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Adjacency preserving mappings on real symmetric matrices

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Abstract. Let S_n denote the space of all $n \times n$ real symmetric matrices. Let $n \geq 2$ and let $\Phi : S_n \to S_m$ be a map preserving adjacency, i.e. if $A, B \in S_n$ and rank $(A - B) = 1$, then rank $(\Phi(A) - \Phi(B)) = 1$. If $\Phi(0) = 0$, we prove that either:
(i) Φ maps S_n into $\mathbb{R}B$, where B is a rank one matrix, or
(ii) there exist $c \in \{-1, 1\}$ and $R \in M_m$ invertible $(m \geq n)$ such that for $A \in S_n$,

$$\Phi(A) = cR \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} R^T.$$

(If $m = n$, the zeros on the right-hand side are absent.)
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1. Introduction

Wen-Ling Huang and Peter Šemrl in [7] characterized adjacency preserving maps from H_n to H_m, where H_n denotes the $n \times n$ hermitian matrices over $\mathbb{C}$. They improved the results going back to Hua ([4, 5]). See also [2, 3, 8, 9, 10, 13, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26]. This article considers adjacency preserving mappings from S_n to S_m, where S_n denotes the $n \times n$ symmetric matrices over $\mathbb{R}$. The authors of [7] suggested this problem in their article. It turns out that the ideas and methods of their paper work in the real case as well (with modifications in some places).

The proof of the complex case uses results by Wen-Ling Huang, Roland Höfer and Zhe-Xian Wan [6], which hold in the real case as well.

The main result of this paper is Theorem 1.

2. Notation

We will consider only matrices over the real number field $\mathbb{R}$. Let $M_n = M_n(\mathbb{R})$ be the space of all $n \times n$ matrices over $\mathbb{R}$. Let S_n denote the linear subspace of all symmetric matrices in M_n, i.e. all $A \in M_n$ such that $A = A^T$, where A^T is the transpose of A. Let $GL(n)$ denote the group of all invertible matrices in M_n. Let

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lin Z denote the real linear span of a set Z (in some vector space). We will often look at matrices in M_n as linear operators on $\mathbb{R}^n$. So for $A \in M_n$, Im $A = A\mathbb{R}^n$ is the *image* of A or the *column space* of A.

If we consider $x, y \in \mathbb{R}^n$ as $n \times 1$ matrices, $xy^T = x \otimes y$ is the rank one matrix with the property $(x \otimes y)z = \langle z, y \rangle\, x$ for $z \in \mathbb{R}^n$.

If $P \in S_n$ and $P^2 = P = P^T \neq 0$, then we call P a *projection*, as it is the orthogonal projection on Im P. Two projections P, Q are orthogonal, $P \perp Q$, iff $PQ = 0$. If x is a unit vector, then $x \otimes x$ is the projection on lin $\{x\}$.

Let $e_1, \ldots e_n$ be the standard basis in $\mathbb{R}^n$ and let $e_i \otimes e_j = E_{ij}$ be the matrix unit, i.e. the matrix with 1 in place (i, j) and zeros elsewhere.

We know that for $R, T \in M_n$, Im $(R+T) \subseteq$ Im $R+$Im T and so rank $(R+T) \leq$ rank $R+$ rank T.

For $A, B \in S_n$ let $d(A, B) =$ rank $(A - B)$. Then (S_n, d) is a metric space. We will often use

Lemma 1. *Let $A, B, C \in M_n$ and $A + B = C$. Then rank $A =$ rank $B +$ rank C iff Im $A =$ Im $B \oplus$ Im C.*

Two matrices A, B are *adjacent* if $d(A, B) = 1$, i.e. rank $(A-B) = 1$. If $d(A, B) = k$, there is a sequence of consecutively adjacent matrices $A_0 = A, A_1, \ldots, A_k = B$ (see Proposition 5.5 in [24]). Conversely, if there is such a sequence, it is straightforward that $d(A, B) \leq k$.

Let $A, B \in S_n$ be adjacent. The line $l(A, B)$ joining A and B is the set consisting of A, B and all $Y \in S_n$, which are adjacent to both A and B. By Corollary 5.9 in [24],

$$l(A, B) = \{A + \lambda(B - A) | \lambda \in \mathbb{R}\}\,.$$

If $P \in S_n$ is a projection, let $PS_nP = \{PAP | A \in S_n\} = \{C \in S_n | PCP = C\}$.

Proposition 1. *For $A, B, S \in S_n$, $R \in GL(n)$, and $c \in \mathbb{R}\setminus\{0\}$ we have $d(A + S, B + S) = d(A, B) = d(RAR^T, RBR^T) = d(cA, cB)$. Consequently, these are equivalent:*

i) A is adjacent to B;

ii) $A + S$ is adjacent to $B + S$;

iii) RAR^T is adjacent to RBR^T;

iv) cA is adjacent to cB.

Corollary 1. *Let $\Phi : S_n \to S_m$ be a map preserving adjacency, i.e. A is adjacent to B implies $\Phi(A)$ is adjacent to $\Phi(B)$. Let $\Psi(A) = \Phi(A) - \Phi(0)$ for $A \in S_n$. Then Ψ is adjacency preserving and $\Psi(0) = 0$.*

Theorem 1 (MAIN THEOREM)**.** *Let m, n be natural numbers, $n \geq 2$. Let $\Phi : S_n \to S_m$ be a map preserving adjacency, with $\Phi(0) = 0$. Then either:*

i) There is a rank one matrix $B \in S_m$ and a function $f : S_n \to \mathbb{R}$ such that for $A \in S_n$

$$\Phi(A) = f(A)B.$$

In this case we say Φ *is a* **degenerate** *adjacency preserving map.*

ii) We have $c \in \{-1, 1\}$, $R \in GL(m)$ *such that for* $A \in S_n$,

$$\Phi(A) = cR \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} R^T.$$

In this case we say Φ *is a* **standard** *map. (Obviously, in this case* $m \geq n$*. If* $m = n$*, the zeros on the right-hand side of the formula are absent.)*

3. Preliminary results

We borrow Lemma 2.1. in [6]:

Lemma 2. *Let* $G \in S_n$ *and let* l *be a line in* S_n*. Then either:*

i) There is k *such that* $d(G, X) = k$ *for all* $X \in l$ *or*

ii) There is a point $K \in l$ *such that* $d(G, X) = d(G, K) + 1$ *for all* $X \in l$, $X \neq K$.

Lemma 3. *Let* $A \in S_n$ *be adjacent to both* R *and* λR*, where* $R \in S_n$ *has rank one and* $\lambda \neq 1$*. Then* $A = \mu R$ *for some* $\mu \in \mathbb{R}$, $\mu \neq 1, \lambda$.

Proof. Since $\lambda \neq 1$, R and λR are adjacent and A is contained in the line $l(R, \lambda R)$. So $A = R + \mu'(R - \lambda R) = \mu R$ and $\mu \neq \lambda, 1$. □

The following lemma is slightly more general than Lemma 2.3. in [7].

Lemma 4. *Let* $P \in M_n$ *be an idempotent and* $A, B \in M_n$ *such that* $P = A + B$ *and rank* $P =$ *rank* $A +$ *rank* B*. Then* A, B *are idempotents and* $AB = BA = 0$.

Proof. By Lemma 1, $\mathrm{Im}\ P = \mathrm{Im}\ A \oplus \mathrm{Im}\ B$. So if $Px = 0$, $Ax = Bx = 0$ and thus $\ker P \subset \ker A$. For $y \in \mathrm{Im}\ A \subset \mathrm{Im}\ P$, $Py = y = Ay + By$, hence $y - Ay = By$. Since $y - Ay \in \mathrm{Im}\ A$, we have $By = 0$. Thus $BA = 0$ and $A^2 = A$. By symmetry, $AB = 0$ and $B^2 = B$. □

Lemma 5. *Let* $P_1, P_2, \ldots, P_k \in S_n$ *be mutually orthogonal rank one projections and* $P = P_1 + \ldots + P_k$*. Let* $\xi(1), \ldots, \xi(n)$ *be an orthonormal system in* $\mathbb{R}^n$ *such that* $P_i(\xi(i)) = \xi(i)$ *for* $i = 1, \ldots, k$*. Then* $P_i(\xi(j)) = \delta_{ij}\xi(j)$*. Let* V *be the orthogonal matrix defined by* $Ve_i = \xi(i)$ *for* $i = 1, \ldots, n$*, so that* $\xi(i)$ *is the* $i-$*th column of* V*. Then* $V^T P_i V = E_{ii}$ *for* $i = 1, \ldots, k$*. If* $A \in PS_nP = \{C \in S_n | PCP = C\}$*, then*

$$V^T AV = \begin{bmatrix} q(A) & 0 \\ 0 & 0 \end{bmatrix},$$

where $q(A) \in S_k$*. We have* $q(P_i) = E_{ii}$ *for* $i = 1, \ldots, k$ *and* $q(P) = E_{11} + \ldots + E_{kk}$.

The mapping $q : PS_nP \to S_k$ *is linear, bijective, and* $q(AB) = q(A)q(B)$ *for* $A, B, AB \in PS_nP$*. So* $q(A^2) = q(A)^2$ *and* q *is a Jordan isomorphism. It preserves* *the distance d and thus adjacency. Also $q(ABA) = q(A)q(B)q(A)$ for all $A, B \in PS_nP$. All these properties are shared by the mappings $h : S_k \to S_n$ and $q^{-1} : S_k \to PS_nP$, where*

$$h(B) = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

and $q^{-1}(B) = Vh(B)V^T$.

Lemma 6. *Let k, n be natural numbers with $3 \le k \le n$. Let $\lambda_1, \ldots, \lambda_k$ be nonzero real numbers and $P_1, \ldots, P_k \in S_n$ mutually orthogonal rank one projections. Let $A = \sum_{j=1}^{k} \lambda_j P_j$. Let $B \in S_n$ have rank B = rank $A = k$ and let B be adjacent to $A - \lambda_i P_i$ for all i. Assume that $d(B, \lambda_i P_i) = k - 1$ for all i. Then $B = A$.*

Proof. By Lemma 1, Im B = Im $(\lambda_i P_i) \oplus$ Im $(B - \lambda_i P_i)$. So Im $P_i \subset$ Im B for all i. If $P = P_1 + \ldots + P_k$, then Im $P \subset$ Im B and rank $P = k$, so Im P = Im B and consequently $PB = B = BP$. Thus $A, B \in PS_nP$. Using notation from Lemma 5, $q(A), q(B) \in S_k$ and $q(B)$ is adjacent to $q(A) - \lambda_i q(P_i)$, $d(q(B), \lambda_i q(P_i)) = k - 1$. Also, $q(P) = E_{11} + \ldots + E_{kk} = I_k$ and $q(A), q(B)$ have maximal rank as elements in S_k. Thus we may assume that $k = n$ and A, B are invertible in S_n, $P_1 + \ldots + P_n = I$.

Now $1 = \text{rank } (B - A + \lambda_i P_i) = \text{rank } (A^{-1}B - I + \lambda_i A^{-1} P_i)$. But $A^{-1} = \sum \lambda_i^{-1} P_i$, so $\lambda_i A^{-1} P_i = P_i$. Let $C = B^{-1}A \in M_n$. Then $C \in GL(n)$ and $1 = \text{rank } (C^{-1} - (I - P_i)) = \text{rank } (I - C(I - P_i))$.

Now $I = (I - C(I - P_i)) + C(I - P_i)$ and rank $C(I - P_i)$ = rank $(I - P_i) = n - 1$. By Lemma 4, $C(I - P_i)$ is an idempotent.

Let $f_1, \ldots, f_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $P_i f_i = f_i$. Then for $j \ne i$, $(I - P_i)f_j = f_j$, so $Cf_j = C(I - P_i)f_j = C(I - P_i)C(I - P_i)f_j = C(I - P_i)Cf_j$. Since C is invertible,

$$(I - P_i)Cf_j = f_j \text{ for } j \ne i.$$

Let $Cf_j = \sum_{m=1}^{n} a_m f_m$. Then $(I - P_i)Cf_j = Cf_j - P_i Cf_j = Cf_j - a_i f_i = \sum_{m \ne i} a_m f_m = f_j$. So $a_m = 0$ for $m \ne i, j$ and $a_j = 1$. Thus $Cf_j = f_j + a_i f_i$. Since $n \ge 3$, there exists k, $1 \le k \le n$, $k \ne i, j$. So $Cf_j = f_j + a_k f_k$ also. Thus $Cf_j = f_j$ for all j and $C = I$. This implies $A = B$. □

Lemma 7. *Let $A, B \in S_m$ and let rank $A = 1$. If rank $(A + \lambda B) = 1$ for every $\lambda \in \mathbb{R}$, then $B = 0$.*

Proof. If rank $B \ge 2$, then there exists a nonsingular 2×2 submatrix in B. For $\lambda \ne 0$, we have rank $(A + \lambda B)$ = rank $(B + \frac{1}{\lambda}A) \ge 2$ for λ large enough, since the chosen submatrix of $(B + \frac{1}{\lambda}A)$ will be nonsingular. Therefore, rank $B \le 1$.

If $B \ne 0$, then B is adjacent to 0. Also, $A + B$ is adjacent to 0 and B, so $A + B \in l(B, 0)$. Thus $A + B = \mu B$ and $A + (1 - \mu)B = 0$ – a contradiction. □

Lemma 8. *Let $A, B \in S_n$ have rank n $(n \ge 2)$, with $A \ne B$. There exists a natural number k and invertible matrices $A = A_0, A_1, \ldots, A_k = B$ such that the neighbours in this sequence are adjacent and there is a matrix $C_j \in l(A_j, A_{j+1})$ with rank $C_j = n - 1$ for $j = 0, \ldots, k - 1$.*

Proof. This is a consequence of Lemmas 2.5 and 2.6 in [6] and it is stated in the proof of Lemma 3.1 in the same paper. □

Lemma 9. *Let $\Phi : S_n \to S_m$ be an adjacency preserving map. Let $A, B \in S_n$ be adjacent. Then $\Phi(l(A, B)) \subset l(\Phi(A), \Phi(B))$. The restriction of Φ to $l(A, B)$ is injective.*

Proof. If $\lambda_1 \neq \lambda_2$ and $C_i = A + \lambda_i(B - A) \in l(A, B)$ $(i = 1, 2)$, then C_1 is adjacent to C_2 and therefore $\Phi(C_1)$ is adjacent to $\Phi(C_2)$, thus $\Phi(C_1) \neq \Phi(C_2)$. □

Lemma 10. *Let $\Phi : S_n \to S_m$ $(n \geq 2)$ be a map preserving adjacency and $\Phi(0) = 0$. Let $max\{rank\ \Phi(A)|A \in GL(n)\} = k$. If $k \geq 2$ and for every singular $A \in S_n$ we have rank $\Phi(A) < k$, then rank $\Phi(B) = k$ for every invertible $B \in S_n$.*

Proof. Let $A, B \in S_n \cap GL(n)$ with $A \neq B$ and let rank $\Phi(A) = k$. By Lemma 8, there exists a natural number r and invertible matrices $A = A_0, A_1, \ldots, A_r = B$ such that the neighbours in this sequence are adjacent and for $j = 0, \ldots, r-1$ there is a matrix $C_j \in l(A_j, A_{j+1})$ with rank $C_j = n - 1$. Hence rank $\Phi(C_j) < k$. Now rank $\Phi(A) = k$, rank $\Phi(A_1) \leq k$, rank $\Phi(C_0) < k$. Lemma 2 (for $G = 0$) tells us that $\Phi(C_0)$ is the only point on the line $l(\Phi(A), \Phi(A_1))$ with rank less than k. Since $C_0 \neq A_1$, Lemma 9 tells us that $\Phi(C_0) \neq \Phi(A_1)$. So rank $\Phi(A_1) = k$. Proceeding in this way we find rank $\Phi(A_j) = k$ for all j, so rank $\Phi(B) = k$. □

Lemma 11. *Let $\Phi : S_n \to S_m$ $(n \geq 2)$ be a map preserving adjacency. If there are $A, B \in S_n$ with $d(\Phi(A), \Phi(B)) = n$, then $d(\Phi(X), \Phi(Y)) = d(X, Y)$ for all $X, Y \in S_n$ and Φ is injective.*

Proof. For $n = m$ this was proved (in even greater generality) by Wen-ling Huang (Corollary 3.1 in [9]).

We know that $d(X, Y) = k \geq 1$ implies the existence of a sequence $X = X_0, X_1, \ldots, X_k = Y$ of consecutively adjacent matrices. If $\Psi : S_n \to S_m$ is adjacency preserving, the neighbours in the sequence $\Psi(X_0), \Psi(X_1), \ldots, \Psi(X_k)$ are also adjacent and therefore $d(\Psi(X), \Psi(Y)) \leq k$. So

$$d(\Psi(X), \Psi(Y)) \leq d(X, Y)$$

for any adjacency preserving map Ψ.

Now the map Ψ, defined by $\Psi(X) = \Phi(X + A) - \Phi(A)$ for $X \in S_n$ is adjacency preserving by Proposition 1 and $\Psi(0) = 0$. We note that rank $(\Psi(B - A)) = d(\Phi(B), \Phi(A)) = n$.

If $Z \in S_n$ is singular, rank $(\Psi(Z)) = d(\Psi(Z), \Psi(0)) \leq d(Z, 0) =$ rank $Z \leq n - 1$. Lemma 10 tells us that rank $(\Psi(X)) = n$ for every $X \in S_n \cap GL(n)$. In particular, if $d(C, A) = n$, i.e. rank $(C - A) = n$, then $n =$ rank $(\Psi(C - A)) =$ rank $(\Phi(C) - \Phi(A)) = d(\Phi(C), \Phi(A))$.

Let $X, Y \in S_n$ be such that $d(X, Y) = n$. For λ large enough, $d(\lambda I, A) =$ rank $(\lambda I - A) = n$ and $d(\lambda I, X) = n$. If we set $C = \lambda I$ above, we see $d(\Phi(\lambda I)\, \Phi(A)) = n$. We may substitute λI for A, A for B in the previous argument and get $d(\Phi(\lambda I), \Phi(X)) = n$. Repeating this procedure we get $d(\Phi(X), \Phi(Y)) = n$.

We have proven that $d(X, Y) = n$ implies $d(\Phi(X), \Phi(Y)) = n$. Suppose now $d(Z, W) =$ rank $(Z - W) = k < n$, with $k \geq 1$. There is U orthogonal such that $Z - W = U(\lambda_1 E_{11} + \ldots + \lambda_k E_{kk})U^T$, with $\lambda_1, \ldots, \lambda_k$ nonzero. Let $G = W - U(E_{k+1,k+1} + \ldots + E_{nn})U^T$. Then $d(G, W) =$ rank $(G - W) = n - k$ and

$(Z-W)+(W-G)=Z-G$ is invertible. Since Φ does not increase the metric d, $n = d(Z,G) = d(Z,W)+d(W,G) \geq d(\Phi(Z),\Phi(W))+d(\Phi(W),\Phi(G)) \geq d(\Phi(Z),\Phi(G)) = n$. So $d(Z,W) = d(\Phi(Z),\Phi(W))$.

If $\Phi(X) = \Phi(Y)$ and $X \neq Y$, then $d(X,Y) \geq 1$, so $d(\Phi(X),\Phi(Y)) \geq 1$ – a contradiction. □

Lemma 12. *Let $m > n \geq 2$ and let $A_1, B_1 \in S_n$ with $A_1 \neq B_1$. If $A, B \in S_m$ are such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and C is adjacent to both A and B, then there is $C_1 \in S_n$ such that

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. The matrices $A-C$ and $C-B$ have rank one. So $A-B = (A-C)+(C-B)$ has rank one or two. If A is adjacent to B, then C lies one the line $l(A,B)$, so $C = A + \lambda(B-A)$ has the desired form.

If $A-B$ has rank two, then Im $(A-B) =$ Im $(A-C) \oplus$ Im $(C-B)$ by Lemma 1. So Im $(A-C) \subset$ Im $(A-B)$ and $C = A-(A-C)$ has the desired form. □

Lemma 13. *Let $m > n \geq 2$ and let $\Phi : S_n \to S_m$ be an adjacency preserving map with $\Phi(0) = 0$. Let*

$$\Phi(I) = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$$

where $K \in S_n$ has rank n. Then for all $A \in S_n$,

$$\Phi(A) = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A_1 \in S_n$.

Proof. Since $n = d(\Phi(I),\Phi(0))$, Lemma 11 tells us that d preserves the distance. Suppose $P \in S_n$ is a projection of rank one. Then $d(0,P) = 1$, $d(I,P) = n-1$, so $d(\Phi(I),\Phi(P)) = n-1$ and $d(0,\Phi(P)) = 1$. Thus

$$n = \text{rank } \Phi(I) = \text{rank } \Phi(P) + \text{rank } (\Phi(I) - \Phi(P)).$$

By Lemma 1, Im $\Phi(I) =$ Im $\Phi(P) \oplus$ Im $(\Phi(I)-\Phi(P))$, so Im $\Phi(P) \subset$ Im $\Phi(I)$ and $\Phi(P)$ has the desired form.

If $A = \lambda P$, then A lies on the line $l(0,P)$, so $\Phi(A)$ lies on the line $l(0,\Phi(P))$, so $\Phi(A) = \mu\Phi(P)$ has the desired form.

Now we use the induction on the rank of A. Suppose we have proved the lemma for all matrices of rank $k \geq 1$. Let rank $A = k+1$. There is U orthogonal and nonzero numbers $\lambda_1, \ldots, \lambda_{k+1}$ such that $A = U(\lambda_1 E_{11} + \ldots + \lambda_{k+1} E_{k+1,k+1})U^T$. The matrix A is adjacent to $B = U(\lambda_2 E_{22} + \ldots + \lambda_{k+1} E_{k+1,k+1})U^T$ and to $C = U(\lambda_1 E_{11} + \ldots + \lambda_k E_{kk})U^T$. So $\Phi(A)$ is adjacent to

$$\Phi(B) = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \Phi(C) = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $B_1, C_1 \in S_n$ and $B_1 \neq C_1$. By Lemma 11, $\Phi(B) \neq \Phi(C)$. We use Lemma 12. □

4. Adjacent matrices in S_2

Lemma 14. *Let $\Phi : S_2 \to S_2$ be a map such that A is adjacent to B iff $\Phi(A)$ is adjacent to $\Phi(B)$. Then Φ is injective.*

Proof. If there are $A, B \in S_2$ such that $d(\Phi(A), \Phi(B)) = 2$, then, by Lemma 11, Φ is injective.

Suppose now that $d(\Phi(X), \Phi(Y)) \leq 1$ for all $X, Y \in S_2$. We will show this is impossible. Since E_{11} and E_{22} are not adjacent, $\Phi(E_{11})$ and $\Phi(E_{22})$ are not adjacent. Therefore $\Phi(E_{11}) = \Phi(E_{22})$. Similarly, $\Phi(2E_{11}) = \Phi(E_{22}) = \Phi(E_{11})$. On the other hand, E_{11} is adjacent to $2E_{11}$, so $\Phi(E_{11})$ is adjacent to $\Phi(2E_{11}) = \Phi(E_{11})$ – a contradiction. □

We denote by Q the quadratic form on $\mathbb{R}^n$, defined by

$$Q(x) = x_n^2 - x_1^2 - x_2^2 - \ldots - x_{n-1}^2.$$

Then $Q(x-y)$ is the **Lorentz separation** of x and y. A bijective linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ is a **Lorentz transformation** if $Q(Lx) = Q(x)$ for all $x \in \mathbb{R}^n$. All Lorentz transformations on $\mathbb{R}^n$ form the **Lorentz group**. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **Weyl transformation** if there are: $\alpha \in \mathbb{R}\setminus\{0\}$, a Lorentz transformation L and $b \in \mathbb{R}^n$ such that $f(x) = \alpha Lx + b$ for all $x \in \mathbb{R}^n$.

The following theorem is due to Alexandrov [1]. We quote it from Lester [11] p. 929, who rediscovered it.

Theorem 2. *(cf. Theorem 3.1 in [7]). Let D be an open connected subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $f : D \to \mathbb{R}^n$ be an injective mapping such that $Q(x-y) = 0$ iff $Q(f(x) - f(y)) = 0$. Then f is the restriction of a conformal mapping.*

Any conformal mapping on $\mathbb{R}^n$ is a Weyl transformation (see [11], p. 929 or [14], pp. 132-133) and that is all we will need:

Corollary 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 3)$ be an injective mapping such that $Q(x-y) = 0$ iff $Q(f(x) - f(y)) = 0$.*
Then f is a Weyl transformation.

We have the linear bijection $T : \mathbb{R}^3 \to S_2$, defined by

$$Tx = \begin{bmatrix} x_3 + x_1 & x_2 \\ x_2 & x_3 - x_1 \end{bmatrix}.$$

Now $\det(Tx - Ty) = \det(T(x-y)) = Q(x-y)$. Therefore:

$$Tx \text{ is adjacent to } Ty \text{ iff } x \neq y \text{ and } Q(x-y) = 0. \tag{1}$$

The following, including Propositions 2 and 3, is reconstructed from the book [16] on Hyperbolic geometry by Ramsey and Richtmyer, pp. 246-250, and the book [12] on the Lorentz group by Paërl, pp. 5-7.

Let $J = -E_{11} - E_{22} + E_{33}$. Then for $x \in \mathbb{R}^3$ and $L \in M_3$ we have $Q(x) =< Jx, x >$ and $Q(Lx) =< JLx, Lx >=< L^T JLx, x >$. So L is a Lorentz transformation iff $L^T JL = J$ and this implies $\det L = \pm 1$. A Lorentz matrix $L = [l_{ij}] \in M_3$ maps the vector $(0,0,1)^T$ into $(l_{13}, l_{23}, l_{33})^T$ and so $l_{33}^2 - l_{13}^2 - l_{23}^2 = 1$, thus $l_{33}^2 \geq 1$. By definition, a Lorentz matrix $L = [l_{ij}] \in M_3$ is a **restricted Lorentz matrix** if $\det L = 1$ and $l_{33} \geq 1$.

Proposition 2. *If $L \in M_3$ is a restricted Lorentz matrix, then there is a matrix $P_1 \in M_2$ with $\det P_1 = 1$ such that*

$$T(Lx) = P_1(Tx)P_1^T$$

for all $x \in \mathbb{R}^3$.

Proof. For $b \in \mathbb{R}$ we have the restricted Lorentz matrices

$$H(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh b & \sinh b \\ 0 & \sinh b & \cosh b \end{pmatrix}$$

and

$$R(b) = \begin{pmatrix} \cos b & -\sin b & 0 \\ \sin b & \cos b & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $L = [l_{ij}]$ be a restricted Lorentz matrix. We find $b \in \mathbb{R}$ such that $l_{33} = \cosh b$. So $H(b)((0,0,1)^T) = (0, \sinh b, \cosh b)^T$. Since

$$(\sinh b)^2 = (\cosh b)^2 - 1 = {l_{33}}^2 - 1 = {l_{13}}^2 + {l_{23}}^2,$$

there is $\alpha \in \mathbb{R}$ such that

$$R(\alpha)H(b)((0,0,1)^T)) = (l_{13}, l_{23}, l_{33})^T = L((0,0,1)^T).$$

If $U = H(b)^{-1}R(\alpha)^{-1}L$, then U is a Lorentz matrix with $\det L = 1$ and $U((0,0,1)^T) = (0,0,1)^T$. We claim that $u_{31} = u_{32} = 0$. In fact, since $U((1,0,0)^T) = (u_{11}, u_{21}, u_{31})^T$, we have ${u_{31}}^2 - {u_{21}}^2 - {u_{11}}^2 = -1$. But

$$U((1,0,1)^T) = U((1,0,0)^T) + U((0,0,1)^T) = (u_{11}, u_{21}, u_{31} + 1)^T,$$

so $(u_{31} + 1)^2 = {u_{31}}^2 + 1$ and thus $u_{31} = 0$. We observe that U, restricted to the xy-plane, is a linear isometry. Since $\det U = 1$, $U = R(\gamma)$ for some γ. Thus

$$L = R(\alpha)H(b)R(\gamma).$$

Let

$$W(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

and

$$V(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}.$$

Then $W(t)(Tx)W(t)^T = T(H(2t)x)$ and $V(s)(Tx)V(s)^T = T(R(2s)x)$ for all $x \in \mathbb{R}^3$. If $P_1 = V(\frac{\alpha}{2})W(\frac{b}{2})V(\frac{\gamma}{2})$, then $T(Lx) = P_1(Tx)P_1^T$.

□

Proposition 3. *For any Lorentz matrix $L \in M_3$ there is $c_1 \in \{-1, 1\}$ and a matrix $P \in M_2$ with $|\det P| = 1$, such that*

$$T(Lx) = c_1 P(Tx)P^T$$

for all $x \in \mathbb{R}^3$.

Proof. We note that $K = -E_{11} + E_{22} + E_{33} = K^{-1}$ is a Lorentz matrix with $\det K = -1$. For $B = E_{12} + E_{21} \in S_2$ we have $\det B = -1$ and $T(Kx) = B(Tx)B^T$.

If $L \in M_3$ is any Lorentz matrix and $r \in \{-1, 1\}$, then $\det(rL) = r^3 \det L = r \det L$ and $\det(rLK) = -r \det L$. Thus there is $r \in \{-1, 1\}$ such that rL or rLK is a restricted Lorentz matrix.

If rL is a restricted Lorentz matrix, then by Proposition 2 there is a matrix $P_1 \in M_2$ with $\det P_1 = 1$ such that

$$rT(Lx) = T(rLx) = P_1(Tx)P_1^T.$$

If rLK is a restricted Lorentz matrix, then by Proposition 2 there is a matrix $P_1 \in M_2$ with $\det P_1 = 1$ such that

$$rT(Lx) = T(rLx) = T(rLK(Kx)) = P_1(T(Kx))P_1^T = P_1(B(Tx)B^T)P_1^T,$$

where $\det(P_1 B) = -1$.

□

Corollary 3. *Let $\Phi : S_2 \to S_2$ be a map such that A is adjacent to B iff $\Phi(A)$ is adjacent to $\Phi(B)$. Then there exist $c \in \{-1, 1\}$, $R \in GL(2)$ such that*

$$\Phi(A) = cRAR^T + \Phi(0) \quad (A \in S_2).$$

Proof. We consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$f(x) = T^{-1}\Phi(Tx),$$

where $T : \mathbb{R}^3 \to S_2$ is the linear bijection defined above. By Lemma 14, Φ is injective and so f is injective. If $x \neq y$ and $Q(x - y) = 0$, then by (1) Tx is adjacent to Ty, so $\Phi(Tx) = Tf(x)$ is adjacent to $\Phi(Ty) = Tf(y)$. Since $f(x) \neq f(y)$, $Q(f(x) - f(y)) = 0$.

If $f(x) \neq f(y)$ and $Q(f(x) - f(y)) = 0$, then by (1) $Tf(x) = \Phi(Tx)$ is adjacent to $Tf(y) = \Phi(Ty)$, so Tx is adjacent to Ty and $Q(x - y) = 0$. If $f(x) = f(y)$, then $x = y$.

We see that $Q(x - y) = 0$ iff $Q(f(x) - f(y)) = 0$. By Corollary 2, there exist $\alpha \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^3$ and a Lorentz matrix $L \in GL(3)$ such that $f(x) = \alpha Lx + b$ for all $x \in \mathbb{R}^3$, hence

$$\Phi(Tx) = Tf(x) = \alpha T(Lx) + Tb.$$

By Proposition 3, there are $c_1 \in \{-1, 1\}$ and $P \in GL(2)$ such that

$$\Phi(Tx) = \alpha c_1 P(Tx)P^T + Tb,$$

i.e.

$$\Phi(A) = cRAR^T + \Phi(0)$$

for $A \in S_2$, where $c \in \{-1, 1\}$ and $R \in GL(2)$.

(Alternatively, instead of using Proposition 3, we could assume $\Phi(0) = 0$. Since $\Phi(Tx) = Tf(x) = \alpha T(Lx)$ is a linear map, we have a linear preserver problem and we could use Theorem 2.2.1 in [26].) □

Proposition 4. *Let $\Phi : S_2 \to S_2$ be an adjacency preserving mapping. Suppose $d(\Phi(G), \Phi(H)) = 2$ for some $G, H \in S_2$. Then there are $c \in \{-1, 1\}$, $R \in GL(2)$ and $S \in S_2$ such that*

$$\Phi(A) = cRAR^T + \Phi(0).$$

Proof. By Lemma 11, $d(\Phi(X), \Phi(Y)) = d(X, Y)$ for all $X, Y \in S_2$. So $\Phi(X)$ is adjacent to $\Phi(Y)$ iff X is adjacent to Y. We use Corollary 3. □

5. Proof of Theorem 1.4

Let $n \geq 2$ and let $\Phi : S_n \to S_m$ be a mapping preserving adjacency, $\Phi(0) = 0$. Theorem 1 claims that Φ is either a degenerate or a standard map.

Lemma 15. *Theorem 1 is true if $n = 2$.*

Proof. If $m = 1$, Φ is a degenerate map. Let $m \geq 2$. We consider two cases.

Case I. Let $d(\Phi(A), \Phi(B)) \leq 1$ for all A, B.

Then rank $\Phi(A) \leq 1$ for all A. Since E_{11} is adjacent to 0, $\Phi(E_{11})$ is adjacent to $\Phi(0) = 0$, so rank $\Phi(E_{11}) = 1$. Let $A \in S_2$. Then $d(\Phi(A), \Phi(E_{11})) \leq 1$. So $\Phi(A) = \Phi(E_{11})$ or $\Phi(A)$ is adjacent to $\Phi(E_{11})$. In the latter case, if $\Phi(A) \neq 0$, then $\Phi(A)$ is adjacent to 0, so $\Phi(A) \in l(0, \Phi(E_{11}))$, thus $\Phi(A) = \lambda\Phi(E_{11})$. So $\Phi(A) = \lambda\Phi(E_{11})$ in any case. Thus Φ is a degenerate map.

Case II. We have $A, B \in S_2$ such that $d(\Phi(A), \Phi(B)) = 2$.

If $m = 2$, then Proposition 4 ends the proof. Let $m > 2$. By Lemma 11, Φ preserves the distance and it is injective. So $d(\Phi(I), 0) = 2 = \text{rank}\,\Phi(I)$. Since $\Phi(I) \in S_m$, there is $U \in M_m$ orthogonal such that

$$U\Phi(I)U^T = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Let $\Psi(A) = U\Phi(A)U^T$ for $A \in S_2$. Then Ψ is distance preserving and $\Psi(0) = 0$. By Lemma 13,

$$\Psi(A) = \begin{bmatrix} \Psi_1(A) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Psi_1(A) \in S_2$ and $\Psi_1(0) = 0$.

Obviously, $d(\Psi(A), \Psi(B)) = d(\Psi_1(A), \Psi_1(B)) = d(A, B)$. By Corollary 3, there are $c \in \{-1, 1\}$ and $R \in GL(2)$ such that $\Psi_1(A) = cRAR^T$. Let

$$W = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \in GL(m).$$

Then

$$\Psi(A) = cW \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} W^T$$

and

$$\Phi(A) = cU^T W \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} (U^T W)^T.$$

□

Lemma 16. *Let $n \geq 2$ and let $\Phi : S_n \to S_m$ be a map preserving adjacency, with $\Phi(0) = 0$. Let*

$$\Phi(I) = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in S_m$$

where $I_n \in M_n$ is the identity matrix. Then we can find $U \in M_n$ orthogonal such that for all $A \in S_n$ we have

$$\Phi(A) = \begin{bmatrix} UAU^T & 0 \\ 0 & 0 \end{bmatrix}.$$

The proof is almost the same as the proof of Step 4.2 in [7].

Lemma 17. *Let $\Phi : S_n \to S_m$ $(m, n \geq 3)$ be an adjacency preserving map and $\Phi(0) = 0$. Suppose that for every projection $P \in S_n$ with rank $P = n - 1$ there is a rank one projection Q such that $\Phi(PS_nP) \subset \mathbb{R}Q$. Then Φ is a degenerate adjacency preserving map.*

The proof is almost the same as the proof of Step 4.3 in [7].

Lemma 18. *Let $\Phi : S_n \to S_m$ $(m, n \geq 3)$ be an adjacency preserving map with $\Phi(0) = 0$. Assume that for every projection P with rank $P = n - 1$ the restriction of Φ to PS_nP is a standard map. Then Φ is a standard adjacency preserving map.*

The proof is practically the same as the proof of Step 4.4 in [7].

Lemma 19. *The statement of Theorem 1 is true for $n = 3$.*

Proof. Although the proof follows Step 4.5 in [7], in our case the calculation at the end is simpler. Let $P \in S_3$ be any projection of rank 2. By Lemma 15, the mapping Φ restricted to PS_3P is either standard or degenerate. If Φ restricted to PS_3P is degenerate for all projections $P \in S_3$ of rank 2, Lemma 17 tells us that Φ is degenerate. If Φ restricted to PS_3P is standard for all such P, then Lemma 18 tells us that Φ is a standard map.

Suppose there exist two projections P and Q of rank 2 such that Φ restricted to PS_3P is degenerate and Φ restricted to QS_3Q is standard. We will show this is impossible. As in Step 4.5 in [7] we may assume that

- $Q = E_{11} + E_{22} = E_2$;
- Φ is linear;

- for $A \in S_2$ we have
$$\Phi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = A = h(A)$$
for $m = 2$ or
$$\Phi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = h(A) \in S_m$$
for $m \geq 3$;
- $\Phi(E_{33}) = E_{22}$.

If P_1 is a rank two projection in S_3, then, by Lemma 15, Φ restricted to $P_1S_3P_1$ is either standard or degenerate. Now $\Phi(E_{22} + E_{33}) = 2E_{22}$, so Φ restricted to $(E_{22} + E_{33})S_2(E_{22} + E_{33})$ cannot be standard and is therefore degenerate. So

$$\Phi(E_{23} + E_{32}) = \alpha E_{22},$$

with $\alpha \neq 0$.

Since $\Phi(E_{11} + E_{33}) = E_{11} + E_{22}$ has rank two, the restriction of Φ to $(E_{11} + E_{33})S_3(E_{11} + E_{33})$ is a standard map. As before, there are $c_1 \in \{-1, 1\}$ and $W_1 \in GL(m)$ such that for $A \in (E_{11} + E_{33})S_3(E_{11} + E_{33})$ we have

$$\Phi(A) = c_1 W_1 h(A) W_1^T.$$

But $\Phi(E_{11}) = E_{11}$ and $\Phi(E_{33}) = E_{22}$. So

$$c_1 W_1(e_1 \otimes e_1)W_1^T = c_1(W_1 e_1) \otimes (W_1 e_1) = e_1 \otimes e_1.$$

This implies $c_1 = 1$ and $We_1 = \pm e_1$. By exchanging W with $-W$ if necessary we may assume $We_1 = e_1$. Similarly, $We_3 = de_2$, where $d \in \{-1, 1\}$. This implies

$$\Phi(E_{13}+E_{31}) = W(e_1 \otimes e_3 + e_3 \otimes e_1)W^T = We_1 \otimes We_3 + We_3 \otimes We_1 = d(E_{12}+E_{21}).$$

Let $A = [1, 1, 1]^T [1, 1, 1] = E_{11}+E_{22}+E_{33}+(E_{12}+E_{21})+(E_{13}+E_{31})+(E_{23}+E_{32})$. Since A has rank 1, A is adjacent to 0, so $\Phi(A)$ is adjacent to 0 and has rank one. We calculate $\Phi(A) = E_{11}+(2+\alpha)E_{22}+(1+d)(E_{12}+E_{21})$ and $\det \Phi(A) = 2+\alpha-(1+d)^2 = \alpha - 2d = 0$, since $d^2 = 1$. So $\Phi(E_{23} + E_{32}) = \alpha E_{22} = 2dE_{22}$.

Let now $B = [0, d, -1]^T [0, d, -1] = d^2E_{22} + E_{33} - d(E_{23} + E_{32})$. Then B has rank one and is adjacent to 0. But $\Phi(B) = (1 - d^2)E_{22} = 0$ – a contradiction. □

We can end the proof of Theorem 1 as in [7].

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