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A note on $\delta\alpha - I - open\ sets$ and $semi^{*} - I - open\ sets$

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Abstract. In this paper, we investigated some properties of a $\delta\alpha - I - open\ set$ [6] and a $semi^{*} - I - open\ set$ [6] in ideal topological spaces. Moreover, the relationships of other related classes of sets are investigated. Also, a new decomposition of continuous functions is obtained by using $\delta - \beta - I - continuous$ and $S^{*} - continuous$ functions.
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1. Introduction and preliminaries

Ideals in topologial spaces have been considered since 1930. This topic has won its importance by Vaidyanathaswamy [13]. In this paper, we investigated some properties of a $\delta\alpha - I - open\ set$ [6] and a $semi^{*} - I - open\ set$ [6] . Moreover, the relationships of other related classes of sets are investigated. A new decomposition of continuous functions is obtained by using $\delta - \beta - I - continuous$ and $S^{*} - continuous$ functions.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [12] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called $\delta - open$ [12] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. The complement of a $\delta - open$ set is called $\delta - closed$. A point $x \in X$ is called a $\delta - cluster$ point of A if $Int(Cl(U)) \cap A \neq \varnothing$ for each open set U containing x. The set of all $\delta - cluster$ points of A is called the $\delta - closure$ of A and is denoted by $Cl_{\delta}(A)$. The $\delta - interior$ of A is the union of all regular open sets of X contained in A and it is denoted by $Int_{\delta}(A)$. A is $\delta - open$ if $Int_{\delta}(A) = A$. $\delta - open$ sets form a topology τ^{δ}.

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies: (i) $A \in I$ and $B \subset A$ implies $B \in I$, (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^{*}$: $P(X) \to P(X)$ called a local function [10] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^{*}(I) = \{x \in X : U \cap A \notin I\ for\ every\ U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. We simply write A^{*} instead of $A^{*}(I, \tau)$. X^{*} is often

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a proper subset of X. The hypothesis $X = X^*$ [7] is equivalent to the hypothesis $\tau \cap I = \varnothing$. For every ideal topological space, there exists a topology $\tau^*(I)$ or briefly τ^*, finer than τ, generated by $\beta(I,\tau) = \{U \setminus \mathrm{I} : U \in \tau \text{ and } \mathrm{I} \in I\}$, but in general $\beta(I,\tau)$ is not always a topology [8]. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. If I is an ideal on X, then (X,τ,I) is called an ideal topological space. A subset A of an ideal topological space (X,τ,I) is said to be $R_I - open$ [14] if $A = Int(Cl^*(A))$. A point x in an ideal space (X,τ,I) is called a $\delta_I - cluster$ point of A if $Int(Cl^*(U)) \cap A \neq \varnothing$ for each neighborhood U of x. The set of all $\delta_I - cluster$ points of A is called the $\delta_I - closure$ of A and will be denoted by $\delta Cl_I(A)$. A is said to be $\delta_I - closed$ [14] if $\delta Cl_I(A) = A$. The complement of a $\delta_I - closed$ set is called a $\delta_I - open$ set. $\delta_I - interior$ of A will be denoted by $\delta Int_I(A)$. $\delta_I - open$ sets form a topology $\tau^{\delta I}$. Then $\tau^{\delta} \subset \tau^{\delta I} \subset \tau$ holds.

Lemma 1 (See [8]). *Let (X,τ,I) be an ideal topological space and A, B subsets of X.*

(1) If $A \subset B$, then $A^* \subset B^*$.

(2) If $G \in \tau$, then $G \cap A^* \subset (G \cap A)^*$.

(3) $A^* = Cl(A^*) \subset Cl(A)$.

Lemma 2 (See [9]). *Let (X,τ,I) be an ideal topological space and A, B subsets of X such that $B \subset A$. Then $B^*(\tau_{|A}, I_{|A}) = B^*(\tau, I) \cap A$.*

Lemma 3 (See [6]). *Let A be a subset of a space (X,τ,I). Then*

(1) $\delta Cl_I(A) \cap U \subset \delta Cl_I(A \cap U)$, for any $\delta_I - open$ set U in X,

(2) $\delta Int_I(A \cup F) \subset \delta Int_I(A) \cup F$, for any $\delta_I - closed$ set F in X.

Proof. (1) For every $x \in X$, take $x \in \delta Cl_I(A) \cap U$. Then, for every $\delta_I - open$ *set* V containing x, $x \in V \cap U$ is $\delta_I - open$ [14] and hence $V \cap U \cap A \neq \varnothing$. This shows that $x \in \delta Cl_I(A \cap U)$. Therefore we get the result.

(2) It follows from (1). □

Lemma 4 (See [6]). *Let (X,τ,I) be an ideal space and A a subset of X.*

(1) If A is open, then $\delta Cl_I(A) = Cl(A)$,

(2) If A is closed, then $\delta Int_I(A) = Int(A)$.

Definition 1. *A subset A of an ideal topological space (X,τ,I) is called*

(1) $\alpha - open$ [1] *if $A \subset Int(Cl(Int(A)))$,*

(2) preopen [2] *if $A \subset Int(Cl(A))$,*

(3) semiopen [11] *if $A \subset Cl(Int(A))$,*

(4) $semi - I - open$ [5] *if $A \subset Cl^*(Int(A))$,*

(5) $semi^* - I - open$ [6] *if $A \subset Cl(\delta Int_I(A))$,*

(6) $semi^* - I - closed$ *if* $Int(\delta Cl_I(A)) \subset A$,

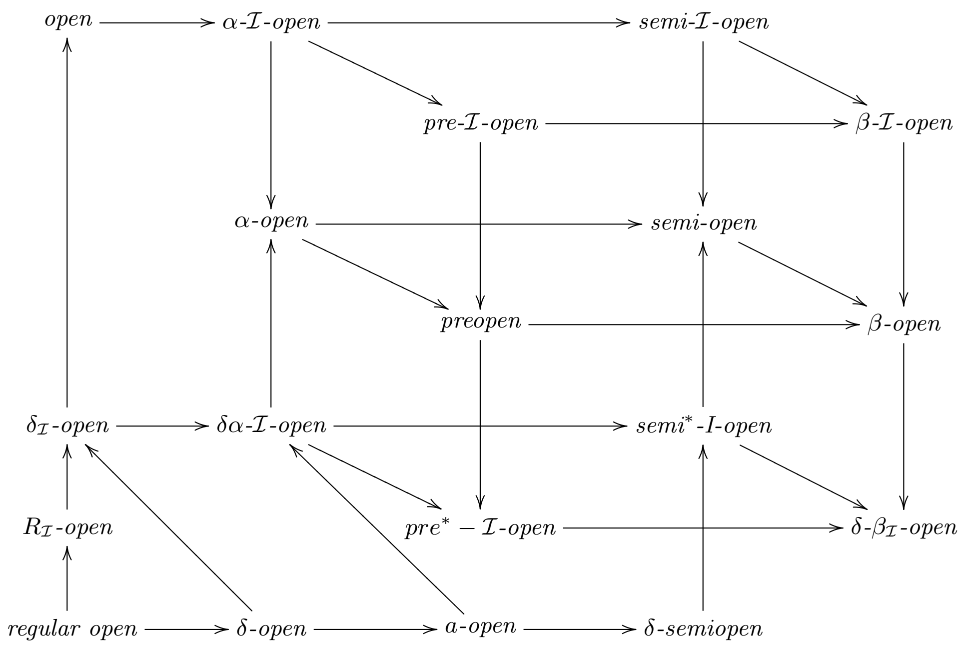
(7) $pre^* - I - open$ [3] *if* $A \subset Int(\delta Cl_I(A))$,

(8) $\delta\beta_I - open$ [6] *if* $A \subset Cl(Int(\delta Cl_I(A)))$,

(9) $\delta\beta_I - closed$ *if* $Int(Cl(\delta Int_I(A))) \subset A$,

(10) $\delta\alpha - I - open$ [6] *if* $A \subset Int(Cl(\delta Int_I(A)))$,

(11) $a - open$ [4] *if* $A \subset Int(Cl(Int_\delta(A)))$.



Diagram

The family of all $\delta\alpha - I - open$ (resp. $semi^* - I - open$, $pre^* - I - open$, $\delta\beta_I - open$) sets of (X, τ, I) is denoted by $\delta\alpha IO(X)$ (resp. $S^*IO(X)$, $P^*IO(X)$, $\delta\beta IO(X)$).

We denote the $\delta_I - boundary$ of A, $\delta_I - F_r(A) = \delta Cl_I(A) - \delta Int_I(A)$.

Theorem 1. *A subset* A *of an ideal topological space* (X, τ, I) *is* $semi^* - I - open$ *if and only if* $Cl(A) = Cl(\delta Int_I(A))$.

Proof. Let A be $semi^* - I - open$. Then we have $A \subset Cl(\delta Int_I(A))$ and therefore $Cl(A) \subset Cl(\delta Int_I(A))$ and hence $Cl(\delta Int_I(A)) \subset Cl(A)$ always hold. Then $Cl(A) = Cl(\delta Int_I(A))$.

Conversely, by $A \subset Cl(A) = Cl(\delta Int_I(A))$, A is $semi^* - I - open$. □

Theorem 2. *A subset A of an ideal topological space (X, τ, I) is $semi^* - I - open$ if and only if for every $\delta_I - open$ set U, $U \subset A \subset Cl(U)$.*

Proof. *Necessity:* suppose that A is $semi^* - I - open$, i.e., $A \subset Cl(\delta Int_I(A))$. If we take $U = \delta Int_I(A)$, we have $Cl(U) = Cl(\delta Int_I(A))$ and $U \subset A$. Thus we have $U \subset A \subset Cl(U)$.

Sufficiency: Suppose that $U \subset A \subset Cl(U)$, for every $\delta_I - open$ set U. If we take $U = A$, then A is $semi^* - I - open$. □

Theorem 3. *Let A be a subset of an ideal topological space (X, τ, I). The following are equivalent;*

(1) A is $semi^* - I - open$,

(2) A is $\delta\beta_I - open$ and $\delta Int_I(\delta_I - F_r(A)) = \varnothing$.

Proof. (1)⟹(2) Let A be $semi^* - I - open$. Then we have

$$Int(\delta Cl_I(A)) \subset \delta Cl_I(A) \subset Cl(\delta Int_I(A)),$$

(by $\delta Int_I(A)$ is also an open set and Lemma 4). Thus

$$\begin{aligned} \delta Int_I(\delta_I - F_r(A)) &= \delta Int_I(\delta Cl_I(A) \cap (X - \delta Int_I(A))) \\ &= \delta Int_I(\delta Cl_I(A)) - Cl(\delta Int_I(A)) \end{aligned}$$

and then $\delta Int_I(\delta_I - F_r(A)) = \varnothing$.

(2)⟹(1) Let A be $\delta\beta_I - open$ and $\delta Int_I(\delta_I - F_r(A)) = \varnothing$. Then

$$A \subset Cl(Int(\delta Cl_I(A))) \subset Cl(\delta Int_I(A)).$$

A is $semi^* - I - open$. □

Theorem 4. *Let (X, τ, I) be an ideal topological space. Then*

$$\delta\alpha IO(X) = S^*IO(X) \cap P^*IO(X).$$

Proof. Let $A \in \delta\alpha IO(X)$. Then $A \in S^*IO(X)$ and $A \in P^*IO(X)$.

Conversely, let $A \in S^*IO(X) \cap P^*IO(X)$. Then $A \in S^*IO(X)$ and $A \in P^*IO(X)$. Since $A \in S^*IO(X)$, by Theorem 3, $\delta Int_I(\delta_I - F_r(A)) = \varnothing$. Since

$$\delta Int_I(\delta_I - F_r(A)) = \delta Int_I(\delta Cl_I(A)) \cap \delta Int_I(X - \delta Int_I(A)),$$

then $Int(\delta Cl_I(A)) \subset Cl(\delta Int_I(A))$. Since $A \in P^*IO(X)$, we have

$$A \subset Int(\delta Cl_I(A)) \subset Int(Cl(\delta Int_I(A)))$$

and therefore, $A \in \delta\alpha IO(X)$. □

Theorem 5 (see [6])**.** *Let (X, τ, I) be an ideal topological space. Then, the family of $\delta\alpha - I - open$ sets is a topology for X.*

We denote this topology with $\tau^{\delta\alpha I}$.

Theorem 6. *Let A and B be subsets of an ideal topological space (X, τ, I). Then the following statements hold;*

(1) $A \in \tau^{\delta\alpha I}$ if and only if $V \subset A \subset Int(Cl(V))$, for every $\delta_I -$ open set V,

(2) If $A \in \tau^{\delta\alpha I}$ and $A \subset B \subset Int(Cl(A))$, then $B \in \tau^{\delta\alpha I}$.

Proof. (1) Straightforward.
(2) Since $A \in \tau^{\delta\alpha I}$, we have

$$\begin{aligned} B \subset Int(Cl(A)) &\subset Int(Cl(Int(Cl(\delta Int_I(A))))) \\ &\subset Int(Cl(\delta Int_I(A))) \subset Int(Cl(\delta Int_I(B))). \end{aligned}$$

Thus $B \in \tau^{\delta\alpha I}$. □

Theorem 7. *Let (X, τ, I) be an ideal topological space. If A is a semi$^* - I -$ open and pre$^* - I -$ open set, then $A \cap B$ is a $\delta\beta_I -$ open set.*

Proof. Let A be *semi*$^* - I -$*open*, i.e., $A \subset Cl(\delta Int_I(A))$ and B be *pre*$^* - I -$*open*, i.e., $B \subset Int(\delta Cl_I(B))$. Then

$$\begin{aligned} A \cap B &= Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(B)) \\ &= Cl(Int(\delta Int_I(A))) \cap Int(Int(\delta Cl_I(B))) \\ &\subset Cl(Int(\delta Int_I(A)) \cap Int(\delta Cl_I(B))) \\ &\subset Cl(Int(\delta Int_I(A) \cap \delta Cl_I(B))) \\ &\subset Cl(Int(\delta Cl_I(A \cap B))) \end{aligned}$$

□

Theorem 8. *Let (X, τ, I) be an ideal topological space. If A is a pre$^* - I -$ open and B is a $\delta\alpha - I -$ open set, then $A \cap B$ is a pre$^* - I -$ open set.*

Proof. Let A be *pre*$^* - I -$*open*, i.e., $A \subset Int(\delta Cl_I(A))$ and B $\delta\alpha - I -$*open*, i.e., $B \subset Int(Cl(\delta Int_I(B)))$. Then

$$\begin{aligned} A \cap B &= Int(\delta Cl_I(A)) \cap Int(Cl(\delta Int_I(B))) \\ &= Int(Int(\delta Cl_I(A)) \cap Cl(\delta Int_I(B))) \\ &\subset Int(Cl(\delta Cl_I(A) \cap \delta Int_I(B))) \\ &\subset Int(\delta Cl_I(\delta Cl_I(A \cap B))) = Int(\delta Cl_I(A \cap B)). \end{aligned}$$

□

Theorem 9. *Let (X, τ, I) be an ideal topological space. The following are equivalent;*

(1) The $\delta_I -$ closure of every $\delta_I -$ open subset of X is $\delta_I -$ open,

(2) $Cl(\delta Int_I(A)) \subset Int(\delta Cl_I(A))$ for every subset A of X,

(3) $S^*IO(X) \subset P^*IO(X)$,

(4) The $\delta_I - closure$ of every $\delta\beta_I - open$ subset is $\delta_I - open$,

(5) $\delta\beta IO(X) \subset P^*IO(X)$.

Proof. (1)$\Longrightarrow$(2) Suppose that $\delta_I - closure$ of every $\delta_I - open$ subset of X is $\delta_I - open$. Then the set $Cl(\delta Int_I(A))$ is $\delta_I - open$. Thus,

$$Cl(\delta Int_I(A)) = Int(Cl(\delta Int_I(A))) \subset Int(\delta Cl_I(A)).$$

(2)$\Longrightarrow$(3) Let $A \in S^*IO(X)$. By (2), we have

$$A \subset Cl(\delta Int_I(A)) \subset Int(\delta Cl_I(A)).$$

Thus, $A \in P^*IO(X)$.

(3)$\Longrightarrow$(4) Let $A \in \delta\beta IO(X)$. Then $\delta Cl_I(A)$ is $semi^* - I - open$. By (3), $\delta Cl_I(A)$ is $pre^* - I - open$. Hence $\delta Cl_I(A) \subset Int(\delta Cl_I(A))$ and therefore $\delta Cl_I(A)$ is $\delta_I - open$.

(4)$\Longrightarrow$(5) Let $A \in \delta\beta IO(X)$. By (4), $\delta Cl_I(A) = Int(\delta Cl_I(A))$. Hence $A \subset \delta Cl_I(A) = Int(\delta Cl_I(A))$ and therefore A is $pre^* - I - open$.

(5)$\Longrightarrow$(1) Let A be $\delta_I - open$. Then $\delta Cl_I(A)$ is $\delta\beta_I - open$. By (5), $\delta Cl_I(A)$ is $pre^* - I - open$. Hence $\delta Cl_I(A) \subset Int(\delta Cl_I(A))$ and therefore $\delta Cl_I(A)$ is $\delta_I - open$. □

Definition 2. *A subset A in an ideal topological space (X, τ, I) is called $\delta_I - dense$ if $\delta Cl_I(A) = X$.*

Theorem 10. *Let (X, τ, I) be an ideal topological space. The following are equivalent;*

(1) $P^*IO(X) \subset S^*IO(X)$,

(2) *Every $\delta_I - dense$ subset is $semi^* - I - open$,*

(3) *$\delta Int_I(A)$ is $\delta_I - dense$ for every $\delta_I - dense$ subset A,*

(4) *$\delta Int_I(\delta_I - F_r(A)) = \varnothing$ for every subset A,*

(5) $\delta\beta IO(X) \subset S^*IO(X)$,

(6) *$\delta Int_I(\delta_I - F_r(A)) = \varnothing$ for every subset $\delta_I - dense$ subset A.*

Proof. (1)$\Longrightarrow$(2) It follows that every $\delta_I - dense$ set is $pre^* - I - open$.

(2)$\Longrightarrow$(3) Let A be a $\delta_I - dense$ set. Then A is $semi^* - I - open$. Thus, $Cl(\delta Int_I(A)) \supset \delta Cl_I(A) = X$ and hence $\delta Int_I(A)$ is $\delta_I - dense$.

(3)$\Longrightarrow$(4) Let $A \subset X$. We have

$$X = \delta Cl_I(A) \cup (X - \delta Cl_I(A)) = \delta Cl_I(A) \cup \delta Int_I(X - A).$$

This implies that $A \cup \delta Int_I(X - A)$ is $\delta_I - dense$. Thus, $\delta Int_I(A \cup \delta Int_I(X - A))$ is $\delta_I - dense$.

$$\delta Int_I(A \cup \delta Int_I(X - A)) \cap \delta Int_I((X - A) \cup \delta Int_I(A)) = X - (\delta_I - F_r(A)).$$

Since $X - (\delta_I - F_r(A))$ is an intersection of two $\delta_I - dense$ $\delta_I - open$, then $X - (\delta_I - F_r(A))$ is $\delta_I - dense$.

(4)$\Longrightarrow$(6) Obvious.

(6)$\Longrightarrow$(3) Let A be $\delta_I - dense$. By (6),

$$\delta Int_I(\delta_I - F_r(A)) = \delta Int_I(X - \delta Int_I(A)) = X - Cl(\delta Int_I(A)) = \varnothing.$$

Thus, $\delta Int_I(A)$ is $\delta_I - dense$.

(4)$\Longrightarrow$(5) Let $A \in \delta\beta IO(X)$. By (4) and Theorem 3, A is $semi^* - I - open$.

(5)$\Longrightarrow$(1) Obvious. □

Theorem 11. *Let (X, τ, I) be an ideal topological space. The following are equivalent;*

(1) $P^*IO(X) \subset S^*IO(X)$,

(2) $Int(\delta Cl_I(A \cap B)) = Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(B))$ for every $A, B \subset X$,

(3) $Cl(\delta Int_I(A \cup B)) = Cl(\delta Int_I(A)) \cup Cl(\delta Int_I(B))$ for every $A, B \subset X$.

Proof. (1)$\Longrightarrow$(2) Let $P^*IO(X) \subset S^*IO(X)$ and $A, B \subset X$. By Theorem 10, $\delta Int_I(\delta_I - F_r(A)) = \varnothing$ for every subset A. Since

$$\begin{aligned}\delta Int_I(\delta_I - F_r(A)) &= \delta Int_I(\delta Cl_I(A) \cap (X - \delta Int_I(A)) \\ &= \delta Int_I(\delta Cl_I(A)) - \delta Cl_I(\delta Int_I(A)),\end{aligned}$$

$\delta Int_I(\delta Cl_I(A)) \subset \delta Cl_I(\delta Int_I(A))$ and therefore

$$\delta Int_I(\delta Cl_I(A)) = \delta Int_I(\delta Cl_I(\delta Int_I(A))).$$

This implies that

$$\begin{aligned}Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(B)) &= Int(Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(B))) \\ &\subset Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(B)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(B)) &\subset Cl(\delta Int_I(A) \cap Int(\delta Cl_I(B))) \\ &\subset Cl(\delta Int_I(A) \cap \delta Cl_I(B)) \subset \delta Cl_I(A \cap B).\end{aligned}$$

Since $Int(\delta Cl_I(A \cap B)) \subset Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(B))$, we have

$$Int(\delta Cl_I(A \cap B)) = Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(B)).$$

(2)$\Longrightarrow$(1) Suppose that (2) holds. Then

$$\begin{aligned}\delta Int_I(\delta_I - F_r(A)) &= \delta Int_I(\delta Cl_I(A) \cap \delta Cl_I(X - A)) \\ &\subset Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(X - A)) \\ &= Int(\delta Cl_I(A \cap (X - A))) = \varnothing.\end{aligned}$$

By Theorem 10, we have $P^*IO(X) \subset S^*IO(X)$.

(2)$\Longleftrightarrow$(3) Take complement. □

Theorem 12. *Let (X, τ, I) be an ideal topological space. The following are equivalent;*

(1) $P^*IO(X) \subset S^*IO(X)$ *and the* $\delta_I - closure$ *of every* $\delta_I - open$ *subset of* X *is* $\delta_I - open$,

(2) $Int(\delta Cl_I(A)) = Cl(\delta Int_I(A))$, *for every subset* A *in* X,

(3) $Cl(Int(\delta Cl_I(A))) = Int(Cl(\delta Int_I(A)))$, *for every subset* A *in* X,

(4) $\delta\beta IO(X) \subset \delta\alpha IO(X)$,

(5) $S^*IO(X) \subset \delta\alpha IO(X)$ *and* $P^*IO(X) \subset \delta\alpha IO(X)$,

(6) $P^*IO(X) = S^*IO(X)$,

(7) A *is* $semi^* - I - open$ *if and only if* $\delta Cl_I(A)$ *is* $\delta_I - open$.

Proof. (1)$\Longrightarrow$(2) It follows from Theorems 9 and 10.

(2)$\Longrightarrow$(3) Let $A \subset X$. Since $Int(\delta Cl_I(A)) = Cl(\delta Int_I(A))$ is $\delta_I - clopen$ ($\delta_I - open$ and $\delta_I - closed$), then $Cl(Int(\delta Cl_I(A))) = Int(Cl(\delta Int_I(A)))$.

(3)$\Longrightarrow$(4) Let $A \in \delta\beta IO(X)$. Then we have

$$A \subset Cl(Int(\delta Cl_I(A))) = Int(Cl(\delta Int_I(A))),$$

i.e, $A \in \delta\alpha IO(X)$.

(4)$\Longrightarrow$(5) and (5)$\Longrightarrow$(6) Straightforward.

(6)$\Longrightarrow$(7) Let A be $semi^* - I - open$. Then we have $\delta Cl_I(A)$ is $semi^* - I - open$ and therefore $pre^* - I - open$. Thus $\delta Cl_I(A) \subset Int(\delta Cl_I(A))$ and $\delta Cl_I(A)$ is $\delta_I - open$. Conversely, let $\delta Cl_I(A)$ be $\delta_I - open$. Therefore, we have $A \subset \delta Cl_I(A) = Int(\delta Cl_I(A))$, i.e., A is $pre^* - I - open$ and hence $semi^* - I - open$.

(7)$\Longrightarrow$(1) Let A be $\delta_I - open$. Then $\delta Cl_I(A)$ is $\delta_I - open$. Let A be a $\delta_I - dense$ set. Then $\delta Cl_I(A)$ is $\delta_I - open$. By hypothesis *(7)*, A is $semi^* - I - open$. Therefore, by Theorem 10, $P^*IO(X) \subset S^*IO(X)$. □

Theorem 13. *Let* A *be a subset of an ideal topological space* (X, τ, I). *Then the following are equivalent;*

(1) A *is regular open,*

(2) A *is* $\delta\alpha - I - open$ *and* $\delta\beta_I - closed$,

(3) A *is* $pre^* - I - open$ *and* $semi^* - I - closed$.

Proof. (1)$\Longrightarrow$(2) Straightforward.

(2)$\Longrightarrow$(1) Let A be a $\delta\alpha - I - open$ and $\delta\beta_I - closed$. Then we have $A = Int(Cl(\delta Int_I(A)))$. Hence A is regular open.

(1)$\Longleftrightarrow$(3) It follows from Theorem 2 in [6]. □

Lemma 5. *Let* (X, τ, I) *be an ideal topological space and* $(U, \tau_{|U}, I_{|U})$ *a subspace of* (X, τ, I).

(1) If A is open in $(U, \tau_{|U}, I_{|U})$, then $\delta Cl_{I|U}(A) = Cl_U(A)$, where $\delta Cl_{I|U}(A)$; $\delta_I -$ closure in $(U, \tau_{|U}, I_{|U})$.

(2) If A is closed in $(U, \tau_{|U}, I_{|U})$, then $\delta Int_{I|U}(A) = Int_U(A)$, where $\delta Int_{I|U}(A)$; $\delta_I - open$ in $(U, \tau_{|U}, I_{|U})$.

Proof. (1) Since every $\delta_I - open$ set in $(U, \tau_{|U}, I_{|U})$ is open in U, we have

$$Cl_U(A) \subset \delta Cl_{I|U}(A).$$

Conversely, let $x \notin Cl_U(A)$. Then there exists an open set V in $(U, \tau_{|U})$ containing x such that $V \cap A = \varnothing$. Since A is open in $(U, \tau_{|U})$, we have $A \cap Int_U(Cl_U(V)) = \varnothing$. By the fact that $Int_U(Cl^*_U(V)) \subset Int_U(Cl_U(V))$, we obtain $A \cap Int_U(Cl^*_U(V)) = \varnothing$. This implies that $x \notin \delta Cl_{I|U}(A)$. Thus $\delta Cl_{I|U}(A) = Cl_U(A)$.

(2) This follows from (1). □

Theorem 14. *If $A \in P^*IO(X)$ and $B \in S^*IO(X)$, then $A \cap B \in S^*IO(A)$.*

Proof. Let $B \in S^*IO(X)$. By Theorem 2, there exists a G $\delta_I - open$ set in X such that $G \subset B \subset Cl(G)$. From this it follows that $A \cap G \subset A \cap B \subset A \cap Cl(G)$. Since $A \in P^*IO(X)$, we have

$$\begin{aligned} A \cap G \subset A \cap B &\subset Int(\delta Cl_I(A)) \cap Cl(G) \\ &\subset Cl(\delta Cl_I(A) \cap G) \subset Cl(\delta Cl_I(A \cap G)), \quad \text{(Lemma 3)} \\ &\subset \delta Cl_I(\delta Cl_I(A \cap G))) = \delta Cl_I(A \cap G). \end{aligned}$$

Hence

$$(A \cap G) \cap A \subset (A \cap B) \cap A \subset \delta Cl_I(A \cap G) \cap A$$

implies that

$$A \cap G \subset A \cap B \subset \delta Cl_{I|A}(A \cap G) = Cl_A(A \cap G), \quad \text{(Lemma 5)}$$

Therefore, since $A \cap G$ is $\delta_I - open$ in A, $A \cap B \in S^*IO(A)$. □

Theorem 15. *If $A \in P^*IO(X)$ and $B \in S^*IO(X)$, then $A \cap B \in P^*IO(B)$.*

Proof.

$$\begin{aligned} B \cap A \subset B \cap Int(\delta Cl_I(A)) &= Int_B(B \cap Int(\delta Cl_I(A))) \\ &\subset Int_B(Cl(\delta Int_I(B)) \cap Int(\delta Cl_I(A))) \\ &\subset Int_B(Cl(\delta Int_I(B) \cap \delta Cl_I(A))) \\ &\subset Int_B(\delta Cl_I(\delta Cl_I(B \cap A))) = Int_B(\delta Cl_I(B \cap A)). \end{aligned}$$

So,

$$\begin{aligned} B \cap A \subset Int_B(\delta Cl_I(B \cap A)) \cap B &= Int_B(\delta Cl_I(B \cap A) \cap B) \\ &= Int_B(\delta Cl_{I|B}(B \cap A)). \end{aligned}$$

This implies that $A \cap B \in P^*IO(B)$. □

Definition 3. *A space* (X, τ) *is extremally disconnected* [15] *if the closure of every open set in* X *is open.*

Theorem 16. *If a space* (X, τ, I) *is extremally disconnected and* $A, B \in S^*IO(X)$*, then* $A \cap B \in S^*IO(X)$.

Proof. Let $A, B \in S^*IO(X)$. Then $A \cap B \subset Cl(\delta Int_I(A)) \cap Cl(\delta Int_I(B))$.
Extremal disconnectedness of X implies openness of

$$Cl(\delta Int_I(B)) = Cl(Int(\delta Int_I(B))).$$

Hence

$$\begin{aligned} A \cap B &\subset Cl(\delta Int_I(A)) \cap Cl(\delta Int_I(B)) \subset Cl(\delta Int_I(A) \cap Cl(\delta Int_I(B))) \\ &\subset Cl(Cl(\delta Int_I(A) \cap \delta Int_I(B))) = Cl(\delta Int_I(A \cap B). \end{aligned}$$

So, $A \cap B \in S^*IO(X)$. □

Remark 1. *The extremally disconnected condition of Theorem 16 cannot be dropped as shown in the following example.*

Example 1. *Let* $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ *and* $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. *Then* $A = \{a, c\}$, $B = \{b, c\} \in S^*IO(X)$, *but* $A \cap B = \{c\} \notin S^*IO(X)$ *because of* (X, τ, I) *is not an extremally disconnected space.*

2. S^*-*sets in ideal topological spaces and decomposition of continuity*

Definition 4. *A subset* A *in an ideal topological space* (X, τ, I) *is called an* $S^* - set$ *if* $A = U \cap V$, *where* U *is open and* V *is* $semi^* - I - closed$ *and*

$$Int(\delta Cl_I(V)) = Cl(\delta Int_I(V)).$$

The family of all $S^* - sets$ of an ideal topological space (X, τ, I) will be denoted by $S^*(X)$.

Definition 5. *(1) A subset* V *in an ideal topological space* (X, τ, I) *is called a* $strongly - t - I - set$ [3]*if* $Int(\delta Cl_I(V)) = Int(V)$.

(2) A subset A *in an ideal topological space* (X, τ, I) *is called a* $strongly\ B - I - set$ [3] *if* $A = U \cap V$, *where* U *is open and* V *is a* $strongly - t - I - set$.

Remark 2. *The notions of a* $semi^* - I - closed$ *set and a* $strongly - t - I - set$*are equivalent.*

Remark 3. *Every* $S^* - set$ *is a* $strongly\ B - I - set$, *but the converse is not true.*

Example 2. *Let* $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ *and* $I = \{\varnothing\}$. *Then* $\{a\}$ *is a* $strongly\ B - I - set$ *set, but it is not an* $S^* - set$ *since* $Int(\delta Cl_I(\{a\})) \neq Cl(\delta Int_I(\{a\}))$.

Theorem 17 (See [6]). *Let A be a subset of an ideal space (X, τ, I). Then*

$$s\delta Cl_I(A) = A \cup Int(\delta Cl_I(A)), \quad (s\delta Cl_I(A);\ a\ semi^* - I - closure\ of A)$$

Theorem 18. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is an $S^* - set$, then $A = U \cap s\delta Cl_I(A)$ for some open set U.*

Proof. Let $A \in S^*(X)$. Then $A = U \cap V$, where U is open and V is $semi^* - I - closed$ and $Int(\delta Cl_I(V)) = Cl(\delta Int_I(V))$. Since $A \subset V$, $s\delta Cl_I(A) \subset s\delta Cl_I(V) = V$. Therefore,

$$U \cap s\delta Cl_I(A) \subset U \cap V = A \subset U \cap s\delta Cl_I(A)$$

and hence the proof is completed. □

Definition 6. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then A is called a $\delta_{I_*} - set$ if $\delta Int_I(A)$ is $\delta_I - closed$.*

Theorem 19. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is a $\delta_{I_*} - set$ and $semi^* - I - open$, then it is $\delta_I - open$.*

Proof. Let A be a $\delta_{I_*} - set$ and $semi^* - I - open$. Then $A \subset Cl(\delta Int_I(A)) = \delta Int_I(A)$ and hence A is $\delta_I - open$. □

Theorem 20. *The following are equivalent for a subset A of an ideal topological space (X, τ, I).*

(1) A is open,

(2) A is $\alpha - open$ and $S^* - set$,

(3) A is preopen and $S^* - set$,

(4) A is $pre^* - I - open$ and $S^* - set$,

(5) A is $\delta\beta_I - open$ and $S^* - set$.

Proof. We prove only (5)⇒(1), other implications are obvious.

(5)⇒(1) Let A be a $\delta\beta_I - open$ and a $S^* - set$. Then we have $A \subset Cl(Int(\delta Cl_I(A)))$ and $A = U \cap V$, where U is *open* and V is $semi^* - I - closed$ and $Int(\delta Cl_I(V)) = Cl(\delta Int_I(V))$. Therefore, we obtain

$$\begin{aligned} A &= A \cap U \subset Cl(Int(\delta Cl_I(A))) \cap U \\ &= Cl(Int(\delta Cl_I(U \cap V))) \cap U \\ &\subset Cl(Int(\delta Cl_I(U))) \cap Cl(Int(\delta Cl_I(V))) \cap U \\ &= U \cap Cl(Int(\delta Cl_I(V))) = U \cap Cl(Cl(\delta Int_I(V))) \\ &= U \cap Cl(\delta Int_I(V)) = U \cap Int(\delta Cl_I(V)) = U \cap Int(V) \end{aligned}$$

and hence A is an *open* set. □

Definition 7. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called*

(1) $\alpha -$ *continuous* [1] *if* $f^{\ -1}(V)$ *is* $\alpha -$ *open for each* $V \in \sigma$,

(2) pre $-$ *continuous* [2] *if* $f^{\ -1}(V)$ *is preopen for each* $V \in \sigma$,

(3) $pre^{*} - I -$ *continuous* [3] *if* $f^{\ -1}(V)$ *is* $pre^{*} - I -$ *open for each* $V \in \sigma$,

(4) $\delta - \beta - I -$ *continuous* [6] *if* $f^{\ -1}(V)$ *is* $\delta\beta_I -$ *open for each* $V \in \sigma$,

(5) $S^{*} -$ *continuous if* $f^{\ -1}(V)$ *is an* $S^{*} -$ *set for each* $V \in \sigma$.

Now, we can give the decomposition of continuity.

Theorem 21. *The following are equivalent for a function* $f : (X, \tau, I) \rightarrow (Y, \sigma)$;

(1) f *is continuous,*

(2) f *is* $\alpha -$ *continuous and* $S^{*} -$ *continuous*,

(3) f *is pre* $-$ *continuous and* $S^{*} -$ *continuous*,

(4) f *is* $pre^{*} - I -$ *continuous and* $S^{*} -$ *continuous*,

(5) f *is* $\delta - \beta - I -$ *continuous and* $S^{*} -$ *continuous.*

Proof. It follows from Theorem 20. □

Remark 4. *By the following examples* $\delta - \beta - I -$ *continuity and* $S^{*} -$ *continuity are independent notions.*

Example 3. *Let* $X = \{a, b, c, d\}, \tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ *and* $I = \{\varnothing\}$ *and* $\sigma = \{\varnothing, X, \{a\}\}$. *Define a function* $f : (X, \tau, I) \rightarrow (Y, \sigma)$ *such that* $f(x) = x$. *Then* f *is* $\delta - \beta - I -$ *continuous, but it is not* $S^{*} -$ *continuous since* $\{a\} \in \delta\beta IO(X)$, *but* $\{a\} \notin S^{*}(X)$.

Example 4. *Let* $X = \{a, b, c, d\}, \tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ *and* $I = \{\varnothing\}$ *and* $\sigma = \{\varnothing, X, \{d\}\}$. *Define a function* $f : (X, \tau, I) \rightarrow (Y, \sigma)$ *such that* $f(x) = x$. *Then* f *is* $S^{*} -$ *continuous, but it is not* $\delta - \beta - I -$ *continuous since* $\{d\} \in S^{*}(X)$, *but* $\{d\} \notin \delta\beta IO(X)$.

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