MATHEMATICAL COMMUNICATIONS 447
Math. Commun. **16**(2011), 447–453.

Almost periodicity and discrete almost periodicity in semiflows

Alica Miller[1,*]

[1] *Department of Mathematics, University of Louisville, Louisville, KY, 40 292 USA*

Received July 6, 2010; accepted February 2, 2011

Abstract. A theory of semiflows with a discrete acting topological semigroup was developed in the 2000 paper by D. Ellis, R. Ellis and M. Nerurkar ([2]). A theory for the case of an arbitrary acting topological semigroup has still to be developed. This paper can be considered as the beginning of an attempt in that direction. We discuss almost periodicity and G-almost periodicity of points in a semiflow and end the paper with an adaptation to the semiflows of the 1998 theorem of Pestov for flows about independence of almost periodicity of a point upon the topology of the acting group ([6]).

AMS subject classifications: 37B05, 54H20

Key words: almost periodic, discretely almost periodic, semiflow, syndetic

1. Introduction and preliminaries

In our notation $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, 3, \dots\}$. All the notions or facts that we used but did not define or state in this paper can be found in [1, 4, 7].

A nonempty set T with an associative binary operation is called a *semigroup*. If in addition there is an identity element for the operation, then T is called a *unital semigroup*. (In the literature it is more often called a *monoid*.) The operation is usually denoted by $\cdot$, which is often omitted. The identity element is usually denoted by e. For any $a \in T$, the map $L_a : T \to T$, $t \mapsto at$ (resp. $R_a : T \to T$, $t \mapsto ta$) is called a *left translation* (resp. *right translation*) by the element a.

A *topological semigroup* (resp. *topological unital semigroup*) is a semigroup (resp. unital semigroup) T equipped with a Hausdorff topology such that the operation on T is jointly continuous.

Definition 1. *A subset A of a topological semigroup T is said to be syndetic (resp. discretely syndetic) if there is a compact (resp. finite) subset K of T such that $T = KA$.*

Definition 2. *A subset A of a topological semigroup T is said to be G-syndetic (resp. discretely G-syndetic) if there is a compact (resp. finite) subset K of T such that for every $t \in T$ we have $tK \cap A \neq \emptyset$. Equivalently, for every $t \in T$ there is a $k \in K$ such that $tk \in A$. In other words, $T = \cup_{k\in K} R_k^{-1}(A)$.*

*Corresponding author. *Email address:* a0mill01@louisville.edu (A. Miller)

http://www.mathos.hr/mc ©2011 Department of Mathematics, University of Osijek

In this definition the letter G stands for Gottschalk who was the first to investigate this notion (see [3]).

The following two examples illustrate that neither syndetic implies G-syndetic, nor the vice-versa.

Example 1. *Let* $T =< 2,5 >$ *be the discrete unital subsemigroup of* $(\mathbb{N},+)$*, generated by the elements* 2 *and* 5*. Hence* $T = \{0,2,4,5,6,\dots\}$*. Then the subset* $A = \{4,5,6,\dots\}$ *of* T *is G-syndetic, but not syndetic.*

Example 2. *Let* $T = [0,1]$ *with the topology induced from* $\mathbb{R}$ *and the operation* $xy = \max\{x,y\}$*. Then the subset* $A = [0,1)$ *is syndetic, but not G-syndetic since for* $t = 1$ *we cannot find any* $k \in [0,1]$ *such that* $tk \in A$*.*

Thus the notions of syndetic and G-syndetic subsets of a topological semigroup are independent of each other. Hence the notions (resp. statements) that depend on syndeticity can be defined (resp. stated) in two ways. Also if a definition or a statement uses syndeticity more than once, then a combination of these two notions can be used (see, for example, [3], Theorem 4).

Remark 1. *The above two types of syndeticity are also considered in my paper [5], where some other properties of topological semiflows are investigated.*

The next two propositions are illustrations of properties of G-syndetic subsets and subsemigroups in a topological semigroup. Analogous properties with G-syndetic replaced with syndetic also hold.

Proposition 1. *Let* T *be a topological semigroup,* A *a G-syndetic subset of* T*,* C *a compact subset of* T *and* B *a subset of* T *such that* $A \subset \cup_{c\in C} R_c^{-1}(B)$*. Then* B *is a G-syndetic subset of* T*.*

Proof. There is a compact subset K of T such that $T = \cup_{k\in K} R_k^{-1}(A)$. Hence $T = \cup_{k\in K}[R_k^{-1}(R_c^{-1}(B)] = \cup_{kc\in KC} R_{kc}^{-1}(B)$. Since KC is compact, B is syndetic. □

Proposition 2. *Let* T *be a topological semigroup,* S *a closed G-syndetic subsemigroup of* T*,* $A \subset S$*. If* A *is G-syndetic in* S*, it is G-syndetic in* T*.*

Proof. Let K be a compact subset of T such that $T = \cup_{k\in K} R_k^{-1}(S)$. For any $s \in S$, let $R'_s : S \to S$ be the map $x \mapsto xs$ from S to S. Suppose A is G-syndetic in S. Then there is a compact subset C of S such that $S = \cup_{c\in C} {R'}_c^{-1}(A)$. Hence

$$\begin{aligned} T &= \cup_{k\in K} R_k^{-1}(S) && (1) \\ &= \cup_{k\in K}[R_k^{-1}(\cup_{c\in C} {R'}_c^{-1}(A))] \\ &\subset \cup_{k\in K}[R_k^{-1}(\cup_{c\in C} R_c^{-1}(A))] && (2) \\ &= \cup_{kc\in KC} R_{kc}^{-1}(A). \end{aligned}$$

Since KC is compact, A is a G-syndetic subset of T. □

Question 1. *Is the opposite direction true? In the case of discrete semigroups, it is, as shown by Neil Hindman (personal communication).*

In the next two sections we will introduce the notion of a semiflow with an arbitrary topological semigroup acting, and the notion of almost periodic and G-almost periodic points in it. We will end the paper with an adaptation to semiflows of a theorem of Pestov for flows about independence of almost periodicity upon the topology of the acting group.

2. The notion of a semiflow

Definition 3. *Let X be a set and T a semigroup. A semigroup action of T on X is a mapping $\pi : T \times X \to X$, such that*

$$\pi(s, \pi(t, x)) = \pi(st, x)$$

for all $s, t \in T$, $x \in X$. If T is a unital semigroup, with identity e, then a unital semigroup action of T on X is a semigroup action $\pi : T \times X \to X$, which also satisfies

$$\pi(e, x) = x$$

for all $x \in X$. If T is a group, then a unital semigroup action of T on X is called a group action of T on X.

We usually write $t.x$ or tx instead of $\pi(t, x)$. The conditions from the above definition then have the following form:

$$s.(t.x) = (st).x,$$
$$e.x = x.$$

If $A \subset T$ and $Y \subset X$, we denote $AY = \{ay \mid a \in A,\ y \in Y\}$.

Definition 4. *A semiflow (resp. unital semiflow) is a triple (X, T, π), where X is a Hausdorff topological space, T is a Hausdorff topological semigroup (resp. Hausdorff topological unital semigroup) and $\pi : T \times X \to X$ is a jointly continuous semigroup (resp. unital semigroup) action of T on X. If (X, T, π) is a unital semiflow and T a topological group, then (X, T, π) is called a flow.*

Instead of (X, T, π) we will often write (X, T).

Example 3. *Let X be a Hausdorff topological space and $f : X \to X$ a continuous map. Then the semiflow (resp. unital semiflow) $(X, \mathbb{N}^*, \pi)$ (resp. $(X, \mathbb{N}, \pi)$), where $\pi(n, x) = f^n(x)$ for all $n \in \mathbb{N}^*$ (resp. $n \in \mathbb{N}$) and all $x \in X$, is denoted by (X, f). If f is a homeomorphism of X, the flow $(X, \mathbb{Z}, \pi)$, where $\pi(n, x) = f^n(x)$ for all $n \in \mathbb{Z}$, $x \in X$, is also denoted by (X, f). The phrases "semiflow (X, f)", "unital semiflow (X, f)" or "flow (X, f)" serve to indicate if the action is by the elements of $\mathbb{N}^*$, $\mathbb{N}$ or $\mathbb{Z}$, respectively.*

Definition 5. *Let (X, T, π) be a semiflow. For every $t \in T$ the map $\pi_t : X \to X$, defined by $\pi_t(x) = \pi(t, x)$ for every $x \in X$, is called the transition map defined by t. For every $x \in X$ the map $\pi^x : T \to X$, defined by $\pi^x(t) = \pi(t, x)$ for every $t \in T$, is called the orbital map defined by x.*

We say that a semiflow (X,T) is *compact* if X is compact and *abelian* if T is abelian.

Definition 6. *Let (X,T,π) be a semiflow (resp. unital semiflow).*

(i) Let $\emptyset \neq A \subset X$. We say that A is invariant if $TA \subset A$.

(ii) If $\emptyset \neq A$ is both closed and invariant, then the restriction $\pi : T \times A \to A$ defines a jointly continuous semigroup (resp. unital semigroup) action of T on A. The resulting semiflow (resp. unital semiflow) is called a subsemiflow (resp. unital subsemiflow) of the semiflow (resp. unital semiflow) (X,T).

Definition 7. *Given a semiflow (X,T), the orbit of a point $x \in X$ is the set*

$$Tx = \{tx \mid t \in T\}.$$

The set $\overline{Tx}$ is called the orbit-closure of x.

Note that if (X,T) is a unital semiflow, then $x \in Tx$. In the case of nonunital semiflows, it is not necessarily true that $x \in \overline{Tx}$.

Proposition 3. *Let (X,T) be a semiflow and $x \in X$. The set $\overline{Tx}$ is invariant.*

Proof. Let $y \in \overline{Tx}$ $(t \in T)$. Then $y = \lim_\alpha t_\alpha x$ for some net (t_α) in T and so $ty = t(\lim_\alpha t_\alpha x) = \lim_\alpha t(t_\alpha x) = \lim_\alpha (tt_\alpha)x$. Hence $ty \in \overline{Tx}$. □

Definition 8. *(i) The semiflow (X,T) is called minimal if there is no proper subset $M \subset X$ which is nonempty, closed and invariant. Equivalently, $\overline{Tx} = X$ for all $x \in X$.*

(ii) A subset M of X is called a minimal set of the semiflow (X,T) if it is nonempty closed invariant and there is no proper subset of M with these properties. Equivalently, M is nonempty closed invariant and the subsemiflow (M,T) is minimal. Another equivalent definition: M is nonempty and $\overline{Tx} = M$ for every $x \in M$. We simply say M is a minimal set if it is clear from the context which semiflow this is related to.

Question 2. *Let (X,T) be a semiflow, $M \subset X$. Consider the following conditions:*

(i) M is a minimal set of (X,T);

(ii) M is nonempty closed and $TU = M$ for every nonempty subset U of M which is open in M;

(ii') M is nonempty closed and $TU \supset M$ for every open subset U of X which contains at least one point of M.

It is easy to see that (ii) $\Leftrightarrow$ (ii'). Are these properties equivalent to (i)? If (X,T) is a flow, then they are (see [4], 2.12).

3. Almost periodicity and discrete almost periodicity in semiflows

Definition 9. *Let (X, T) be a semiflow, $x \in X$ and U a neighborhood of x. Then we define the dwelling set of x in U in the following way:*

$$D(x, U) = \{t \in T \mid tx \in U\}.$$

Definition 10. *Let (X, T) be a semiflow. A point $x \in X$ is called almost periodic (resp. discretely almost periodic) for (X, T) if for any neighborhood U of x the set $D(x, U)$ is syndetic (resp. discretely syndetic). Equivalently, for every neighborhood U of x there is a syndetic (resp. discretely syndetic) subset S of T such that $Sx \subset U$.*

Definition 11. *Let (X, T) be a semiflow. A point $x \in X$ is called G-almost periodic (resp. G-discretely almost periodic) for (X, T) if for any neighborhood U of x the set $D(x, U)$ is G-syndetic (resp. G-discretely syndetic). Equivalently, for every neighborhood U of x there is a G-syndetic (resp. G-discretely syndetic) subset S of T such that $Sx \subset U$.*

Proposition 4. *Let (T, X, π) be an abelian semiflow, $x \in X$. The following are equivalent:*

(i) x is a G-almost periodic point;

(ii) for every neighborhood U of x there is a compact subset K of T such that for every $t \in T$, $(t + K).x \cap U \neq \emptyset$;

(iii) for every neighborhood U of x there is a compact subset K of T such that $Tx \subset \cup_{k \in K} \pi_k^{-1}(U)$.

Proof. (i)$\Longrightarrow$(ii): Suppose x is almost periodic. Let U be a neighborhood of x. Let S be a syndetic subset of T such that $Sx \subset U$. Then there is a compact set $K \subset T$ such that for every $t \in T$ there is $k \in K$ with $t + k \in S$, i.e., such that $(t + K) \cap S \neq \emptyset$. Hence $(t + K)x \cap Sx \neq \emptyset$ and so $(t + K)x \cap U \neq \emptyset$.

(ii)$\Longrightarrow$(iii): Suppose that (ii) holds. Let U be a neighborhood of x and let K be the compact set from (ii). Then for every $t \in T$ there is $k \in K$ such that $(t+k)x \in U$, i.e., $k.(tx) \in U$ (since T is abelian), i.e., $tx \in \pi_k^{-1}(U)$. Hence $Tx \subset \cup_{k \in K} \pi_k^{-1}(U)$.

(iii)$\Longrightarrow$(i): Suppose that (iii) holds and let U be a neighborhood of x. Let K be the compact set from (iii). Then for every $t \in T$ there is $k \in K$ such that $tx \in \pi_k^{-1}(U)$, i.e., $k.(tx) \in U$, i.e., $(t + k)x \in U$, i.e., $t + k \in D(x, U)$. Hence $D(x, U)$ is syndetic. □

Remark 2. *Note that in the case of flows, the above conditions are equivalent with the condition*

(iii') for every neighborhood U of x there is a compact subset K of T such that $Tx \subset KU$.

Proposition 5. *Let (T, X) be a commutative semiflow and $x \in X$ an almost periodic point. Then every point tx $(t \in T)$ is almost periodic.*

Proof. Let U be a neighborhood of tx. Then $\pi_t^{-1}(U)$ is a neighborhood of x. Hence there is a syndetic subset S of T such that $Sx \subset \pi_t^{-1}(U)$. Then $\pi_t(Sx) \subset U$, i.e., $S(tx) \subset U$. Hence tx is almost periodic. □

The next theorem is an adaptation to semiflows of a theorem of Pestov (see [6]) for flows, which in turn was the answer to a question raised by de Vries in [7].

Theorem 1. *Let T be a topological unital semigroup whose left translations are open. Let (X, T, π) be a semiflow and $x \in X$. Then x is almost periodic if and only if it is discretely almost periodic.*

Proof. $\Longleftarrow$): clear.

$\Longrightarrow$): Suppose that x is almost periodic. Let U be a neighborhood of x. We will show that $D(x, U)$ is discretely syndetic.

Since the action π of T on X is continuous, then, using its continuity at the point (e, x), we conclude that there is a neighborhood O of e in T and U' of x in X such that $OU' \subset U$. Hence

$$O \cdot D(x, U') \subset D(x, U). \tag{3}$$

(Indeed, if $o \in O$ and $t \in D(x, U')$, then $tx \in U'$, hence $(ot)x = o(tx) \in OU' \subset U$, hence $ot \in D(x, U)$.)

Since x is almost periodic, there is a compact subset K of T such that

$$T = K \cdot D(x, U'). \tag{4}$$

Since the left translations of T are open, then for each $k \in K$, kO is an open set containing k. Hence $(kO)_{k \in K}$ is an open cover of K. Hence, since K is compact, there is a finite subcover:

$$K \subset FO, \tag{5}$$

where F is a finite subset of K.

Let $t \in T$. Then, by (4),

$$(\exists k \in K) \ \ (\exists t' \in D(x, U')) \ \ t = kt'. \tag{6}$$

By (5), there is $q \in F$ and $o \in O$, such that

$$k = qo. \tag{7}$$

From (6) and (7) we have

$$t = qot'. \tag{8}$$

By (3)

$$ot' \in D(x, U). \tag{9}$$

Hence, by (8) and (9),

$$t \in FD(x, U). \tag{10}$$

Hence x is discretely almost periodic. □

Question 3. *Is the previous theorem true if almost periodicity and discrete almost periodicity are replaced by G-almost periodicity and discrete G-almost periodicity respectively? (If X is assumed to be compact, then it is, see [3], Corollary 1 to Theorem 1.)*

Remark 3. *Some other statements about topological semiflows are proved in my paper [5] for syndetic subsemigroups of the acting topological semigroup, but I am not able to prove them if syndeticity is replaced by G-syndeticity. For example, see Theorem 2.3 in [5]. It is of interest to have both versions since in some contexts G-syndeticity is more natural for semigroups.*

References

[1] J. Auslander, *Minimal Flows and their Extensions*, North-Holland, Amsterdam, 1988.

[2] D. Ellis, R. Ellis, M. Nerurkar, *The topological dynamics of semigroup actions*, Trans. Amer. Math. Soc. **353**(2000), 1279–1320.

[3] W. Gottschalk, *Almost periodic points with respect to transformation semi-groups*, Ann. of Math. **47**(1946), 762–766.

[4] W. Gottschalk, G. Hedlund, *Topological Dynamics*, AMS Colloquium Publications, Providence, RI, 1955.

[5] A. Miller, *Envelopes of syndetic subsemigroups of the acting topological semigroup in a semiflow*, Topology Appl. **158**(2011), 291–297.

[6] V. Pestov, *Independence of almost periodicity upon the topology of the acting group*, Topology Appl. **90**(1998), 223–225.

[7] J. de Vries, *Elements of Topological Dynamics*, Kluwer Academic Publishers Group, Dordrecht, 1993.