is $CP(2m+1) \times HP(k)$. The normal bundle over $CP(2m+1) \times HP(k)$ is ν^r. Thus there is a bounding involution $(M^{4m+2+4k+r}, T)$ fixing $CP(2m+1) \times HP(k)$ for every $r \geq 0$. However, we are interested in whether there is a non-bounding involution fixing $CP(2m+1) \times HP(k)$.

Let us recall some results about the bordism of involutions. Suppose that (M, T) is a closed manifold M with involution T and the fixed point set of T is $F = CP(2m+1) \times HP(k)$. Let ν denote the normal bundle of F in M. From [1] we know that the bordism class of (M, T) is determined by the bordism class of the bundle (F, ν). Further, the real projective space bundle $RP(\nu)$ bounds in the bordism of RP^∞, where the map into RP^∞ classifies the double cover of $RP(\nu)$ by the sphere bundle $S(\nu)$.

The mod 2 cohomology of $CP(2m+1) \times HP(k)$ is

$$H^*(CP(2m+1) \times HP(k); Z_2) = Z_2[\alpha, \beta]/(\alpha^{2m+2} = \beta^{k+1} = 0),$$

where α is the 2-dimensional class coming from $CP(2m+1)$ and β is the 4-dimensional class coming from $HP(k)$. The total Stiefel-Whitney class of $CP(2m+1) \times HP(k)$ is

$$w = (1+\alpha)^{2m+2}(1+\beta)^{k+1}.$$

Let

$$u = 1 + u_1 + u_2 + \cdots + u_r \in H^*(CP(2m+1) \times HP(k); Z_2)$$

denote the total Stiefel-Whitney class of ν^r. Then the cohomology of $RP(\nu^r)$ is

$$Z_2[\alpha,\ \beta,\ c]/(\alpha^{2m+2} = \beta^{k+1} = 0;\ c^r + u_1 c^{r-1} + u_2 c^{r-2} + \cdots + u_r = 0)$$

and the total Stiefel-Whitney class of $RP(\nu^r)$ is

$$\begin{aligned} w(RP(\nu^r)) &= w\{(1+c)^r + u_1(1+c)^{r-1} + \cdots + u_r\} \\ &= (1+\alpha)^{2m+2}(1+\beta)^{k+1}\{(1+c)^r + u_1(1+c)^{r-1} + \cdots + u_r\}, \end{aligned}$$

where $c \in H^1(RP(\nu^r); Z_2)$ is the Stiefel-Whitney class of the double cover of $RP(\nu^r)$ by $S(\nu^r)$ (see [1, p. 75]).

The class of $RP(\nu^r)$ in the bordism of RP^∞ is determined by the characteristic numbers

$$w_{i_1}(RP(\nu)) \cdots w_{i_s}(RP(\nu)) c^t [RP(\nu)],$$

where $i_1 + \cdots + i_s + t = \dim RP(\nu^r) = 4m + 2 + 4k + r - 1$. In order to find the value of such numbers, we have a formula of Conner [2, (3.1)]

$$\begin{aligned} \alpha^i \beta^j c^t [RP(\nu)] &= \alpha^i \beta^j \overline{u}_{4m+2+4k-2i-4j}[CP(2m+1) \times HP(k)] \\ &= \text{coefficient of } \alpha^{2m+1}\beta^k \text{ in } \alpha^i \beta^j \overline{u}_{4m+2+4k-2i-4j}, \end{aligned}$$

where $2i + 4j + t = 4m + 2 + 4k + r - 1$ and $\overline{u} = 1/u$ is the dual Stiefel-Whitney class of ν^r.

For convenience, we introduce the following characteristic classes which were initially introduced in [9].

$$\begin{aligned} w[j] &= \frac{w(RP(\nu^r))}{(1+c)^{r-j}} \\ &= w\{(1+c)^j + u_1(1+c)^{j-1} + \cdots + u_j + u_{j+1}(1+c)^{-1} + \cdots\} \\ &= 1 + w[j]_1 + w[j]_2 + \cdots + w[j]_{4m+2+4k+r-1}, \end{aligned}$$

for which $w[j]_i$ is a polynomial in the classes $w_s(RP(\nu))$ and c. These classes satisfy (see [9])

$$\begin{aligned} w[i]_{2i} &= w_i c^i + \text{ terms with smaller powers of } c, \\ w[i]_{2i+1} &= (w_{i+1} + u_{i+1})c^i + \text{ terms with smaller powers of } c, \\ w[i]_{2i+2} &= u_{i+1}c^{i+1} + \text{ terms with smaller powers of } c. \end{aligned}$$

In particular,

$$\begin{aligned} w[0]_1 &= u_1 + w_1, \\ w[0]_2 &= u_1 c + (w_2 + u_1 w_1 + u_2), \\ w[0]_4 &= u_1 c^3 + (u_2 + w_1 u_1)c^2 + (u_3 + w_2 u_1)c + w_4 + w_3 u_1 + w_2 u_2 + w_1 u_3 + u_4. \end{aligned}$$

Suppose that $(M^{4m+2+4k+r},\ T)$ is an involution fixing $CP(2m+1) \times HP(k)$. When $r \geq 4m+2+4k$, from [5] we know that the involution bounds. When r=0 or

r=1, it is not difficult to prove that every involution bounds. Then we assume that $1 < r < 4m+2+4k$.

The proof of Theorem 2 is divided into two cases: (I) $k=2n$, (II) $k=2n+1$.

(I) $k = 2n$

Proposition 1. *Every involution fixing $CP(2m+1)\times HP(2n)$ bounds.*

Proof. If there is a non-bounding involution fixing $CP(2m+1)\times HP(2n)$, then the normal bundle ν^r is non-bounding. By Corollary 1, we know that a is odd. Then $u_1 = 0,\ u_2 = \alpha,\ w_2 = 0$ and $w[0]_2 = u_1c + (w_2 + u_1w_1 + u_2) = \alpha$. Let $2m+1 = 2^p(2q+1)-1$ $(p \geq 1,\ q \geq 0)$. Then

$$\begin{aligned} w(RP(\nu^r)) &= (1+\alpha)^{2m+2}(1+\beta)^{2n+1}\{(1+c)^r + u_1(1+c)^{r-1} + \cdots + u_r\} \\ &= (1+\alpha^{2^p})^{2q+1}(1+\beta)^{2n+1}\{(1+c)^r + u_1(1+c)^{r-1} + \cdots + u_r\}, \end{aligned}$$

where $\alpha^{2^p(2q+1)} = 0$ and $\beta^{2n+1} = 0$. Thus $(1+\alpha^{2^p})^{2q+1} = 1+\alpha^{2^p}+\cdots+\alpha^{2^p\cdot 2q}$ and $(1+\beta)^{2n+1} = 1+\beta+\cdots+\beta^{2n}$.

If r is odd, then

$$w_{2^{p+1}\cdot 2q+8n+r-1}(RP(\nu^r)) = \alpha^{2^p\cdot 2q}\beta^{2n}(rc^{r-1} + (r-1)u_1c^{r-2} + \cdots + u_{r-1})$$

is the top-dimensional class in $w(RP(\nu^r))$, and

$$\begin{aligned} &w[0]_2^{2^p-1}w_{2^{p+1}\cdot 2q+8n+r-1}(RP(\nu^r))[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}(rc^{r-1} + (r-1)u_1c^{r-2} + \cdots + u_{r-1})[RP(\nu^r)] \\ &\quad = r\alpha^{2m+1}\beta^{2n}c^{r-1}[RP(\nu^r)] \\ &\quad = r\alpha^{2m+1}\beta^{2n}[CP(2m+1)\times HP(2n)] = r, \end{aligned}$$

which is a nonzero characteristic number. Since we know that $RP(\nu^r)$ bounds, this is a contradiction.

If $r = 4h+2$, then $\binom{r}{2} \equiv 1 \pmod 2$, $\binom{r-2}{2} \equiv 0 \pmod 2$ and

$$\begin{aligned} &w[0]_2^{2^p-1}w_{2^{p+1}\cdot 2q+8n+r-2}(RP(\nu^r))c[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}(c^{r-1} + \cdots + u_{r-2}c)[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}c^{r-1}[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}[CP(2m+1)\times HP(2n)] = 1 \neq 0, \end{aligned}$$

we get a contradiction.

If $r = 4h$, then $\binom{r}{2} \equiv 0 \pmod 2$, $\binom{r-2}{2} \equiv 1 \pmod 2$ and

$$\begin{aligned} &w[0]_2^{2^p-2}w_{2^{p+1}\cdot 2q+8n+r-2}(RP(\nu^r))c^3[RP(\nu^r)] \\ &\quad = \alpha^{2m}\beta^{2n}(\alpha c^{r-1} + \cdots + u_{r-2}c^3)[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}c^{r-1}[RP(\nu^r)] \\ &\quad = \alpha^{2m+1}\beta^{2n}[CP(2m+1)\times HP(2n)] \neq 0, \end{aligned}$$

we also get a contradiction.

So every involution fixing $CP(2m+1)\times HP(2n)$ bounds. □

(II) $k = 2n+1$

Suppose that $2m+1 = 2^p(2q+1)-1$ and $2n+1 = 2^{p'}(2q'+1)-1$, where $p \geq 1,\ q \geq 0,\ p' \geq 1$ and $q' \geq 0$. To determine the bordism classification of all involutions fixing $CP(2m+1) \times HP(2n+1)$, we explore the conditions under which the bundle with class $u = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$ bounds.

Lemma 4. *Suppose that ν^r is the normal bundle of the fixed point set of a non-bounding involution fixing $CP(2m+1) \times HP(2n+1)$ with $u = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon = u'(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$, where $u' = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d$. If $\varepsilon = 1$ and $\frac{2^s-2i}{4}$ is odd, then $u'_{4m+8n+4} = \alpha^{2m}\beta^{2n+1}$ and $u_{4m+8n+4} = 0$.*

Proof. Let $\frac{2^s-2i}{4} = 2l-1$ $(l>0)$. Then $i = 2^{s-1}-4l+2 = 2(2^{s-2}-2l+1)$. By Theorem 1, $2m+1 = i+1$ and $8n+4 = 2^s-2i$. Thus $i = 2m$ and $\frac{2^s-2i}{4} = 2n+1$. We assert $u'_{4m+8n+4} \neq 0$. If $u'_{4m+8n+4} = 0$, then $u_{4m+8n+4} = u'_{4m+8n+4} + \alpha^{2m}\beta^{2n+1} = \alpha^{2m}\beta^{2n+1} \neq 0$. So $r \geq 4m+8n+4$. Since $r < 4m+2+8n+4$, we have $r = 4m+8n+4$ or $r = 4m+1+8n+4$.

(1) For $r = 4m+8n+4$, we have $w = (1+\alpha)^{2m+2}(1+\beta)^{2n+2}$, $w_1 = w_{r+1} = w_2 = w_{r+2} = 0$ and

$$\begin{aligned} w[r-1]_{2r} &= u_rc^r + u_rw_1c^{r-1} + w_{r+1}c^{r-1} + u_rw_2c^{r-2} + w_{r+2}c^{r-2} \\ &\quad + \text{terms with smaller powers of } c \\ &= u_rc^r + \text{ terms with smaller powers of } c \\ &= u_r(u_1c^{r-1} + u_2c^{r-2} + \ldots + u_r) \\ &\quad + \text{terms with dimension smaller than } 2r \\ &= u_ru_2c^{r-2} + \text{ terms with smaller power of } c \\ &= \alpha^{2m+1}\beta^{2n+1}c^{r-2} + \text{terms with smaller power of } c. \end{aligned}$$

Then $w[r-1]_{2r}c[RP(\nu^r)] = \alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)] \neq 0$, which is a contradiction.

(2) For $r = 4m+1+8n+4$, we have

$$\begin{aligned} w[r-2]_{2(r-1)} &= u_{r-1}c^{r-1} + \text{ terms with smaller power of } c \\ &= \alpha^{2m}\beta^{2n+1}c^{r-1} + \text{terms with smaller power of } c. \end{aligned}$$

So $w[0]_2w[r-2]_{2(r-1)}[RP(\nu^r)] = \alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)] \neq 0$, which is a contradiction. Thus $u'_{4m+8n+4} \neq 0$, and it contains a monomial $\alpha^{i'}\beta^{j'}$ with $i' \leq 2m+1$, $j' \leq 2n+1$ and $2i'+4j' = 4m+8n+4$. Such a monomial must be $\alpha^{2m}\beta^{2n+1}$. So $u'_{4m+8n+4} = \alpha^{2m}\beta^{2n+1}$ and $u_{4m+8n+4} = 0$. □

Lemma 4 shows that terms of the form α^{odd}, $\alpha^{odd}\beta^{odd}$, $\alpha^{odd}\beta^{even}$, $\alpha^{even}\beta^{odd}$, β^{odd} in u can only be given by u'.

Lemma 5. *If ν^r is the normal bundle of the fixed point set of a non-bounding involution fixing $CP(2m+1) \times HP(2n+1)$ with $u = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$, then b and d are odd.*

Proof. If b and d are even, by Lemma 4, u and w contain only even power of β, where w denotes the total Stiefel-Whitney class of $CP(2m+1)\times HP(2n+1)$. Thus ν^r bounds, which is a contradiction.

By Corrolary 1, we know that a is odd. If b is even and d is odd, then

$$\begin{aligned}u' &= (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d\\ &= (1+\alpha)(1+\alpha^2+\beta)(1+\alpha)^{a-1}(1+\beta)^b(1+\alpha^2+\beta)^{d-1}\\ &= (1+\alpha+\alpha^2+\alpha^3+\beta+\alpha\beta)(\sum\alpha^{even}\beta^{even}).\end{aligned}$$

If b is odd and d is even, then

$$\begin{aligned}u' &= (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d\\ &= (1+\alpha)(1+\beta)(1+\alpha)^{a-1}(1+\beta)^{b-1}(1+\alpha^2+\beta)^d\\ &= (1+\alpha+\beta+\alpha\beta)(\sum\alpha^{even}\beta^{even}).\end{aligned}$$

For both cases, we have $w[0]_2 = u_1c+(w_2+u_1w_1+u_2)=\alpha$ and

$$\begin{aligned}w[0]_4 &= u_1c^3+(u_2+w_1u_1)c^2+(u_3+w_2u_1)c+w_4+w_3u_1+w_2u_2+w_1u_3+u_4\\ &= \alpha c^2+\varepsilon_1\alpha^2+\beta,\end{aligned}$$

where $\varepsilon_1=0$ or 1. Then

$$w[0]_2^{2m+1}w[0]_4^{2n+1}c^{r-1}[RP(\nu^r)] = \alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)]\neq 0,$$

which is a contradiction. So b and d are odd. $\square$

Lemma 6. *Suppose that ν^r is a vector bundle over $CP(2m+1)\times HP(2n+1)$ and the total Stiefel-Whitney class of ν^r has the form $u=(1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$, for which a, b and d are odd. Then for $2m+1\geq 5$, ν^r bounds if and only if*

(1) $2m+1<2^{p'+1}-2,\ 2n+1=2^{p'}(2q'+1)-1$, *where* $p'\geq 1$ *and* $q'\geq 0$,

(2) $2m+1<2^{t+1}-2$, *where* $b-d=2^t(2f+1)$,

(3) $\varepsilon=0$ *or* $\varepsilon=1$ *and* $2m+1\neq 2^{j+1}-1$, *where* 2^j *is the largest power of* 2 *in the common terms of the* 2*-adic expansions of* $2m+1$ *and* $8n+4$.

Proof.

$$\begin{aligned}u &= (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon\\ &= (1+\alpha+\alpha^2+\alpha^3+\alpha^2\beta+\beta^2+\alpha^3\beta+\alpha\beta^2)\hat{u},\end{aligned}$$

where $\hat{u}=(1+\alpha)^{a-1}(1+\beta)^{b-1}(1+\alpha^2+\beta)^{d-1}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$. Since $2m+1\geq 5$, we have $i\neq 2$, $u_1=0$, $u_2=\alpha$, $u_3=0$, $u_4=\varepsilon_1\alpha^2$, $u_5=0$, $u_6=\varepsilon_1\alpha^3$, $u_7=0$ and $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\epsilon_3\beta^2$, where $\epsilon_k=0$ or 1 $(1\leq k\leq 2)$ and $\epsilon_3\equiv\binom{b}{2}+\binom{d}{2}+bd\equiv\binom{b+d}{2}\pmod 2$.

Since $w = (1+\alpha)^{2m+2}(1+\beta)^{2n+2} = (1+\alpha)^{2m+2}(1+\beta^{2^{p'}})^{2q'+1}$, we have

$$w_{2i'+1} = 0,\ w_{2i'} = \binom{2m+2}{i'}\alpha^{i'} \text{ for } i' < 2^{p'+1}$$
$$w_{2^{p'+2}} = \beta^{2^{p'}} + \binom{2m+2}{2^{p'+1}}\alpha^{2^{p'+1}}.$$

Let $\tilde{w}_{2^{p'+2}} = w_{2^{p'+2}} + \binom{2m+2}{2^{p'+1}} u_2^{2^{p'+1}} = \beta^{2^{p'}}$.

If $2m+1 > 2^{p'+1}-2$, we have

$$\begin{aligned}
&\tilde{w}_{2^{p'+2}}^{2q'}(u_8+\epsilon_2 u_2^4)^{2^{p'}-1}u_2^{2m+1-2(2^{p'}-1)}[CP(2m+1)\times HP(2n+1)]\\
&\quad = \beta^{2^{p'}\cdot 2q'}(\alpha^2\beta+\epsilon\beta^2)^{2^{p'}-1}\alpha^{2m+1-2(2^{p'}-1)}[CP(2m+1)\times HP(2n+1)]\\
&\quad = \alpha^{2m+1}\beta^{2n+1}[CP(2m+1)\times HP(2n+1)],
\end{aligned}$$

which is nonzero. Thus the bundle ν^r does not bound.

So we suppose $2m+1 < 2^{p'+1}-2$. The following argument is divided into two cases: (1) $u_8 = \alpha^2\beta+\epsilon_2\alpha^4$, (2) $u_8 = \alpha^2\beta+\epsilon_2\alpha^4+\beta^2$.

(1) $u_8 = \alpha^2\beta+\epsilon_2\alpha^4$

In this case, $\epsilon_3 \equiv \binom{b+d}{2} \equiv 0 \pmod 2$, then $b+d$ is divisible by 4. We write $b+d = 2^k\cdot$ (odd) with $2^k \geq 4$. Then

$$\begin{aligned}
u'' &= \frac{u}{(1+u_2)^a}\\
&= (1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon\\
&= [(1+\beta)^{2^k-1}(1+\alpha^2+\beta)]^d(1+\beta)^{b-(2^k-1)d}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon\\
&= [1+\beta^{2^k}+\alpha^2(1+\beta)^{2^k-1}]^d(1+\beta)^{b+d-2^kd}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon\\
&= [1+\alpha^2+\alpha^2\beta+\cdots+\alpha^2\beta^{2^k-2}+(\alpha^2\beta^{2^k-1}+\beta^{2^k})]^d(1+\beta)^{b+d-2^kd}\\
&\quad\times(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon
\end{aligned}$$

with $b+d-2^kd \equiv 2^k\cdot$ (odd) $-2^k\cdot$ (odd) $\equiv 0 \pmod{2^{k+1}}$.

(i) If $2^k > 2^{p'}$, then the characteristic ring of ν^r (i.e. the subring of $H^*(CP(2m+1)\times HP(2n+1);Z_2)$ generated by the classes u_i and w_i) contains $\alpha,\ \alpha^2\beta,\ \cdots,\ \alpha^2\beta^{2^{p'}-1}$ and $\beta^{2^{p'}}$. So we have a nonzero characteristic number

$$(\beta^{2^{p'}})^{2q'}(\alpha^2\beta^{2^{p'}-1})\alpha^{2m+1-2}[CP(2m+1)\times HP(2n+1)].$$

(ii) If $2^k \leq 2^{p'}$, then $\alpha,\ \alpha^2\beta,\ \cdots,\ \alpha^2\beta^{2^k-2}$ and $\alpha^2\beta^{2^k-1}+\beta^{2^k}$ are characteristic classes. Let $2n+1 = 2^k-1+2^k\cdot l$. We have a nonzero characteristic number for $2m+1 \geq 5$

$$(\alpha^2\beta^{2^k-1}+\beta^{2^k})^l\alpha^2\beta^{2^k-2}\alpha^2\beta\alpha^{2m+1-4}[CP(2m+1)\times HP(2n+1)].$$

These nonzero characteristic numbers show that the bundle is always non-bounding for $u_8=\alpha^2\beta+\epsilon_2\alpha^4$.

(2) $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\beta^2$

In this case, $\epsilon_3\equiv\binom{b+d}{2}\equiv 1 \pmod 2$, so $b+d\equiv 2 \pmod 4$ and $b-d\equiv b+d-2d\equiv 2-2\equiv 0 \pmod 4$. We write $b-d=2^t(2f+1)$ with $2^t\geq 4$. Then

$$\begin{aligned} u''&=\frac{u}{(1+u_2)^a}=(1+\alpha^2+\alpha^2\beta+\beta^2)^d(1+\beta)^{b-d}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^{\varepsilon}\\ &=(1+\alpha^2+\alpha^2\beta+\beta^2)^d(1+\beta)^{2^t(2f+1)}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^{\varepsilon}\end{aligned}$$

and

$$u'''=\frac{u''}{(1+u_2^2+u_8+\epsilon_2u_2^4)^d}=(1+\beta)^{2^t(2f+1)}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^{\varepsilon}.$$

If $\varepsilon=0$ and $2^t<2^{p'}$, the characteristic ring of the bundle is generated by the classes α, $\alpha^2\beta+\beta^2$ and β^{2^t}. If $\varepsilon=0$ and $2^t\geq 2^{p'}$, the characteristic ring of the bundle is generated by the classes α, $\alpha^2\beta+\beta^2$ and $\beta^{2^{p'}}$.

If $\varepsilon=1$, then write $2^j<2m+1<2^{j+1}$, where 2^j is the largest common term of $2m+1$ and $8n+4$ ($8n+4=2^{p'+2}2q'+2^{p'+2}-4=2^{p'+2}2q'+2^{p'+1}+\cdots+2^{j+1}+2^j+\cdots+4$, $j\leq p'$). By Theorem 1, $2m+1=2^j(2g+1)+x<2^{j+1}$. It forces $g=0$, $i=2^j$ and $8n+4=2^s-2^{j+1}+y=2^{s-1}+\cdots+2^{j+1}+y=2^{p'+2}2q'+2^{p'+1}+\cdots+2^{j+1}+2^j+\cdots+4$.

Thus $y=2^j+\cdots+4$, $8n+4=2^{s-1}+\cdots+2^{j+1}+2^j+\cdots+4=2^s-4$ and $2n+1=2^{s-2}-1=2^{p'}-1$. If $2^t<2^{p'}$, then the characteristic ring of ν is generated by the classes α, $\alpha^2\beta+\beta^2$, β^{2^t} and $\alpha^i\beta^{\frac{2^s-2i}{4}}$. If $2^t\geq 2^{p'}$, then $\beta^{2^{p'}}=\beta^{2^t}=0$. The characteristic ring is generated by the classes α, $\alpha^2\beta+\beta^2$ and $\alpha^i\beta^{\frac{2^s-2i}{4}}$.

For $\varepsilon=0$ or 1, write $2n+1=2^t-1+2^tl$. If $2^{t+1}-2<2m+1<2^{p'+1}-2$, we have

$$(\alpha^2\beta+\beta^2)^{2^t-1}(\beta^{2^t})^l\alpha^{2m+1-(2^{t+1}-2)}[CP(2m+1)\times HP(2n+1)]\neq 0,$$

which shows that the bundle is non-bounding.

Now we suppose $2m+1<2^{t+1}-2$.

If the class $\alpha^i\beta^{\frac{2^s-2i}{4}}$ is present (i.e. $\varepsilon=1$), then $\beta^{2^{p'}}=0$, $\alpha^i\beta^{\frac{2^s-2i}{4}}=\alpha^{2^j}\beta^{\frac{2^s-2^{j+1}}{4}}$ $=\alpha^{2^j}\beta^{2^{p'}-2^{j-1}}$ and $(\alpha^{2^j}\beta^{2^{p'}-2^{j-1}})^2=0$. Since $2^t+2^{p'}-2^{j-1}\geq 2^{p'}$, we have $\beta^{2^t}\cdot\alpha^{2^j}\beta^{2^{p'}-2^{j-1}}=0$. The only possible characteristic number involving $\alpha^{2^j}\beta^{2^{p'}-2^{j-1}}$ which could be nonzero would be of the form

$$\alpha^{x'}(\beta(\alpha^2+\beta))^{y'}(\alpha^{2^j}\beta^{2^{p'}-2^{j-1}})[CP(2m+1)\times HP(2n+1)],$$

and the value of this class is the coefficient of $\alpha^{2m+1-2^j-x'}\beta^{2^{j-1}-1}$ in $(\beta(\alpha^2+\beta))^{y'}$, where $2x'+8y'=4m+2+8n+4-2^{j+1}-(2^{p'+2}-2^{j+1})=4m-2$ and $y'\leq 2^{j-1}-1$. The coefficient is

$$\binom{y'}{\frac{2m+1-2^j-x'}{2}}\equiv\binom{y'}{2y'-(2^{j-1}-1)}\equiv\binom{y'}{2^{j-1}-1-y'} \bmod 2.$$

It is nonzero if and only if $y' = 2^{j-1} - 1$, and in this case $2m + 1 = 2^{j+1} - 1$.

If $\varepsilon = 0$, or $\varepsilon = 1$ and $2m + 1 \neq 2^{j+1} - 1$, then the characteristic numbers which could be nonzero would involve only polynomials in α, $\alpha^2\beta + \beta^2$ and $\beta^{2^{k'}}$, where $2^{k'} = \min(2^t, 2^{p'})$. We will show that every characteristic number involving α, $\alpha^2\beta + \beta^2$ and $\beta^{2^{k'}}$ is zero.

Suppose that there exist some $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ such that

$$\alpha^{\tilde{x}}(\beta(\alpha^2 + \beta))^{\tilde{y}}\beta^{2^{k'}\cdot\tilde{z}}[CP(2m+1) \times HP(2n+1)] = \binom{\tilde{y}}{\frac{2m+1-\tilde{x}}{2}} \equiv 1 \pmod 2,$$

where

$$\begin{cases} 2\tilde{x} + 8\tilde{y} + 2^{k'+2} \cdot \tilde{z} = 4m + 2 + 8n + 4, \\ 2\tilde{y} - \frac{2m+1-\tilde{x}}{2} + 2^{k'} \cdot \tilde{z} = 2n + 1. \end{cases}$$

If $\tilde{x} = 2m + 1$, we have $2\tilde{y} + 2^{k'} \cdot \tilde{z} = 2n + 1$, which is impossible since $k' \geq 1$. So $\tilde{x} < 2m + 1$ and $\tilde{x}$ is odd.

Writing $2n + 1 = 2^{k'} - 1 + 2^{k'}l$, we have $\beta^{2^{k'}(l+1)} = 0$. Thus $\tilde{z} \leq l$. Recall that $2m + 1 < 2^{k'+1} - 2$, then $(\alpha^2\beta + \beta^2)^{2^{k'}} = \beta^{2^{k'+1}}$. Suppose $\tilde{y} < 2^{k'}$. We have $4\tilde{y} < 2^{k'+2}$. From

$$\begin{aligned} 4\tilde{y} &= 4n + 2 + 2m + 1 - 2^{k'+1}\tilde{z} - \tilde{x} \\ &= 2^{k'+1} - 2 + 2^{k'+1}l + 2m + 1 - 2^{k'+1}\tilde{z} - \tilde{x} \\ &= 2^{k'+1}(l - \tilde{z}) + 2^{k'+1} - 2 + 2m + 1 - \tilde{x} \\ &\geq 2^{k'+1}(l - \tilde{z}) + 2^{k'+1}, \end{aligned}$$

we know that $\tilde{z} = l$ and $4\tilde{y} = 2^{k'+1} - 2 + 2m + 1 - \tilde{x}$. Thus $\tilde{y} = 2^{k'-1} + \frac{2m+1-\tilde{x}-2}{4}$.

$$\binom{\tilde{y}}{\frac{2m+1-\tilde{x}}{2}} \equiv \binom{\tilde{y}}{2\tilde{y} - (2^{k'} - 1)} \equiv \binom{\tilde{y}}{2^{k'} - 1 - \tilde{y}} \equiv 1 \pmod 2$$

implies $\tilde{y} = 2^{k'} - 1$. So $2m+1 = 2^{k'+1} - 2 + \tilde{x} \geq 2^{k'+1} - 2$, and this is a contradiction. Thus every characteristic number involving α, $\alpha^2\beta + \beta^2$ and $\beta^{2^{k'}}$ is zero and ν^r bounds.

The proof is completed. $\square$

Proposition 2. *For $2m + 1 = 2^p - 1$ and $2n + 1 = 2^{p'} - 1$, every involution fixing $CP(2m + 1) \times HP(2n + 1)$ bounds.*

Proof. If $2m + 1 = 2^p - 1$ and $2n + 1 = 2^{p'} - 1$, then $w = (1 + \alpha^{2^p})(1 + \beta^{2^{p'}}) = 1$. So the bordism class of the normal bundle ν^r is totally determined by the class u.

By R_* we denote the characteristic ring of the map of $RP(\nu^r)$ into RP^∞, i.e. the subring of $H^*(RP(\nu^r); Z_2)$ generated by c and the classes $w_i(RP(\nu^r))$, where

$$w(RP(\nu^r)) = (1 + c)^r + u_1(1 + c)^{r-1} + \cdots + u_r.$$

Since $c \in R_*$, we can solve inductively to obtain $u_i \in R_*$ for $1 \leq i \leq r$. So R_* contains the characteristic ring of ν^r (i.e. the classes $u_1, u_2, \cdots, u_r$). For every partition ω of $4m + 2 + 8n + 4$, we have $u_\omega[CP(2m + 1) \times HP(2n + 1)] = u_\omega c^{r-1}[RP(\nu^r)] = 0$. So ν^r bounds. $\square$

Lemma 7 (See [5]). *Let $(M^n,\ T)$ be a smooth involution on a closed n-dimensional manifold with the fixed point data $(F,\ \nu)=\bigsqcup_r(F^{n-r},\ \nu^r)$. If $f(x_1,\cdots,x_n)$ is a symmetric polynomial over Z_2 in n variables of degree at most n, then*

$$f(x_1,\cdots,x_n)[M^n]=\sum_r\frac{f(1+y_1,\cdots,1+y_r,z_1,\cdots,z_{n-r})}{\prod_1^r(1+y_i)}[F^{n-r}],$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$, and $\sigma_i(z)$ by the Stiefel-Whitney classes $w_i(M)$, $w_i(\nu^r)$, and $w_i(F)$, respectively, and taking the value of the resulting cohomology class on the fundamental homology class of M or F.

Lemma 8 (See [5, p. 317]). *Let $\sigma_j(x_1,\cdots,x_r,x_{r+1},\cdots,x_n)$ be the j-th elementary symmetric function in n variables. Then*

$$\begin{aligned}&\sigma_j(1+y_1,\cdots,1+y_r,z_1,\cdots,z_{n-r})\\&\quad=\sum_{p+q\le j}\binom{r-p}{j-p-q}\sigma_p(y_1,\cdots,y_r)\sigma_q(z_1,\cdots,z_{n-r}).\end{aligned}$$

Proposition 3. *For $2m+1\ge 5$, every involution fixing $CP(2m+1)\times HP(2n+1)$ bounds.*

Proof. If there is a non-bounding involution fixing $CP(2m+1)\times HP(2n+1)$, then the total Stiefel-Whitney class of the normal bundle ν^r has the form

$$\begin{aligned}u&=(1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon\\&=(1+\alpha+\alpha^2+\alpha^3+\alpha^2\beta+\beta^2+\alpha^3\beta+\alpha\beta^2)\hat{u},\end{aligned}$$

where a, b and d are all odd and $\hat{u}=(1+\alpha)^{a-1}(1+\beta)^{b-1}(1+\alpha^2+\beta)^{d-1}(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon$. Since $2m+1\ge 5$, we have $2^s\ge 16$. So $u_1=0,\ u_2=\alpha,\ u_3=0,\ u_4=\varepsilon_1\alpha^2,\ u_5=0,\ u_6=\varepsilon_1\alpha^3,\ u_7=0$ and $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\epsilon_3\beta^2$, where $\epsilon_k=0$ or 1 $(1\le k\le 3)$.

The following argument is divided into two cases: (1) $u_8=\alpha^2\beta+\epsilon_2\alpha^4$, (2) $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\beta^2$.

(1) $u_8=\alpha^2\beta+\epsilon_2\alpha^4$

Just as in Lemma 6, write $b+d=2^k\cdot$ (odd) with $2^k\ge 4$.

(i) If $2^k>2^{p'}$, the characteristic ring of ν^r contains the classes $\alpha,\ \alpha^2,\ \alpha^2\beta,\ \cdots,\ \alpha^2\beta^{2^{p'}-2},\ \alpha^2\beta^{2^{p'}-1}$ and $\beta^{2^{p'}}$.

For $2^{p'}=2$, we have $w_1=w_2=w_3=w_5=w_7=0,\ w_4=\binom{2m+2}{2}\alpha^2$, $w_6=\binom{2m+2}{3}\alpha^3$ and $w_8=\binom{2m+2}{4}\alpha^4+\beta^2$. From Lemma 8, we know

$$\begin{aligned}\sigma_2(1+y,z)&=\binom{r}{2}+\binom{r-1}{1}\sigma_1(y)+\binom{r}{1}\sigma_1(z)+\sigma_2(y)+\sigma_1(y)\sigma_1(z)+\sigma_2(z)\\&=\binom{r}{2}+\alpha.\end{aligned}$$

Let $\sigma_2^{'}(x) = \sigma_2(x) + \binom{r}{2}$. Then $\sigma_2^{'}(1+y,z) = \sigma_2(1+y,z) + \binom{r}{2} = \alpha$.

$$
\begin{aligned}
\sigma_8(1+y,z) = \sum_{p+q\le 8} &\binom{r-p}{8-p-q}\sigma_p(y) \\
\sigma_q(z) = &\binom{r}{8} + \binom{r-1}{7}\sigma_1(y) + \binom{r}{7}\sigma_1(z) + \binom{r-2}{6}\sigma_2(y) + \binom{r-1}{6}\sigma_1(y)\sigma_1(z) \\
&+\binom{r}{6}\sigma_2(z) + \binom{r-3}{5}\sigma_3(y) + \binom{r-2}{5}\sigma_2(y)\sigma_1(z) + \binom{r-1}{5}\sigma_1(y)\sigma_2(z) \\
&+\binom{r}{5}\sigma_3(z) + \binom{r-4}{4}\sigma_4(y) + \binom{r-3}{4}\sigma_3(y)\sigma_1(z) + \binom{r-2}{4}\sigma_2(y)\sigma_2(z) \\
&+\binom{r-1}{4}\sigma_1(y)\sigma_3(z) + \binom{r}{4}\sigma_4(z) + \binom{r-5}{3}\sigma_5(y) + \binom{r-4}{3}\sigma_4(y)\sigma_1(z) \\
&+\binom{r-3}{3}\sigma_3(y)\sigma_2(z) + \binom{r-2}{3}\sigma_2(y)\sigma_3(z) + \binom{r-1}{3}\sigma_1(y)\sigma_4(z) \\
&+\binom{r}{3}\sigma_5(z) + \binom{r-6}{2}\sigma_6(y) + \binom{r-5}{2}\sigma_5(y)\sigma_1(z) + \binom{r-4}{2}\sigma_4(y)\sigma_2(z) \\
&+\binom{r-3}{2}\sigma_3(y)\sigma_3(z) + \binom{r-2}{2}\sigma_2(y)\sigma_4(z) + \binom{r-1}{2}\sigma_1(y)\sigma_5(z) \\
&+\binom{r}{2}\sigma_6(z) + \binom{r-7}{1}\sigma_7(y) + \binom{r-6}{1}\sigma_6(y)\sigma_1(z) + \binom{r-5}{1}\sigma_5(y)\sigma_2(z) \\
&+\binom{r-4}{1}\sigma_4(y)\sigma_3(z) + \binom{r-3}{1}\sigma_3(y)\sigma_4(z) + \binom{r-2}{1}\sigma_2(y)\sigma_5(z) \\
&+\binom{r-1}{1}\sigma_1(y)\sigma_6(z) + \binom{r}{1}\sigma_7(z) + \sigma_8(y) + \sigma_7(y)\sigma_1(z) + \sigma_6(y)\sigma_2(z) \\
&+\sigma_5(y)\sigma_3(z) + \sigma_4(y)\sigma_4(z) + \sigma_3(y)\sigma_5(z) + \sigma_2(y)\sigma_6(z) + \sigma_1(y)\sigma_7(z) + \sigma_8(z) \\
= &\binom{r}{8} + \binom{r-2}{6}\alpha + \varepsilon_1\binom{r-4}{4}\alpha^2 + \binom{r}{4}\binom{2m+2}{2}\alpha^2 + \varepsilon_1\binom{r-6}{2}\alpha^3 \\
&+\binom{r-2}{2}\binom{2m+2}{2}\alpha^3 + \binom{r}{2}\binom{2m+2}{3}\alpha^3 + \alpha^2\beta + \varepsilon_2\alpha^4 \\
&+\varepsilon_1\binom{2m+2}{2}\alpha^4 + \binom{2m+2}{3}\alpha^4 + \binom{2m+2}{4}\alpha^4 + \beta^2.
\end{aligned}
$$

Let

$$
\begin{aligned}
\sigma_8^{'}(x) = \sigma_8(x) &+ \binom{r}{8} + \binom{r-2}{6}\sigma_2^{'}(x) + \varepsilon_1\binom{r-4}{4}\sigma_2^{'}(x)^2 + \binom{r}{4}\binom{2m+2}{2}\sigma_2^{'}(x)^2 \\
&+\varepsilon_1\binom{r-6}{2}\sigma_2^{'}(x)^3 + \binom{r-2}{2}\binom{2m+2}{2}\sigma_2^{'}(x)^3 + \binom{r}{2}\binom{2m+2}{3}\sigma_2^{'}(x)^3 \\
&+\varepsilon_2\sigma_2^{'}(x)^4 + \varepsilon_1\binom{2m+2}{2}\sigma_2^{'}(x)^4 + \binom{2m+2}{3}\sigma_2^{'}(x)^4 + \binom{2m+2}{4}\sigma_2^{'}(x)^4.
\end{aligned}
$$

Then $\sigma_8^{'}(1+y,z) = \alpha^2\beta + \beta^2$. Taking $f(x) = (\sigma_8^{'}(x))^{2q'+1}(\sigma_2^{'}(x))^{2m+1-2}$ with $\deg f = 8(2q'+1) + 2(2m+1-2) = 4m+2+8n+4 < \dim M = 4m+2+8n+4+r$, by

Lemma 7 we have

$$0=f(x)[M]=\frac{(\alpha^2\beta+\beta^2)^{2q'+1}\alpha^{2m+1-2}}{\prod_{i=1}^{r}(1+y_i)}[CP(2m+1)\times HP(2n+1)]=1,$$

which is a contradicition.

For $2^{p'}\geq 4$, we have

$$\begin{aligned}
w_{2i'}&=\binom{2m+2}{i'}\alpha^{i'}, w_{2i'+1}=0\ (0\leq i'<2^{p'+1}),\ w_{2^{p'+2}}=\binom{2m+2}{2^{p'+1}}\alpha^{2^{p'+1}}+\beta^{2^{p'}},\\
u_{4i'}&=\alpha^2\beta^{i'-1}+\varepsilon_{i'}\alpha^{2i'},\ u_{4i'+1}=0,\ u_{4i'+2}=\gamma_i^{'}\alpha^3\beta^{i'-1}+\delta_i^{'}\alpha^{2i'+1},\\
u_{4i'+3}&=0\ (2\leq i'\leq 2^{p'}).
\end{aligned}$$

Using the above method, we get $\sigma_2^{'}(x)$ and $\sigma_8^{'}(x)$ such that $\sigma_2^{'}(1+y,z)=\alpha$ and $\sigma_8^{'}(1+y,z)=\alpha^2\beta$. In the same way, adding a polynomial in $\sigma_2'(x)$ and $\sigma_8'(x)$ to $\sigma_{12}(x)$ to get $\sigma_{12}'(x)$ such that $\sigma_{12}'(1+y,z)=\alpha^2\beta^2$, adding a polynomial in $\sigma_2'(x)$, $\sigma_8'(x)$ and $\sigma_{12}'(x)$ to $\sigma_{16}(x)$ to get $\sigma_{16}'(x)$ such that $\sigma_{16}'(1+y,z)=\alpha^2\beta^3,\ \cdots,$ adding a polynomial in $\sigma_2'(x),\ \sigma_8'(x),\ \cdots,\ \sigma'_{2^{p'+2}-4}(x)$ to $\sigma_{2^{p'+2}}(x)$ to get $\sigma'_{2^{p'+2}}(x)$ such that $\sigma'_{2^{p'+2}}(1+y,z)=\alpha^2\beta^{2^{p'}-1}+\beta^{2^{p'}}$ and taking

$$f(x)=(\sigma'_{2^{p'+2}}(x))^{2q'+1}(\sigma_2'(x))^{2m+1-2},$$

from Lemma 7 we get a contradiction.

(ii) If $4\leq 2^k\leq 2^{p'}$, writing $2n+1=2^k-1+2^kl$, from Lemma 6 we know that the characteristic ring of ν^r contains the classes $\alpha,\ \alpha^2,\ \alpha^2\beta,\ \cdots,\ \alpha^2\beta^{2^k-2}$ and $\alpha^2\beta^{2^k-1}+\beta^{2^k}$. So $u_{4i'}=\alpha^2\beta^{i'-1}+\epsilon_i'\alpha^{2i'}$, $u_{4i'+1}=0$, $u_{4i'+2}=\gamma_i'\alpha^3\beta^{i'-1}+\delta_i'\alpha^{2i'+1}$, $u_{4i'+3}=0\ (2\leq i'\leq 2^k-1)$ and $u_{2^{k+2}}=\alpha^2\beta^{2^k-1}+\beta^{2^k}+\epsilon_{2^k}\alpha^{2^{k+1}}$.

Using the above method, we get a series of symmetric function $\sigma_2'(x)$, $\sigma_8'(x)$, $\ldots,\ \sigma'_{2^{k+2}-4}(x)$ and $\sigma'_{2^{k+2}}(x)$ such that $\sigma_2'(1+y,z)=\alpha,\ \ \sigma_8'(1+y,z)=\alpha^2\beta,\ \ldots,$ $\sigma'_{2^{k+2}-4}(1+y,z)=\alpha^2\beta^{2^k-2}$ and $\sigma'_{2^{k+2}}(1+y,z)=\alpha^2\beta^{2^k-1}+\beta^{2^k}$. Taking

$$f(x)=(\sigma'_{2^{k+2}}(x))^l\sigma'_{2^{k+2}-4}(x)\sigma_8'(x)(\sigma_2'(x))^{2m+1-4},$$

from Lemma 7 we get a contradiction. So $u_8=\alpha^2\beta+\epsilon_2\alpha^4$ does not occur.

(2) $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\beta^2$

From Lemma 6, we need to consider the following cases:

(a) $\varepsilon=1$ and $2m+1=2^{j+1}-1$,

(b) $2^{p'}>2^t\geq 4$ and $2m+1>2^{t+1}-2$, where $b-d=2^t(2f+1)$ with $2^t\geq 4$,

(c) $2\leq 2^{p'}\leq 2^t$ and $2m+1>2^{p'+1}-2$.

In the case (a), $\varepsilon = 1$ implies $2n+1 = 2^{p'} - 1$. By Proposition 2 we know that every involution fixing $CP(2m+1) \times HP(2n+1)$ bounds.

In the case (b), $w_{2i'} = \binom{2m+2}{i'} \alpha^{i'}$ and $w_{2i'+1} = 0$ $(0 \le i' < 2^{t+1})$. $u_{2i'} = \lambda_{i'}(\alpha^2\beta + \beta^2)^{j'}\alpha^{i'-4j'} + \epsilon_{i'}\alpha^{i'}$, where $\lambda_{i'}$ and $\epsilon_{i'}$ are 0 or 1 $(4 < i' < 2^{t+1})$. $u_{2^{t+2}} = \beta^{2^t} + \lambda_{2^{t+1}}(\alpha^2\beta + \beta^2)^{j'}\alpha^{2^{t+1}-4j'} + \epsilon_{2^{t+1}}\alpha^{2^{t+1}}$. Let $\sigma'_2(x)$ and $\sigma'_8(x)$ as in (1)-(i). Then $\sigma'_2(1+y, z) = \alpha$ and $\sigma'_8(1+y, z) = \alpha^2\beta + \beta^2$. We can add a polynomial in $\sigma'_2(x)$ and $\sigma'_8(x)$ to $\sigma_{2^{t+2}}(x)$ to get $\sigma'_{2^{t+2}}(x)$ such that $\sigma'_{2^{t+2}}(1+y, z) = \beta^{2^t}$. Writing $2n+1 = 2^t - 1 + 2^t l$ and taking

$$f(x) = (\sigma'_{2^{t+2}}(x))^l (\sigma'_8(x))^{2^t - 1} (\sigma'_2(x))^{2m+1-(2^{t+1}-2)},$$

from Lemma 7 we get a contradiction.

In the case (c), we have $w_{2i'} = \binom{2m+2}{i'}\alpha^{i'}$, $w_{2i'+1} = 0$ $(0 < i' < 2^{p'+1})$ and $w_{2^{p'+2}} = \beta^{2^{p'}} + \binom{2m+2}{2^{p'+1}}\alpha^{2^{p'+1}}$. For $2^{p'} = 2$ and r odd,

$$\begin{aligned} w(RP(\nu^r)) &= (1+\alpha^{2^p})^{2q+1}(1+\beta^{2})^{2q'+1}\{(1+c)^r + u_1(1+c)^{r-1} + \ldots + u_r\} \\ &= \alpha^{2^p \cdot 2q}\beta^{4q'}(rc^{r-1} + (r-1)u_1c^{r-2} \\ &\quad + \ldots + u_{r-1}) + \text{terms with a smaller dimension.} \end{aligned}$$

Then

$$w_{2^{p+1}\cdot 2q + 16q' + r - 1}(RP(\nu^r)) = \alpha^{2^p \cdot 2q}\beta^{4q'}(rc^{r-1} + (r-1)u_1c^{r-2} + \ldots + u_{r-1})$$

is the top-dimensional class in $w(RP(\nu^r))$. Since $w[0]_2 = u_1c + (w_2 + u_1w_1 + u_2) = \alpha$ and

$$\begin{aligned} w[0]_8 &= u_1c^7 + u_2c^6 + (u_3 + w_2u_1)c^5 + (u_4 + w_2u_2)c^4 + (u_5 + w_4u_1)c^3 \\ &\quad + (u_6 + w_4u_2)c^2 + (u_7 + w_2u_5 + w_4u_3 + w_6u_1)c + u_8 + w_2u_6 \\ &\quad + w_4u_4 + w_6u_2 + w_8 \\ &= \alpha c^6 + \varepsilon_1\alpha^2c^4 + \alpha^2\beta + \varepsilon_2\alpha^4 + \varepsilon_1\binom{2m+2}{2}\alpha^4 + \binom{2m+2}{3}\alpha^4 + \binom{2m+2}{4}\alpha^4 \\ &\quad + \left(\varepsilon_1 + \binom{2m+2}{2}\right)\alpha^3c^2, \end{aligned}$$

we have

$$\begin{aligned} &w[0]_2^{2^p-3} w_{2^{p+1}\cdot 2q+16q'+r-1}(RP(\nu^r)) w[0]_8[RP(\nu^r)] \\ &\quad = \alpha^{2m-1}\beta^{4q'}(rc^{r-1} + (r-1)u_1c^{r-2} + \ldots + u_{r-1})w[0]_8[RP(\nu^r)] \\ &\quad = r\alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)] = r, \end{aligned}$$

which is a nonzero characteristic number. We know that $RP(\nu^r)$ bounds, so this is a contradiction.

For $2^{p'}=2$ and $r=4h+2$, we have $\binom{r}{2}\equiv 1 \pmod 2$, $\binom{r-2}{2}\equiv 0 \pmod 2$ and

$$\begin{aligned}
&w[0]_2^{2^p-3}w_{2^{p+1}\cdot 2q+16q'+r-2}(RP(\nu^r))w[0]_8c[RP(\nu^r)]\\
&\quad=\alpha^{2m+1}\beta^{2n+1}(c^{r-1}+\cdots+u_{r-2}c)[RP(\nu^r)]\\
&\quad=\alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)]\\
&\quad=1\neq 0,
\end{aligned}$$

which is a contradiction.

For $2^{p'}=2$ and $r=4h$, we have $\binom{r}{2}\equiv 0 \pmod 2$, $\binom{r-2}{2}\equiv 1 \pmod 2$ and

$$\begin{aligned}
&w[0]_2^{2^p-4}w_{2^{p+1}\cdot 2q+16q'+r-2}(RP(\nu^r))w[0]_8c^3[RP(\nu^r)]\\
&\quad=\alpha^{2m}\beta^{2n+1}(\alpha c^{r-1}+\cdots+u_{r-2}c^3)[RP(\nu^r)]\\
&\quad=\alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu^r)]\\
&\quad=1\neq 0,
\end{aligned}$$

which is also a contradiction.

For $2^{p'}\geq 4$, we have $u_1=u_3=u_5=u_7=0$, $u_2=\alpha$, $u_4=\varepsilon_1\alpha^2$, $u_6=\varepsilon_1\alpha^3$, $u_8=\alpha^2\beta+\epsilon_2\alpha^4+\beta^2$ and $u_{2i'}=\delta_{i'}(\alpha^2\beta+\beta^2)^{j'}\alpha^{i'-4j'}+\lambda_{i'}\alpha^{i'}$ $(4<i'\leq 2^{p'+1})$, where $\delta_{i'}$ and $\lambda_{i'}$ are 0 or 1.

Using the method in (1), we get the symmetric functions $\sigma_2'(x)$, $\sigma_8'(x)$ and $\sigma_{2^{p'+2}}'(x)$ such that $\sigma_2'(1+y,z)=\alpha$, $\sigma_8'(1+y,z)=\alpha^2\beta+\beta^2$ and $\sigma_{2^{p'+2}}'(1+y,z)=\beta^{2^{p'}}$. Taking

$$f(x)=(\sigma_{2^{p'+2}}'(x))^{2q'}(\sigma_8'(x))^{2^{p'}-1}(\sigma_2'(x))^{2m+1-(2^{p'+1}-2)},$$

from Lemma 7, we get a contradiction.

This completes the proof. □

Proposition 4. *Every involution fixing $CP(3)\times HP(2n+1)$ bounds.*

Proof. If $\varepsilon=1$, then $2n+1=2^{p'}-1$. By Proposition 2, every involution bounds. Thus we need only to consider the case $\varepsilon=0$, i.e. $u=(1+\alpha)^a(1+\alpha^2+\beta)(1+\beta)^{b'}$.

(1) If $\binom{b'-1}{2}\equiv 1 \pmod 2$, then $u_8=\alpha^2\beta$, $b'-1\equiv 2 \pmod 4$ and $b'+1\equiv 0 \pmod 4$. Let $b'+1=2^k(2f+1)$ $(k\geq 2)$.

$$\begin{aligned}
u&=(1+\alpha)^a[(1+\alpha^2+\beta)(1+\beta)^{2^k-1})](1+\beta)^{b'-2^k+1}\\
&=(1+\alpha)^a[1+\beta^{2^k}+\alpha^2(1+\beta)^{2^k-1}](1+\beta^{2^{k+1}})^f\\
&=(1+\alpha)^a(1+\alpha^2+\alpha^2\beta+\cdots+\alpha^2\beta^{2^k-1}+\beta^{2^k})(1+\beta^{2^{k+1}})^f.
\end{aligned}$$

If $2^k>2^{p'}$, the characteristic ring of ν^r is generated by α, $\alpha^2\beta$, $\alpha^2\beta^2$, $\cdots$, $\alpha^2\beta^{2^{p'}-1}$ and $\beta^{2^{p'}}$. Just as (1)-(i) in the proof of Proposition 3, taking

$$f(x)=(\sigma_{2^{p'+2}}'(x))^{2q'+1}\sigma_2'(x)^{2m+1-2},$$

from Lemma 7 we get a contradiction. If $2^k \leq 2^{p'}$, the characteristic ring of ν^r is generated by $\alpha,\ \alpha^2\beta,\ \alpha^2\beta^2,\ \cdots,\ \alpha^2\beta^{2^k-2}, \alpha^2\beta^{2^k-1}+\beta^{2^k}$ and $\beta^{2^{p'}}$. None of these monomials can give a monomial $\alpha^3\beta^{2n+1}$, so ν^r bounds.

(2) If $\binom{b'-1}{2} \equiv 0 \pmod 2$, we write $b'-1 = 2^t(2f+1)$. The characteristic ring of ν^r is generated by $\alpha,\ \alpha^2\beta+\beta^2$ and β^{2^k}, where $k = \min(t,\ p')$.

For $2^{p'} = 2$ and r odd, we have $\binom{2n+2}{2n} \equiv 1 \pmod 2$,

$$\begin{aligned}
w(RP(\nu^r)) &= (1+\alpha^4)(1+\beta)^{2n+2}\{(1+c)^r + u_1(1+c)^{r-1} + \ldots + u_r\} \\
&= (1+\beta)^{2n+2}\{rc^{r-1} + (r-1)u_1c^{r-2} + \ldots + u_{r-1} \\
&\quad + \text{terms with a dimension smaller than } r-1\} \\
&= \beta^{2n}(rc^{r-1} + (r-1)u_1c^{r-2} + \ldots + u_{r-1}) \\
&\quad + \text{ terms with a dimension smaller than } 8n+r-1, \\
w_{8n+r-1}(RP(\nu^r)) &= \beta^{2n}(rc^{r-1} + (r-1)u_1c^{r-2} + \ldots + u_{r-1}), \\
w[0]_2 &= u_1c + (w_2 + u_1w_1 + u_2) = \alpha, \\
w[0]_8 &= u_1c^7 + u_2c^6 + (u_3 + w_2u_1)c^5 + (u_4 + w_2u_2)c^4 + (u_5 + w_4u_1)c^3 \\
&\quad + (u_6 + w_4u_2)c^2 + (u_7 + w_2u_5 + w_4u_3 + w_6u_1)c \\
&\quad + u_8 + w_2u_6 + w_4u_4 + w_6u_2 + w_8 \\
&= \alpha c^6 + \varepsilon_1\alpha^2c^4 + \alpha^2\beta + \varepsilon_1\binom{2m+2}{2}\alpha^4 + \binom{2m+2}{3}\alpha^4 \\
&\quad + \binom{2m+2}{4}\alpha^4 + (\varepsilon_1 + \binom{2m+2}{2})\alpha^3c^2.
\end{aligned}$$

Let

$$\begin{aligned}
(w[0]_8)' &= w[0]_8 + w[0]_2c^6 + \epsilon_1 w[0]_2^2c^4 + (\varepsilon_1 + \binom{2m+2}{2})w[0]_2^3c^2 \\
&\quad + \varepsilon_1\binom{2m+2}{2}w[0]_2^4 + \binom{2m+2}{3}w[0]_2^4 + \binom{2m+2}{4}w[0]_2^4 \\
&= \alpha^2\beta.
\end{aligned}$$

Then $w[0]_2(w[0]_8)'w_{8n+r-1}(RP(\nu^r))[RP(\nu^r)] \neq 0$, which is a contradiction.

For $2^{p'} = 2$ and $r = 4h+2$, we have $\binom{r}{2} \equiv 1 \pmod 2$, $\binom{r-2}{2} \equiv 0 \pmod 2$,

$$\begin{aligned}
w(RP(\nu^r)) &= \beta^{2n}[\binom{r}{2}c^{r-2} + \binom{r-1}{2}u_1c^{r-3} + \binom{r-2}{2}u_2c^{r-4} + \cdots + u_{r-2}] \\
&\quad + \text{ terms with a dimension smaller than } 8n+r-2, \\
w_{8n+r-2}(RP(\nu^r)) &= \beta^{2n}[\binom{r}{2}c^{r-2} + \binom{r-1}{2}u_1c^{r-3} + \binom{r-2}{2}u_2c^{r-4} + \cdots + u_{r-2}], \\
w[0]_2 &= u_1c + (w_2 + u_1w_1 + u_2) = \alpha.
\end{aligned}$$

Just as above, we also get $(w[0]_8)'$ such that $(w[0]_8)' = \alpha^2\beta$. Then

$$\begin{aligned} &w[0]_2(w[0]_8)'w_{8n+r-2}(RP(\nu^r))c[RP(\nu^r)] \\ &\quad= \alpha^3\beta^{2n+1}(c^{r-1}+\cdots+u_{r-2}c)[RP(\nu^r)] \\ &\quad= \alpha^3\beta^{2n+1}c^{r-1}[RP(\nu^r)] \neq 0, \end{aligned}$$

which is a contradiction.

For $2^{p'} = 2$ and $r = 4h$, we have $\binom{r}{2} \equiv 0 \pmod 2$ and $\binom{r-2}{2} \equiv 1 \pmod 2$. Then

$$\begin{aligned} &(w[0]_8)'w_{8n+r-2}(RP(\nu^r))c^3[RP(\nu^r)] \\ &\quad= \alpha^2\beta^{2n+1}(\alpha c^{r-4}+\cdots+u_{r-2})c^3[RP(\nu^r)] \\ &\quad= \alpha^3\beta^{2n+1}c^{r-1}[RP(\nu^r)] \neq 0, \end{aligned}$$

which is also a contradiction.

So for $2^{p'} = 2$, there is no non-bounding involution fixing $CP(3) \times HP(2n+1)$.

For $2^{p'} > 2$, suppose that there exist some $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ such that

$$\alpha^{x'}(\alpha^2\beta+\beta^2)^{y'}\beta^{2^k\cdot z'}[CP(3)\times HP(2n+1)] \neq 0,$$

where $2x' + 8y' + 2^{k+2}\cdot z' = 6 + 8n + 4$, i.e. $x' + 4y' + 2^{k+1}\cdot z' = 3 + 4n + 2$. Then x' is odd. Since $x' \leq 3$, $x' = 1$ or 3. If $x' = 3$, then $2y' + 2^k \cdot z' = 2n+1$, which is impossible. If $x' = 1$, then $\binom{y'}{1} \equiv 1 \pmod 2$, i.e. y' is odd. Thus $1+2(y'-1)+2^k\cdot z' = 2n+1 = 2^{p'}(2q'+1)-1$, which is also impossible. So ν^r bounds.

The proof is completed. □

Proposition 5. *Every involution fixing $CP(1) \times HP(2n+1)$ bounds.*

Proof. In this case, $\alpha^2 = 0$. From Theorem 1 and Lemma 5, we know that every involution fixing $CP(1) \times HP(2n+1)$ has the total Stiefel-Whitney class $u = (1+\alpha)(1+\beta)^{b+d}$, where b and d are odd. So we cannot obtain any odd power of β from u and w and every involution fixing $CP(1) \times HP(2n+1)$ bounds. □

Combining Propositions 1, 3, 4 and 5 together, we have Theorem 2.

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