MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 455–470.

Even number of symmetric positive solutions for the system of higher order boundary value problems on time scales

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Received July 1, 2010; accepted January 5, 2011

Abstract. In this paper, we establish at least two symmetric positive solutions for the system of higher order two-point boundary value problems on symmetric time scales by determining growth conditions and applying fixed point theorem in cone under suitable conditions. At the end of the paper, as an application, we demonstrate our results with examples.

AMS subject classifications: 39A10, 34B15, 34A40.

Key words: Boundary value problem, cone, symmetric positive solution, symmetric time scale.

1. Introduction

In this paper, we consider the system of higher order dynamical equations on symmetric time scales,

$$\begin{cases} y_1^{(\Delta\nabla)^n} + f_1(t, y_1, y_2) = 0, \ t \in [a, b]_{\mathbb{T}} \\ y_2^{(\Delta\nabla)^m} + f_2(t, y_1, y_2) = 0, \ t \in [a, b]_{\mathbb{T}} \end{cases} \tag{1}$$

subject to the two-point boundary conditions

$$\begin{cases} y_1^{(\Delta\nabla)^i}(a) = 0 = y_1^{(\Delta\nabla)^i}(b), \quad i = 0, 1, 2, ..., n-1, \\ y_2^{(\Delta\nabla)^j}(a) = 0 = y_2^{(\Delta\nabla)^j}(b), \quad j = 0, 1, 2, ..., m-1, \end{cases} \tag{2}$$

where $f_i : [a, b]_{\mathbb{T}} \times \mathbb{R}^2 \to [0, \infty)$ are continuous and $f_i(t, y_1, y_2) = f_i(a+b-t, y_1, y_2)$ for $i = 1, 2$, $a \in \mathbb{T}_k$, $b \in \mathbb{T}^k$ for a time scale $\mathbb{T}$, and also $\sigma(a) < \rho(b)$. By using the cone theory techniques, we establish sufficient conditions for the existence of at least two symmetric positive solutions to the BVP (1)-(2).

The development of the theory has gained attention by many researchers; to mention a few, Erbe and Wang [15], Eloe and Henderson [12, 13], Eloe, Henderson and Sheng [14], Henderson and Thompson [20], Avery and Henderson [3, 4, 5], Avery, Davis and Henderson [7], Davis and Henderson [10], Davis, Henderson and Wong

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[11], Anderson [2], Henderson and Wong [19], and Henderson, Murali and Prasad [18].

By an interval time scale, we mean the intersection of a real interval with a given time scale.

$$i.e.\ \ [a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}.$$

An interval time scale $\mathbb{T} = [a,b]_{\mathbb{T}}$ *is said to be a symmetric time scale if*

$$t \in \mathbb{T} \Leftrightarrow a+b-t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $(h > 0)$, then the symmetry definition is always satisfied. In addition, the interval time scale $\mathbb{T} = [1,2] \cup \{3,4,5\} \cup [6,7] \cup \{8\} \cup [9,10] \cup \{11,12,13\} \cup [14,15]$ has the symmetry property. But the time scale $\mathbb{T} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is not a symmetric time scale.

By a *symmetric solution* (y_1, y_2) of the BVP (1)-(2), we mean (y_1, y_2) is a solution of the BVP (1)-(2) and satisfies

$$y_1(t) = y_1(b+a-t) \text{ and } y_2(t) = y_2(b+a-t), \quad t \in [a,b]_{\mathbb{T}}.$$

This paper is organized as follows. In Section 2, we prove some lemmas and inequalities which are needed later. In Section 3, by using the cone theory techniques, we establish sufficient conditions for the existence of at least two symmetric positive solutions to the BVP (1)-(2). The main tool in this paper is an application of the Avery and Henderson's fixed point theorem for the operator leaving a Banach space cone invariant.

2. Green's function and bounds

In this section, we give some lemmas and inequalities for the Green's function of the homogeneous BVP corresponding to (1)-(2).

To obtain a solution $(y_1(t), y_2(t))$ of the BVP (1)-(2) we need the $G_p(t,s)$ (p is a positive integer) which is the Green's function of the BVP,

$$\begin{aligned} (-1)^p y^{(\Delta\nabla)^p} &= 0,\ t \in [a,b]_{\mathbb{T}} \\ y^{(\Delta\nabla)^i}(a) &= 0 = y^{(\Delta\nabla)^i}(b), \quad i = 0,1,...,p-1. \end{aligned}$$

By induction method, the Green's function $G_p(t,s)$ can be recursively expressed as

$$G_j(t,s) = \int_a^b G_{j-1}(t,r) G_1(r,s) \nabla r, \ \text{ for } j = 2,3,...,p, \quad (t,s) \in [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \tag{3}$$

where $G_1(t,s)$ is the Green's function of the BVP,

$$\begin{aligned} -y^{\Delta\nabla} &= 0, \quad t \in [a,b]_{\mathbb{T}}, \\ y(a) &= 0 = y(b), \end{aligned}$$

and it is given by

$$G_1(t,s) = \begin{cases} \frac{(b-s)(t-a)}{(b-a)}, & \mathrm{t} \leq \mathrm{s} \\ \frac{(b-t)(s-a)}{(b-a)}, & \mathrm{s} \leq \mathrm{t} \end{cases}.$$

Further, it is easy to verify that

$$G_p(t,s) \geq 0 \text{ on } (t,s) \in [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}.$$

We derive growth conditions on $f_1,\ f_2$ so that the BVP (1)-(2) has at least two symmetric positive solutions. For this the following are needed.

Let $D = \{v(t) | v : [a,b]_{\mathbb{T}} \to \mathbb{R}$ isa continuous function$\}$. Define the operator $F_j : D \to D$ by

$$F_j v(t) := \int_a^b G_j(t,s) v(s) \nabla s, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}, \quad j \geq 1.$$

By the construction of F_j, it is clear that

$$(-1)^j (F_j v)^{(\Delta\nabla)^j}(t) = v(t), \quad t \in [a,b]_{\mathbb{T}}$$
$$(F_j v)^{(\Delta\nabla)^i}(a) = 0 = (F_j v)^{(\Delta\nabla)^i}(b), \quad i = 0,1,...,j-1, \quad \text{and } j \geq 1.$$

Hence, we see that the BVP (1)-(2) has a solution if and only if the following BVP has a solution,

$$\begin{cases} v_1^{\Delta\nabla} + f_1(t, F_{n-1}v_1, F_{m-1}v_2) = 0, & t \in [a,b]_{\mathbb{T}} \\ v_2^{\Delta\nabla} + f_2(t, F_{n-1}v_1, F_{m-1}v_2) = 0, & t \in [a,b]_{\mathbb{T}}, \end{cases} \tag{4}$$

$$\begin{cases} v_1(a) = 0 = v_1(b), \\ v_2(a) = 0 = v_2(b). \end{cases} \tag{5}$$

Indeed, if (y_1, y_2) is a solution of the BVP (1)-(2), then $(v_1 = y_1^{(\Delta\nabla)^{(n-1)}}, v_2 = y_2^{(\Delta\nabla)^{(m-1)}})$ is a solution of the BVP (4)-(5). Conversely, if (v_1, v_2) is a solution of the BVP (4)-(5), then $(y_1 = F_{n-1}v_1, y_2 = F_{m-1}v_2)$ is a solution of the BVP (1)-(2). In fact, we have the representation

$$y_1(t) = \int_a^b G_{n-1}(t,s) v_1(s) \nabla s, \quad y_2(t) = \int_a^b G_{m-1}(t,s) v_2(s) \nabla s,$$

where

$$v_1(s) = \int_a^b G_1(s,\tau) f_1(\tau, F_{n-1}v_1, F_{m-1}v_2) \nabla\tau,$$
$$v_2(s) = \int_a^b G_1(s,\tau) f_2(\tau, F_{n-1}v_1, F_{m-1}v_2) \nabla\tau.$$

It is also noted that a solution (v_1, v_2) of the BVP (4)-(5) is symmetric;

$$i.e., \quad v_1(t) = v_1(b+a-t) \text{ and } v_2(t) = v_2(b+a-t), \quad t \in [a,b]_{\mathbb{T}},$$

and it gives rise to a symmetric solution (y_1, y_2) of the BVP (1)-(2).

The following lemmas are needed to establish the main result.

Lemma 1. *Let $l \in [\frac{b-a}{8}, \frac{b-a}{2}]_{\mathbb{T}}$. For $(t,s) \in [a+l, b-l]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$, we have*

$$|G_j(t,s)| \geq L_l^j \phi_l^{j-1} \frac{(b-s)(s-a)}{b-a}, \tag{6}$$

where j is a positive integer, $L_l = \frac{l}{b-a}$ and $\phi_l = \int_{a+l}^{b-l} \frac{(b-s)(s-a)}{b-a} \nabla s$.

Proof. First, for $j = 1$ inequality (6) holds provided

$$L_l \leq \min\{\min_{t \leq s} \frac{t-a}{s-a}, \min_{s \leq t} \frac{b-t}{b-s}\} = \frac{l}{b-a}.$$

Next for fixed j, assuming that (6) is true, from (3) we have for $(t,s) \in [a+l, b-l]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$,

$$\begin{aligned}
|G_{j+1}(t,s)| &= |\int_a^b G_j(t,r) G_1(r,s) \nabla r| \\
&\geq |\int_{a+l}^{b-l} G_j(t,r) G_1(r,s) \nabla r| \\
&\geq \int_{a+l}^{b-l} L_l^j \phi_l^{j-1} \frac{(b-r)(r-a)}{b-a} \times L_l \frac{(b-s)(s-a)}{b-a} \nabla r \\
&= L_l^{j+1} \phi_l^j \frac{(b-s)(s-a)}{b-a}.
\end{aligned}$$

Hence, by induction the proof is complete. □

Lemma 2. *For $(t,s) \in [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$, we have*

$$|G_j(t,s)| \leq \phi_0^{j-1} \frac{(b-s)(s-a)}{b-a}, \tag{7}$$

where j is a positive integer and $\phi_0 = \int_a^b \frac{(b-s)(s-a)}{b-a} \nabla s$.

Proof. For $j = 1$ inequality (7) is obvious. Next, for fixed j, assume that (7) is true, then from (3) we have

$$\begin{aligned}
|G_{j+1}(t,s)| &= |\int_a^b G_j(t,r) G_1(r,s) \nabla r| \\
&\leq \int_a^b \phi_0^{j-1} \frac{(b-r)(r-a)}{b-a} \times \frac{(b-s)(s-a)}{b-a} \nabla r \\
&= \phi_0^j \frac{(b-s)(s-a)}{b-a}.
\end{aligned}$$

Hence, by induction the proof is complete. □

Lemma 3. *For $t, s \in [a,b]_{\mathbb{T}}$, the Green's function $G_j(t,s)$ satisfies the symmetric property,*

$$G_j(t,s) = G_j(b+a-t,\ b+a-s). \tag{8}$$

Proof. By the definition of $G_j(t,s)$, $(j \geq 2)$

$$G_j(t,s) = \int_a^b G_{j-1}(t,r) G_1(r,s) \nabla r, \quad \text{for all} \quad t,s \in [a,b]_{\mathbb{T}}.$$

Clearly, we can see, $G_1(t,s) = G_1(a+b-t, a+b-s)$. Now, the proof is by induction. For $j = 2$, the inequality (8) is obvious. Next, assume that (8) is true, for fixed $j \geq 2$, then from (3) we have

$$\begin{aligned} G_{j+1}(t,s) &= \int_a^b G_j(t,r) G_1(r,s) \nabla r \\ &= \int_a^b G_j(a+b-t, a+b-r) G_1(a+b-r, a+b-s) \nabla r \\ &= \int_a^b G_j(a+b-t, r_1) G_1(r_1, a+b-s) \nabla r_1 \\ &= G_{j+1}(a+b-t, a+b-s). \end{aligned}$$

$\square$

Lemma 4. *For $t \in [a,b]_{\mathbb{T}}$, the operator F_j satisfies the symmetric property*

$$F_j y(t) = F_j y(b+a-t).$$

Proof. By definition of F_j,

$$\begin{aligned} F_j y(t) &= \int_a^b G_j(t,s) v(s) \nabla s \\ &= \int_a^b G_j(a+b-t, a+b-s) v(s) \nabla s \\ &= \int_a^b G_j(a+b-t, s_1) v(s_1) \nabla s_1 \\ &= F_j y(b+a-t). \end{aligned}$$

$\square$

3. Existence of multiple symmetric positive solutions

In this section, we establish the existence of at least two symmetric positive solutions for the system of higher order BVP (1)-(2), by using Avery and Henderson's fixed point theorem.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c', we define the set

$$P(\psi, c') = \{y \in P | \psi(y) < c'\},$$

In obtaining multiple symmetric positive solutions of the BVP (1)-(2), the following Avery and Hendersons Fixed Point Theorem will be fundamental.

Theorem 1 (See [6]). *Let P be a cone in a real Banach space B. Suppose α and γ are increasing nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that for some positive numbers c' and k,*

$$\gamma(y) \le \theta(y) \le \alpha(y) \quad and \quad \| y \| \le k\gamma(y) \quad for\ all \quad y \in \overline{P(\gamma, c')}.$$

Suppose there exist positive numbers a' and b' with $a' < b' < c'$ such that

$$\theta(\lambda y) \le \lambda\theta(y) \quad 0 \le \lambda \le 1 \quad and \quad y \in \partial P(\theta, b').$$

Further, let $T : \overline{P(\gamma, c')} \to P$ be a completely continuous operator such that

$(B1)$ $\gamma(Ty) > c'$ *for all* $y \in \partial\overline{P(\gamma, c')}$,

$(B2)$ $\theta(Ty) < b'$ *for all* $y \in \partial\overline{P(\theta, b')}$,

$(B3)$ $P(\alpha, a') \neq \emptyset$ *and* $\alpha(Ty) > a'$ *for all* $y \in \partial\overline{P(\alpha, a')}$ *with* $\theta(Ty) > b'$.

Then, T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that

$$\theta(y_1) < b', \quad with \quad \alpha(y_1) > a',$$

and

$$\gamma(y_2) < c' \quad with \quad \theta(y_2) > b'. \qquad \square$$

To apply the above theorem we define the following.

Let $C_0 = \{(v_1, v_2) | v_1, v_2 : [a, b]_{\mathbb{T}} \to \mathbb{R}$ are continuous functions$\}$ be the Banach space equipped with the norm

$$\| (v_1, v_2) \| = \| v_1 \|_0 + \| v_2 \|_0,$$

where

$$\| v \|_0 = \max_{t \in [a,b]_{\mathbb{T}}} |v(t)|.$$

For a fixed $k_0 \in [\frac{b-a}{8}, \frac{b-a}{2}]_{\mathbb{T}}$, define the cone $P \subset C_0$ by

$$\begin{aligned} P = \{(v_1, v_2) \in C_0 | v_1(t) = v_1(b + a - t) \ \text{ and } \ v_2(t) = v_2(b + a - t), \\ v_1(t) \ge 0 \ \text{ and } \ v_2(t) \ge 0, \\ v_1^{\Delta\nabla}(t) \le 0 \ \text{ and } \ v_2^{\Delta\nabla}(t) \le 0, \quad t \in [a, b]_{\mathbb{T}}, \\ \min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \ge \tfrac{k_0}{t_1 - a} \| (v_1, v_2) \| \}, \end{aligned}$$

where $t_1 = \frac{b+a}{2}$. We let $t_0 = a + k_0$ and $t_0 \le t_1$. Now, define the nonnegative continuous increasing functionals γ, θ and α on P by

$$\begin{aligned} \theta(v_1, v_2) &= \max_{t \in [a, a+k_0]_{\mathbb{T}} \cup [b-k_0, b]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_0)| + |v_2(t_0)|, \\ \gamma(v_1, v_2) &= \min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_0)| + |v_2(t_0)|, \\ \alpha(v_1, v_2) &= \max_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_1)| + |v_2(t_1)|. \end{aligned}$$

We observe that for any $(v_1, v_2) \in P$,

$$\gamma(v_1, v_2) = \theta(v_1, v_2) \le \max_{t\in[a+k_0,b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = \alpha(v_1, v_2), \tag{9}$$

$$\begin{aligned} \| (v_1, v_2) \| &\le \frac{b-a}{2k_0} \min_{t\in[a+k_0,b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \qquad (10)\\ &\le \frac{b-a}{2k_0} \max_{t\in[a,a+k_0]_{\mathbb{T}}\cup[b-k_0,b]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \\ &= \frac{b-a}{2k_0}\theta(v_1, v_2) = \frac{b-a}{2k_0}\gamma(v_1, v_2), \end{aligned}$$

and also

$$\begin{aligned} \| (v_1, v_2) \| &\le \frac{b-a}{2k_0} \min_{t\in[a+k_0,b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \\ &\le \frac{b-a}{2k_0} \max_{t\in[a+k_0,b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \\ &= \frac{b-a}{2k_0}\alpha(v_1, v_2). \end{aligned}$$

We are now ready to present the main result of this section.

Theorem 2. *Suppose there exist $0 < a' < b' < c'$ such that f_1 and f_2 satisfy the following conditions:*

$(A1)$ $|f_1(t, u_{n-1}, w_{m-1})| > \frac{c'}{\phi_0}$ *for all* $(t, |u_{n-1}|, |w_{m-1}|)$ *in* $[a, b]_{\mathbb{T}} \times [L_l^{n-1}\phi_l^{n-1}c', \frac{(b-a)c'}{2k_0}\phi_0^{n-1}] \times [L_l^{m-1}\phi_l^{m-1}c', \frac{(b-a)c'}{2k_0}\phi_0^{m-1}]$,

or

$|f_2(t, u_{n-1}, w_{m-1})| > \frac{c'}{\phi_0}$ *for all* $(t, |u_{n-1}|, |w_{m-1}|)$ *in* $[a, b]_{\mathbb{T}} \times [L_l^{n-1}\phi_l^{n-1}c', \frac{(b-a)c'}{2k_0}\phi_0^{n-1}] \times [L_l^{m-1}\phi_l^{m-1}c', \frac{(b-a)c'}{2k_0}\phi_0^{m-1}]$.

$(A2)$ $|f_i(t, u_{n-1}, w_{m-1})| < \frac{b'}{2\phi_0}$ *for all* $(t, |u_{n-1}|, |w_{m-1}|)$ *in* $[a, b]_{\mathbb{T}} \times [L_l^{n-1}\phi_l^{n-1}b', \frac{(b-a)b'}{2k_0}\phi_0^{n-1}] \times [L_l^{m-1}\phi_l^{m-1}b', \frac{(b-a)b'}{2k_0}\phi_0^{m-1}]$ *for* $i = 1, 2$.

$(A3)$ $|f_1(t, u_{n-1}, w_{m-1})| > \frac{a'}{\phi_0}$ *for all* $(t, |u_{n-1}|, |w_{m-1}|)$ *in* $[a, b]_{\mathbb{T}} \times [L_l^{n-1}\phi_l^{n-1}a', \frac{(b-a)a'}{2k_0}\phi_0^{n-1}] \times [L_l^{m-1}\phi_l^{m-1}a', \frac{(b-a)a'}{2k_0}\phi_0^{m-1}]$,

or

$|f_2(t, u_{n-1}, w_{m-1})| > \frac{a'}{\phi_0}$ *for all* $(t, |u_{n-1}|, |w_{m-1}|)$ *in* $[a, b]_{\mathbb{T}} \times [L_l^{n-1}\phi_l^{n-1}a', \frac{(b-a)a'}{2k_0}\phi_0^{n-1}] \times [L_l^{m-1}\phi_l^{m-1}a', \frac{(b-a)a'}{2k_0}\phi_0^{m-1}]$.

Then the BVP (1)-(2) *has at least two symmetric positive solutions.*

Proof. Define the completely continuous operator $T : C_0 \to C_0$ by

$$T(v_1, v_2) := \Big(\int_a^b G_1(t,s) f_1(s, F_{n-1}v_1, F_{m-1}v_2)\nabla s, \\ \int_a^b G_1(t,s) f_2(s, F_{n-1}v_1, F_{m-1}v_2)\nabla s\Big). \tag{11}$$

And also we denote

$$T_1(v_1, v_2) := \int_a^b G_1(t,s) f_1(s, F_{n-1}v_1, F_{m-1}v_2)\nabla s,$$

$$T_2(v_1, v_2) := \int_a^b G_1(t,s) f_2(s, F_{n-1}v_1, F_{m-1}v_2)\nabla s.$$

It is obvious that a fixed point of T is a solution of the BVP (4)-(5). We seek two fixed points $(x_1, x_2), (y_1, y_2) \in P$ of T. First, we show that $T : P \to P$. Let $(v_1, v_2) \in P$. Clearly, $T_1(v_1, v_2)(t) \geq 0$, $T_2(v_1, v_2)(t) \geq 0$ for $t \in [a, b]_{\mathbb{T}}$, and $T_1^{\Delta\nabla}(v_1, v_2)(t) \leq 0$, $T_2^{\Delta\nabla}(v_1, v_2)(t) \leq 0$ for $t \in [a, b]_{\mathbb{T}}$. Further, since $G_1(t,s) = G_1(b+a-t, b+a-s)$. Hence, it follows that $T_1(v_1, v_2)(t) = T_1(v_1, v_2)(b+a-t)$, $T_2(v_1, v_2)(t) = T_2(v_1, v_2)(b+a-t)$, for $t \in [a, b]_{\mathbb{T}}$. Also, noting that $T_1(v_1, v_2)(a) = 0 = T_1(v_1, v_2)(b), T_2(v_1, v_2)(a) = 0 = T_2(v_1, v_2)(b)$ and $\| T(v_1, v_2) \| = |T_1(v_1, v_2)(t_1)| + |T_1(v_1, v_2)(t_1)|$, we have

$$\begin{aligned}
&\min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (|T_1(v_1, v_2)(t)| + |T_2(v_1, v_2)(t)|) \\
&\quad = \min_{t \in [a+k_0, t_1]_{\mathbb{T}}} (|T_1(v_1, v_2)(t)| + |T_2(v_1, v_2)(t)|) \\
&\quad \geq \min_{t \in [a+k_0, t_1]_{\mathbb{T}}} \frac{t-a}{t_1-a} \| T(v_1, v_2) \| \\
&\quad = \frac{k_0}{t_1-a} \| T(v_1, v_2) \| = \frac{2k_0}{b-a} \| T(v_1, v_2) \| .
\end{aligned}$$

Thus, $T : P \to P$.

Next, it obvious that $\theta(0, 0) = 0$. Further, for any $(v_1, v_2) \in P$, by (9)-(10), respectively, we have $\gamma(v_1, v_2) = \theta(v_1, v_2) \leq \alpha(v_1, v_2)$ and $\| (v_1, v_2) \| \leq \frac{b-a}{2k_0}\gamma(v_1, v_2)$. Also, for any $0 \leq \lambda \leq 1$ and $(v_1, v_2) \in P$. We have

$$\begin{aligned}
\theta(\lambda(v_1, v_2)) &= \max_{[a, a+k_0]_{\mathbb{T}} \cup [b-k_0, b]_{\mathbb{T}}} | \lambda(| v_1(t) | + | v_2(t) |) | \\
&= \lambda \max_{[a, a+k_0]_{\mathbb{T}} \cup [b-k_0, b]_{\mathbb{T}}} (| v_1(t) | + | v_2(t) |) = \lambda\theta(v_1, v_2).
\end{aligned}$$

It remains to verify conditions $(B1)-(B3)$ of Theorem 1. To show that condition $(B1)$ holds, let $(v_1, v_2) \in \partial\overline{P(\gamma, c')}$. So

$$\gamma(v_1, v_2) = \min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (| v_1(t) | + | v_2(t) |).$$

For $t \in [a+k_0, b-k_0]_{\mathbb{T}}$, it is clear from (10) that

$$c' = \min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (| v_1(t) | + | v_2(t) |) \leq \| (v_1, v_2) \| \leq \frac{b-a}{2k_0} c'.$$

Using Lemma 2, and for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\leq \frac{(b-a)c'}{2k_0}\int_a^b G_{n-1}(t,s)\nabla s \\ &\leq \frac{(b-a)c'}{2k_0}\phi_0^{n-2}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s = \frac{(b-a)c'}{2k_0}\phi_0^{n-1}. \end{aligned}$$

Similarly,

$$F_{m-1}v_2 \leq \frac{(b-a)c'}{2k_0}\phi_0^{m-1}.$$

Using Lemma 1, we obtain

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\geq \int_{a-l}^{b-l} G_{n-1}(t,s)\nabla s \\ &\geq L_l^{n-1}\phi_l^{n-2}c'\int_{a-l}^{b-l}\frac{(b-s)(s-a)}{b-a}\nabla s \\ &= L_l^{n-1}\phi_l^{n-1}c'. \end{aligned}$$

Similarly,

$$F_{m-1}v_2 \geq L_l^{m-1}\phi_l^{m-1}c'.$$

We may now use condition $(A1)$ to obtain

$$\begin{aligned} \gamma(T(v_1,v_2)) = &\min_{t\in[a+k_0,b-k_0]_{\mathbb{T}}} |(\int_a^b G_1(t,s)f_1(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s| \\ &+ |\int_a^b G_1(t,s)f_2(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s|) \\ &> \frac{c'}{\phi_0}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s = c'. \end{aligned}$$

Therefore, we have shown that $\gamma(T(v_1,v_2)) > c'$ for all $(v_1,v_2) \in \partial\overline{P(\gamma,c')}$.

Next, we shall verify condition $(B2)$ holds, let $(v_1,v_2) \in \partial\overline{P(\theta,b')}$. So

$$\theta((v_1,v_2)) = \max_{t\in[a,a+k_0]_{\mathbb{T}}\cup[b-k_0,b]_{\mathbb{T}}}(|\,|v_1(t)|+|v_2(t)|)\,|\,.$$

For $t \in [a,a+k_0]_{\mathbb{T}} \cup [b-k_0,b]_{\mathbb{T}}$. It is clear from (10) that

$$b' = \max_{t\in[a,a+k_0]_{\mathbb{T}}\cup[b-k_0,b]_{\mathbb{T}}}(|\,|v_1(t)|+|v_2(t)|)\,| \leq \|\,(v_1,v_2)\,\| \leq \frac{(b-a)b'}{2k_0}.$$

Using Lemma 2, we obtain for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\leq \frac{(b-a)b'}{2k_0}\int_a^b G_{n-1}(t,s)\nabla s \\ &\leq \frac{(b-a)b'}{2k_0}\phi_0^{n-2}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s \\ &= \frac{(b-a)b'}{2k_0}\phi_0^{n-1}. \end{aligned}$$

Similarly,

$$F_{m-1}v_2 \leq \frac{(b-a)b'}{2k_0}\phi_0^{m-1}.$$

Using Lemma 1, we obtain for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\geq b'\int_{a-l}^{b-l} G_{n-1}(t,s)\nabla s \\ &\geq L_l^{n-1}\phi_l^{n-2}b'\int_{a-l}^{b-l} \frac{(b-s)(s-a)}{b-a}\nabla s \\ &= L_l^{n-1}\phi_l^{n-1}b'. \end{aligned}$$

Similarly,

$$F_{m-1}v_2 \geq L_l^{m-1}\phi_l^{m-1}b'.$$

We may now use condition $(A2)$ to obtain

$$\begin{aligned} \theta(T(v_1,v_2)) = \max_{t\in[a,a+k_0]_{\mathbb{T}}\cup[b-k_0,b]_{\mathbb{T}}} &\Big(\int_a^b G_1(t,s)f_1(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s, \\ &\int_a^b G_1(t,s)f_2(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s\Big) \\ < \frac{b'}{2\phi_0}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s &+ \frac{b'}{2\phi_0}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s = b'. \end{aligned}$$

Therefore, we have shown that $\theta(T(v_1,v_2)) < b'$ for all $(v_1,v_2) \in \partial\overline{P(\theta,b')}$.

Finally, we show that $(B3)$ holds. Clearly, $\frac{a'}{2} \in P(\alpha,a') \neq \emptyset$. Now, let $(v_1,v_2) \in \partial\overline{P(\alpha,a')}$. So $\alpha(v_1,v_2) = \max_{t\in[a+k_0,b-k_0]_{\mathbb{T}}}(\mid v_1(t)\mid + \mid v_2 \mid)$. For $t \in [a+k_0,b-k_0]_{\mathbb{T}}$. It is clear from (10) that

$$a' = \max_{t\in[a+k_0,b-k_0]_{\mathbb{T}}}(\mid v_1(t)\mid + \mid v_2(t)\mid) \leq \| (v_1,v_2) \| \leq \frac{(b-a)a'}{2k_0}.$$

Using Lemma 2, we obtain for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\leq \frac{(b-a)a'}{2k_0}\int_a^b G_{n-1}(t,s)\nabla s \\ &\leq \frac{(b-a)a'}{2k_0}\phi_0^{n-2}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s = \frac{(b-a)a'}{2k_0}\phi_0^{n-1}. \end{aligned}$$

Similarly,

$$F_{m-1}v_2 \leq \frac{(b-a)a'}{2k_0}\phi_0^{m-1}.$$

Using Lemma 1, we obtain for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} F_{n-1}v_1 &= \int_a^b G_{n-1}(t,s)v_1(s)\nabla s \\ &\geq a'\int_{a-l}^{b-l} G_{n-1}(t,s)\nabla s \\ &\geq L_l^{n-1}\phi_l^{n-2}a'\int_{a-l}^{b-l} \frac{(b-s)(s-a)}{b-a}\nabla s = L_l^{n-1}\phi_l^{n-1}a'. \end{aligned}$$

Similarly,

$$F_{n-1}v_2 \geq L_l^{m-1}\phi_l^{m-1}a'.$$

We may now use condition $(A3)$ to obtain

$$\begin{aligned} \alpha(T(v_1,v_2)) &= \max_{t\in[a+k_0,b-k_0]_{\mathbb{T}}}\Big(\int_a^b G_1(t,s)f_1(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s \\ &\qquad \int_a^b G_1(t,s)f_2(F_{n-1}v_1(s),F_{m-1}v_2(s))\nabla s\Big) \\ &> \frac{a'}{\phi_0}\int_a^b \frac{(b-s)(s-a)}{b-a}\nabla s = a'. \end{aligned}$$

Therefore, we have shown that $\alpha(T(v_1,v_2)) > a'$ for all $(v_1,v_2) \in \partial\overline{P(\alpha,a')}$.

We have proved that all the conditions of Theorem 1 are satisfied and so there exist at least two symmetric positive solutions $(v_1,v_2),(w_1,w_2) \in \overline{P(\gamma,c')}$ for the BVP (4)-(5). Therefore, the BVP (1)-(2) has at least two symmetric positive solutions

$(y_1, y_2), (x_1, x_2)$ of the form

$$\begin{aligned} y_1(t) &= F_{n-1}v_1(t) = \int_a^b G_{n-1}(t,s)v_1(s)\nabla s, \\ y_2(t) &= F_{m-1}v_2(t) = \int_a^b G_{m-1}(t,s)v_2(s)\nabla s, \\ x_1(t) &= F_{n-1}w_1(t) = \int_a^b G_{n-1}(t,s)w_1(s)\nabla s, \\ x_2(t) &= F_{m-1}w_2(t) = \int_a^b G_{m-1}(t,s)w_2(s)\nabla s. \end{aligned}$$

This completes the proof of the theorem. □

4. Examples

As an application, we demonstrate our results with examples.

Example 1

Consider the system of differential equations,

$$\begin{cases} y_1^{\Delta\nabla} + f_1(t, y_1, y_2) = 0, & t \in [0,1]_{\mathbb{T}} \\ y_2^{\Delta\nabla} + f_2(t, y_1, y_2) = 0, & t \in [0,1]_{\mathbb{T}} \end{cases} \tag{12}$$

subject to the two-point boundary conditions

$$\begin{cases} y_1(0) = 0 = y_1(1), \\ y_2(0) = 0 = y_2(1), \end{cases} \tag{13}$$

where

$$f_1(t, y_1, y_2) = \begin{cases} 5(t-\frac{1}{2})^2 \sin(2y_1y_2+1)\frac{\pi}{4}, & t \in [0,1]_{\mathbb{T}},\ y_1 \in [0, \frac{1}{2}],\ y_2 \in [0, \frac{1}{4}] \\ \frac{127y_1-142}{12}, & y_1 \in [\frac{1}{2}, 2], \\ \frac{37y_1+157}{7}, & y_1 \in [2, 9], \\ 35y_1 - 245, & y_1 \in [9, 13], \\ \frac{200y_1+599}{39}, & y_1 \in [13, 55], \end{cases}$$

$$f_2(t, y_1, y_2) = \begin{cases} 112y_1^2y_2^2, & y_1 \in [\frac{1}{20}, 1],\ y_2 \in [\frac{1}{4}, 1], \\ -101y_2 + 213, & y_2 \in [2, 8], \\ 812y_2 - 7091, & y_2 \in [9, 36], \end{cases}$$

on a time scale,

$$\mathbb{T} = \left[0, \frac{1}{8}\right] \cup \left\{\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}\right\} \cup \left[\frac{7}{8}, 1\right]$$

and

$$[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}.$$

Clearly, $f_i,\ i = 1,2$ satisfies the symmetry property

$$i.e.,\quad f_i(t, y_1, y_2) = f_i(1-t, y_1, y_2),\quad \text{for all } t \in [0,1]_{\mathbb{T}},\quad \text{for } i = 1,2.$$

Let $G(t,s)$, be the Green's function of the following boundary value problem

$$\left.\begin{aligned} -y^{\Delta\nabla}(t) &= 0, \quad t \in [0,1]_{\mathbb{T}}, \\ y(0) &= 0 = y(1), \end{aligned}\right\} \tag{14}$$

which is given by

$$G(t,s) = \begin{cases} (1-s)t, & t \le s, \\ (1-t)s, & s \le t, \end{cases}$$

for all $t, s \in [0,1]_{\mathbb{T}}$.

The operator $T(v_1, v_2)$ satisfies the symmetry property, i.e.

$$T(v_1, v_2)(t) = T(v_1, v_2)(1-t), \quad \text{for all } t \in [0,1]_{\mathbb{T}}.$$

After computation

$$\phi_0 = \int_0^1 G(s,s)\nabla s = \frac{1}{6}.$$

If we choose the positive constants a', b' and c' as $\frac{1}{4}$, $\frac{3}{2}$ and 7 respectively, then all the conditions of Theorem 2 are satisfied. Hence by Theorem 2,the boundary value problem (12), (13) has at least two symmetric positive solutions.

Example 2

Consider the system of differential equations

$$\begin{cases} y_1^{(\Delta\nabla)^2} + f_1(t, y_1, y_2) = 0, & t \in [0,1]_{\mathbb{T}} \\ \quad y_2^{\Delta\nabla} + f_2(t, y_1, y_2) = 0, & t \in [0,1]_{\mathbb{T}} \end{cases} \tag{15}$$

subject to the two-point boundary conditions

$$\begin{cases} y_1(0) = 0 = y_1(1), \\ y_1^{\Delta\nabla}(0) = 0 = y_1^{\Delta\nabla}(1), \\ y_2(0) = 0 = y_2(1), \end{cases} \tag{16}$$

where

$$f_1(t,y_1,y_2)=\begin{cases} 2688y_1y_2, & y_1\in[\frac{1}{4},1],\quad y_2\in[\frac{1}{96},\frac{1}{6}],\\ -17y_1+45, & y_1\in[2,8],\\ 308y_1-2247, & y_1\in[9,36],\\ -4303y_1+86295, & y_1\in[40,100],\\ 60267y_1-3012865, & y_1\in[200,800],\end{cases}$$

$$f_2(t,y_1,y_2)=\begin{cases} y_2^2e^{(t-\frac{1}{2})^2}, & t\in[0,1]_{\mathbb{T}},\ y\in[0,\frac{1}{4}],\\ \frac{(800-2e^{\frac{1}{4}})y_2-200+3e^{\frac{1}{4}}}{40}, & y_2\in[\frac{1}{4},\frac{3}{2}],\\ \frac{50(y_2+4)}{11}, & y_2\in[\frac{3}{2},7],\end{cases}$$

on a time scale

$$\mathbb{T}=\left[0,\frac{1}{8}\right]\cup\left\{\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4}\right\}\cup\left[\frac{7}{8},1\right]$$

and

$$[0,1]_{\mathbb{T}}=[0,1]\cap\mathbb{T}.$$

Clearly, $f_i,\ i=1,2$ satisfies the symmetry property, i.e.

$$f_i(t,y_1,y_2)=f_i(1-t,y_1,y_2),\quad \text{for all } t\in[0,1]_{\mathbb{T}}\ \text{ for } i=1,2.$$

Let $G_1(t,s)$, be the Green's function of the following boundary value problem

$$\left.\begin{array}{c} -y^{\Delta\nabla}(t)=0,\quad t\in[0,1]_{\mathbb{T}},\\ y(0)=0=y(1),\end{array}\right\}, \tag{17}$$

which is given by

$$G_1(t,s)=\begin{cases}(1-s)t, & t\le s,\\ (1-t)s, & s\le t,\end{cases}$$

for all $t,s\in[0,1]_{\mathbb{T}}$. Now we define

$$G_2(t,s)=\int_0^1 G_1(t,r)G_1(r,s)\nabla r,$$

and it is the Green's function of the boundary value problem

$$\left.\begin{array}{c} y^{(\Delta\nabla)^2}(t)=0,\quad t\in[0,1]_{\mathbb{T}},\\ y(0)=0=y(1),\\ y^{\Delta\nabla}(0)=0=y^{\Delta\nabla}(1).\end{array}\right\} \tag{18}$$

The operator $T(v_1,v_2)$ satisfies the symmetry property, i.e.

$$T(v_1,v_2)(t)=T(v_1,v_2)(a+b-t),\quad \text{for all } t\in[0,1]_{\mathbb{T}}.$$

After computation

$$\phi_0 = \int_0^1 G_1(s,s)\nabla s = \frac{1}{6}.$$

If we choose the positive constants a', b' and c' as $\frac{1}{4}$, 2 and 9, respectively, then all the conditions of Theorem 2 are satisfied. Hence by Theorem 2, the boundary value problem (15), (16) has at least two symmetric positive solutions.

Acknowledgement

One of the authors (P. Murali) is thankful to CSIR, India for awarding SRF. The authors thank the referees for their valuable suggestions.

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