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A Nakano type inequality for mixed forms on complex Finsler manifolds

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Abstract. In this paper, we obtain a Nakano type inequality for vertical valued $\partial_{\overline{h}}$ -harmonic mixed forms with compact support on the total space of the holomorphic tangent bundle of a complex Finsler manifold.
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1. Introduction

The Bochner technique and vanishing theorems on strongly pseudoconvex complex Finsler manifolds were initiated and intensively studied by C. Zhong and T. Zhong in [19, 21, 22, 23] and [24], whose aims were to study the horizontal complex Laplacian and vanishing theorem for complex type forms on strongly pseudoconvex complex Finsler manifolds. Some possible extensions concerning the study of the vertical complex Laplacian and a vanishing theorem of Bochner type for holomorphic sections on the total space of a complex Finsler bundle are studied in [7, 9]. The aim of this paper is to continue the study mentioned above. Here we study a Nakano type inequality for vertical valued $\partial_{\overline{h}}$-harmonic mixed forms with compact support on pseodoconvex complex Finsler manifolds.

Firstly, following [1, 3] and [14], we briefly recall some basic facts on complex Finsler manifolds. Next, by analogy with the case of hermitian foliations [18], we can consider the $(p+r, q+s)$ complex type forms on the total space of the holomorphic tangent bundle of a complex Finsler manifold [15], as $(p, q+r+s)$ mixed type forms. If we assume that these forms have compact support, as in the Kähler geometry, we can consider classical operators $*, d, d^*, L, \Lambda$ and we get some decompositions of these operators with respect to mixed type forms. In the last section, taking into account that any complex Finsler metric defines a hermitian metric on the holomorphic vertical bundle, we prove a Nakano type inequality with respect to the complex Rund connection, for vertical valued $\partial_{\overline{h}}$-harmonic mixed forms with compact support. The methods used here are closely related to those used by [18]. We hope that further developments can be done to obtain Nakano type vanishing theorems [11, 17], for the horizontal cohomology groups of a complex Finsler bundle (manifold) defined in [6, 8] or other vanishing theorems of Bochner type for holomorphic line bundles [19].

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Preliminaries and settings

Let $\pi : T^{1,0}M \to M$ be the holomorphic tangent bundle of an n-dimensional complex manifold M. Denote by $(\pi^{-1}(U),(z^k,\eta^k))$, $k=1,\dots,n$ the induced complex coordinates on $T^{1,0}M$, where $(U,(z^k))$ is a local chart domain of M. At local change charts on $T^{1,0}M$, the transformation rules of these coordinates are given by

$$z'^k = z'^k(z)\ ,\ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j, \tag{1}$$

where $\frac{\partial z'^k}{\partial z^j}$ are holomorphic functions and $\det(\frac{\partial z'^k}{\partial z^j}) \neq 0$.

It is well known that $T^{1,0}M$ has a natural structure of $2n$-dimensional complex manifold, because the transition functions $\frac{\partial z'^k}{\partial z^j}$ are holomorphic.

Denote by $\widetilde{M} = T^{1,0}M - \{o\}$, where o is the zero section of $T^{1,0}M$, and we consider $T_{\mathbb{C}}\widetilde{M} = T^{1,0}\widetilde{M} \oplus T^{0,1}\widetilde{M}$ the complexified tangent bundle of $\widetilde{M}$, where $T^{1,0}\widetilde{M}$ and $T^{0,1}\widetilde{M} = \overline{T^{1,0}\widetilde{M}}$ are the holomorphic and antiholomorphic tangent bundles of $\widetilde{M}$, respectively.

Let $V^{1,0}\widetilde{M} = \ker \pi_*$ be the holomorphic vertical bundle over $\widetilde{M}$ and $\mathcal{V}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of v-type*.

A given supplementary subbundle $H^{1,0}\widetilde{M}$ of $V^{1,0}\widetilde{M}$ in $T^{1,0}\widetilde{M}$, i.e.

$$T^{1,0}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \tag{2}$$

defines a *complex nonlinear connection* on $\widetilde{M}$, briefly c.n.c., and we denote by $\mathcal{H}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of h-type*.

By conjugation over all, we get a decomposition of the complexified tangent bundle, namely

$$T_{\mathbb{C}}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \oplus H^{0,1}\widetilde{M} \oplus V^{0,1}\widetilde{M}.$$

The elements of the conjugates are called *vector fields of* $\overline{h}$*-type* and $\overline{v}$*-type*, respectively.

If $N_k^j(z,\eta)$ are the local coefficients of the c.n.c. then the following set of complex vector fields

$$\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j\frac{\partial}{\partial \eta^j}\}\,,\,\{\frac{\partial}{\partial \eta^k}\}\,,\,\{\frac{\delta}{\delta \overline{z}^k} = \frac{\partial}{\partial \overline{z}^k} - N_{\overline{k}}^{\overline{j}}\frac{\partial}{\partial \overline{\eta}^j}\}\,,\,\{\frac{\partial}{\partial \overline{\eta}^k}\} \tag{3}$$

are called the local adapted bases of $\mathcal{H}^{1,0}(\widetilde{M})$, $\mathcal{V}^{1,0}(\widetilde{M})$, $\mathcal{H}^{0,1}(\widetilde{M})$ and $\mathcal{V}^{0,1}(\widetilde{M})$, respectively. The dual adapted bases are given by

$$\{dz^k\}\,,\,\{\delta\eta^k = d\eta^k + N_j^k dz^j\}\,,\,\{d\overline{z}^k\}\,,\,\{\delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\}. \tag{4}$$

Throughout this paper, we consider the abbreviated notations

$$\partial_k = \frac{\partial}{\partial z^k}\,,\,\dot{\partial}_k = \frac{\partial}{\partial \eta^k}\,,\,\delta_k = \frac{\delta}{\delta z^k}\,,\,d^k = dz^k\,,\,\delta^k = \delta\eta^k,$$

$$\partial_{\overline{k}} = \frac{\partial}{\partial \overline{z}^k}\ ,\ \dot{\partial}_{\overline{k}} = \frac{\partial}{\partial \overline{\eta}^k}\ ,\ \delta_{\overline{k}} = \frac{\delta}{\delta \overline{z}^k}\ ,\ d^{\overline{k}} = d\overline{z}^k\ ,\ \delta^{\overline{k}} = \delta\overline{\eta}^k.$$

Let M be a strongly pseudoconvex complex Finsler manifold [1], i.e. M is endowed with a complex Finsler metric $F : T^{1,0}M \to \mathbb{R}_+ \cup \{0\}$ satisfying:

(1) F^2 is smooth on $\widetilde{M}$;

(2) $F(z, \eta) > 0$ for all $(z, \eta) \in \widetilde{M}$ and $F(z, \eta) = 0$ if and only if $\eta = 0$;

(3) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for all $(z, \eta) \in T^{1,0}M$ and $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$;

(4) the complex hessian

$$(G_{j\overline{k}}) = (\dot{\partial}_j \dot{\partial}_{\overline{k}}\ (F^2)) \tag{5}$$

is positive definite on $\widetilde{M}$.

Let $(G^{\overline{m}j})$ be the inverse of $(G_{j\overline{m}})$. According to [1, 3] and [14], a c.n.c. on (M, F) depending only on the complex Finsler metric F is the Chern-Finsler c.n.c., locally given by

$$\overset{CF}{N^j_k} = G^{\overline{m}j} \partial_k\ \dot{\partial}_{\overline{m}}\ (F^2), \tag{6}$$

and it has an important property, namely

$$\overset{CF}{R^i_{kj}} = \delta_k\ \overset{CF}{N^i_j} - \delta_j\ \overset{CF}{N^i_k} = 0. \tag{7}$$

In the sequel, we will consider the adapted frames and coframes with respect to the Chern-Finsler c.n.c. and the hermitian metric structure G on $\widetilde{M}$ given by the Sasaki type lift of the fundamental tensor $G_{j\overline{k}}$, locally given by

$$G = G_{j\overline{k}} d^j \otimes d^{\overline{k}} + G_{j\overline{k}} \delta^j \otimes \delta^{\overline{k}}. \tag{8}$$

The fundamental form of G is $\omega = \omega^h + \omega^v$, where

$$\omega^h = \sqrt{-1} G_{j\overline{k}} d^j \wedge d^{\overline{k}}\ ,\ \omega^v = \sqrt{-1} G_{j\overline{k}} \delta^j \wedge \delta^{\overline{k}}. \tag{9}$$

2. Operators on mixed forms

According to [15], the set $\mathcal{A}(\widetilde{M})$ of complex valued differential forms on $\widetilde{M}$ is given by the direct sum

$$\mathcal{A}(\widetilde{M}) = \oplus_{p,q,r,s=\overline{0,n}} \mathcal{A}^{p,q,r,s}(\widetilde{M}), \tag{10}$$

where $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ or simply $\mathcal{A}^{p,q,r,s}$ is the set of all $(p+q+r+s)$-forms which can be non zero only when these act on p vector fields of h-type, on q vector fields of

$\overline{h}$-type, on r vector fields of v-type, and on s vector fields of $\overline{v}$-type. The elements of $\mathcal{A}^{p,q,r,s}$ are called (p,q,r,s)*-forms* on $\widetilde{M}$.

With respect to the adapted coframes $\{d^k, d^{\overline{k}}, \delta^k, \delta^{\overline{k}}\}$ of $T^*_{\mathbb{C}}\widetilde{M}$ a form $\varphi \in \mathcal{A}^{p,q,r,s}$ is locally given by

$$\varphi = \frac{1}{p!q!r!s!}\varphi_{I_p\overline{J_q}K_r\overline{H_s}}d^{I_p} \wedge d^{\overline{J_q}} \wedge \delta^{K_r} \wedge \delta^{\overline{H_s}}, \tag{11}$$

where I_p denotes the ordered p-tuple $(i_1 \ldots i_p)$, J_q the ordered q-tuple $(j_1 \ldots j_q)$, K_r the ordered r-tuple $(k_1 \ldots k_r)$, H_s the ordered s-tuple $(h_1 \ldots h_s)$ and $d^{I_p} = d^{i_1} \wedge \ldots \wedge d^{i_p}$, $d^{\overline{J_q}} = d^{\overline{j_1}} \wedge \ldots \wedge d^{\overline{j_q}}$, $\delta^{K_r} = \delta^{k_1} \wedge \ldots \wedge \delta^{k_r}$ and $\delta^{\overline{H_s}} = \delta^{\overline{h_1}} \wedge \ldots \wedge \delta^{\overline{h_s}}$, respectively. For φ given by (11) its conjugate is locally given by

$$\overline{\varphi} = \frac{1}{p!q!r!s!}\varphi_{\overline{I_p}J_q\overline{K_r}H_s}d^{\overline{I_p}} \wedge d^{J_q} \wedge \delta^{\overline{K_r}} \wedge \delta^{H_s} \tag{12}$$

and it is obvious that the conjugation is a real-linear isomorphism between $\mathcal{A}^{p,q,r,s}$ and $\mathcal{A}^{q,p,s,r}$.

We notice that these forms are the $(p+r, q+s)$ *complex type* and according to [15], if (M,F) is a complex Finsler manifold endowed with the Chern-Finsler c.n.c., then by (7) the exterior differential d admits the decomposition

$$d\mathcal{A}^{p,q,r,s} \subset \mathcal{A}^{p+1,q,r,s} \oplus \mathcal{A}^{p,q+1,r,s} \oplus \mathcal{A}^{p,q,r+1,s} \oplus \mathcal{A}^{p,q,r,s+1} \oplus$$

$$\oplus \mathcal{A}^{p+1,q+1,r-1,s} \oplus \mathcal{A}^{p+1,q,r-1,s+1} \oplus \mathcal{A}^{p+1,q+1,r,s-1} \oplus \mathcal{A}^{p,q+1,r+1,s-1}, \tag{13}$$

which allows us to define eight morphisms of complex vector spaces if we consider the different components, namely

$$\begin{aligned}
&\partial_h : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r,s}, && \partial_v : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r+1,s}\\
&\partial_{\overline{h}} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r,s}, && \partial_{\overline{v}} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1}\\
&\partial_1 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r-1,s}, && \partial_2 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r-1,s+1}\\
&\partial_3 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r,s-1}, && \partial_4 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r+1,s-1}.
\end{aligned}$$

We remark that these operators and classical operators ∂ and $\overline{\partial}$ that appear in the decomposition $d = \partial + \overline{\partial}$ of the differential on a complex manifold are related by

$$\partial = \partial_h + \partial_v + \partial_3 + \partial_4\ ,\ \overline{\partial} = \partial_{\overline{h}} + \partial_{\overline{v}} + \partial_1 + \partial_2. \tag{14}$$

In the sequel, similarly to [18], the (p,q,r,s)-forms on $\widetilde{M}$ can be considered as $(p, q+r+s)$ *mixed type* forms on $\widetilde{M}$, denoted by $\mathcal{A}^{p;q,r,s}(\widetilde{M})$.

Let $\mathcal{A}^{p;q,r,s}_c(\widetilde{M}) \subset \mathcal{A}^{p;q,r,s}(\widetilde{M})$ be the subspace consisting of those mixed forms with compact support on $\widetilde{M}$. As in the Kähler geometry, we can consider the classical scalar product with respect to the hermitian metric structure from (8) and the operators $d, *, d^*, L, \Lambda$ (see [4], [17]), and it is important here to obtain decompositions of these operators with respect to the mixed type forms.

Thus, from (9) we have $L = L^h + L^v$, where L^h denotes the left exterior multiplication by ω^h and has mixed type $(1,1)$, and similarly, L^v denotes the left exterior multiplication by ω^v and has mixed type $(0,2)$.

By (13) the operator d has the decomposition in two parts of the respective mixed types $(1,0)$ and $(0,1)$, namely $d = d_{1,0} + d_{0,1}$, where

$$d_{1,0} = \partial_h + \partial_1 + \partial_2 + \partial_3 \ , \ d_{0,1} = \partial_{\overline{h}} + \partial_v + \partial_{\overline{v}} + \partial_4. \tag{15}$$

Thus, the mixed type differential forms of a complex Finsler manifold are organized as a double cochain complex.

Next, we define the operator # by composition of $*$ with the complex conjugation. Thus # sends forms of the mixed type $(p, q+r+s)$ to forms of mixed type $(n-p, 3n-q-r-s)$. According to [23], for $\varphi \in \mathcal{A}_c^{p;q,r,s}(\widetilde{M})$ locally given by (11) we have

$$\#\varphi = \frac{\varphi^{\#}_{I_{n-p}\overline{J_{n-q}}K_{n-r}\overline{H_{n-s}}}}{(n-p)!(n-q)!(n-r)!(n-s)!} d^{I_{n-p}} \wedge d^{\overline{J_{n-q}}} \wedge \delta^{K_{n-r}} \wedge \delta^{\overline{H_{n-s}}}, \tag{16}$$

where

$$\begin{aligned}
\varphi^{\#}_{I_{n-p}\overline{J_{n-q}}K_{n-r}\overline{H_{n-s}}} &= \frac{\varepsilon(p,q,r,s)}{p!q!r!s!} G_{I_p I_{n-p}\overline{J_q}\ \overline{J_{n-q}}} G_{K_r K_{n-r}\overline{H_s}\ \overline{H_{n-s}}} \overline{\varphi^{\overline{I_p}J_q\overline{K_r}H_s}}, \\
\varphi^{\overline{I_p}J_q\overline{K_r}H_s} &= \varphi_{I'_p\overline{J'_q}K'_r\overline{H'_s}} G^{\overline{I_p}I'_p} G^{\overline{J'_q}J_q} G^{\overline{K_r}K'_r} G^{\overline{H'_s}H_s}, \\
G_{I_p I_{n-p}\overline{J_q}\ \overline{J_{n-q}}} &= G_{i_1\overline{j_1}} \dots G_{i_p\overline{j_q}} G_{i_{p+1}\overline{j_{q+1}}} \dots G_{i_n\overline{j_n}} \ , \ G^{\overline{I_p}I'_p} = G^{\overline{i_1}i'_1} \dots G^{\overline{i_p}i'_p}
\end{aligned}$$

and

$$\varepsilon(p,q,r,s) = (-1)^{n+(q+r+s)(n-p)+(r+s)(n-q)+s(n-r)}.$$

In the following we will renounce to give local expressions of the operators considered below.

Hence, using # we can write scalar products as

$$\langle \varphi, \psi \rangle = \int_{\widetilde{M}} \varphi \wedge \#\psi \tag{17}$$

for any mixed forms φ, $\psi \in \mathcal{A}_c^{p;q,r,s}(\widetilde{M})$.

As in the classical theory, (see [4], [18]), it follows

$$\#^{-1}\varphi = (-1)^{\deg \varphi}\#\varphi \ , \ d^* = -\#d\#. \tag{18}$$

Thus

$$d^* = d^*_{1,0} + d^*_{0,1} \ , \ d^*_{1,0} = -\#d_{1,0}\# \ , \ d^*_{0,1} = -\#d_{0,1}\#, \tag{19}$$

where the terms have mixed types $(-1,0)$ and $(0,-1)$.

Taking into account the decomposition of $d_{1,0}$ and $d_{0,1}$ from (15), we obtain

$$\partial^*_h = -\#\partial_h\# \, , \, \partial^*_1 = -\#\partial_1\# \, , \, \partial^*_2 = -\#\partial_2\# \, , \, \partial^*_3 = -\#\partial_3\#, \tag{20}$$

$$\partial^*_{\overline{h}} = -\#\partial_{\overline{h}}\# \, , \, \partial^*_v = -\#\partial_v\# \, , \, \partial^*_{\overline{v}} = -\#\partial_{\overline{v}}\# \, , \, \partial^*_4 = -\#\partial_4\#. \tag{21}$$

It also follows

$$\Lambda = \Lambda^h + \Lambda^v \ , \ \Lambda^h = \#^{-1} L^h \# \ , \ \Lambda^v = \#^{-1} L^v \#. \tag{22}$$

Finally, by the same considerations as in [18], we obtain

$$\Lambda^v d_{1,0} - d_{1,0}\Lambda^v = 0 \ , \ \Lambda^h d_{0,1} - d_{0,1}\Lambda^h = -\sqrt{-1} d^*_{1,0} \tag{23}$$

and by equating the terms of the same components in the second relation of (23) we get

$$\Lambda^h \partial_{\overline{h}} - \partial_{\overline{h}}\Lambda^h = -\sqrt{-1}\partial^*_h \tag{24}$$

and other similar identities with respect to the considered operators.

3. A Nakano type inequality

Let $V^{1,0}\widetilde{M}$ be the holomorphic vertical bundle of a strongly pseudoconvex complex Finsler manifold (M, F). Then # and the operators from (23) make sense on $V^{1,0}\widetilde{M}$-valued forms by componentwise application. It is well known (see [1, 2, 14]) that the complex Finsler metric F defines a hermitian metric on $V^{1,0}\widetilde{M}$ by

$$\mathcal{G}^v(X, Y) = G_{j\overline{k}} X^j \overline{Y}^k \tag{25}$$

for any $X = X^j \ \dot{\partial}_j \ , Y = Y^k \ \dot{\partial}_k \in \Gamma(V^{1,0}\widetilde{M})$.

Then the product of the $V^{1,0}\widetilde{M}$-valued forms with compact support $\varphi = \varphi^j \otimes \ \dot{\partial}_j$ and $\psi = \psi^k \otimes \ \dot{\partial}_k$ is given by

$$(\varphi, \psi) = \int_{\widetilde{M}} G_{j\overline{k}} \langle \varphi^j, \psi^k \rangle \tag{26}$$

and we shall denote by " + " the adjointness with respect to this inner product.

Let us consider $\omega = (\omega^i_j)$ the $(1,0)$-connection form of the complex Rund connection [16], locally given by

$$\omega^i_j = L^i_{jk} d^k \ , \ L^i_{jk} = G^{\overline{m}i} \delta_k (G_{j\overline{m}}) \tag{27}$$

and it is considered as a $(1,0)$ mixed type form.

The curvature of the complex Rund connection admits a decomposition in two parts of the mixed type $(1,1)$, namely

$$\Omega = R + P \ , \ R^i_j = R^i_{jk\overline{l}} d^k \wedge d^{\overline{l}} \ , \ P^i_j = P^i_{jk\overline{l}} d^k \wedge \delta^{\overline{l}}, \tag{28}$$

where, locally $R^i_{jk\overline{l}} = -\delta_{\overline{l}}(L^i_{jk})$ and $P^i_{jk\overline{l}} = - \ \dot{\partial}_{\overline{l}} \ (L^i_{jk})$, (for details see [14]).

Taking into account that for smooth functions on $\widetilde{M}$ we have $d_{1,0} = \partial_h$ it follows

$$\omega^i_j = G^{\overline{m}i} d_{1,0}(G_{j\overline{m}}). \tag{29}$$

If we set

$$\widetilde{D}\varphi^i = d_{1,0}\varphi^i + \omega^i_j \wedge \varphi^j, \tag{30}$$

by the same calculations as in [18], we get

$$d^{*+}_{1,0} = \widetilde{D}. \tag{31}$$

Now the covariant exterior derivative with respect to the complex Rund connection is just

$$D = \widetilde{D} + d_{0,1}, \tag{32}$$

where $\widetilde{D}$ is the term of the mixed type $(1,0)$ of D.

By general properties of connections, we have

$$D^2\varphi^i = \Omega^i_j \wedge \varphi^j, \tag{33}$$

where

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k = d_{0,1}\omega^i_j. \tag{34}$$

From (28) and (34) we deduce $\Omega^i_j = \partial_{\overline{h}}\omega^i_j + \partial_{\overline{v}}\omega^i_j$. Because the curvature forms Ω^i_j are the mixed type $(1,1)$, by (32) and (33) we get $\widetilde{D}^2 = 0$, $d^2_{0,1} = 0$ and

$$\widetilde{D}d_{0,1} + d_{0,1}\widetilde{D} = e(\Omega), \tag{35}$$

where $e(\Omega)$ is the operator defined by the right-hand side of (33).

The relation (30) give the decomposition $\widetilde{D} = \widetilde{D}_h + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3$ where

$$\widetilde{D}_h\varphi^i = \partial_h\varphi^i + \omega^i_j \wedge \varphi^j \, , \; \widetilde{D}_1\varphi^i = \partial_1\varphi^i \, , \; \widetilde{D}_2\varphi^i = \partial_2\varphi^i \, , \; \widetilde{D}_3\varphi^i = \partial_3\varphi^i. \tag{36}$$

Thus, from (31) and (32) we deduce

$$\partial^{*+}_h = \widetilde{D}_h \;\; , \;\; D_h = \widetilde{D}_h + \partial_{\overline{h}}. \tag{37}$$

Taking only nonzero components in (35) we get

$$\widetilde{D}_h\partial_{\overline{h}} + \partial_{\overline{h}}\widetilde{D}_h = e(R) \; , \; \widetilde{D}_h\partial_{\overline{v}} + \partial_{\overline{v}}\widetilde{D}_h = e(P). \tag{38}$$

Now let us consider $\partial^+_{\overline{h}}$ and $\mathcal{H}(V^{1,0}\widetilde{M}) = \ker\partial_{\overline{h}} \cap \ker\partial^+_{\overline{h}}$ be the space of $\partial_{\overline{h}}$-*harmonic* $V^{1,0}\widetilde{M}$-*valued forms* with compact support. For such form φ of the mixed type $(p, q+r+s)$ by similar calculations as in [18] we get

$$\begin{aligned} 0 &\leq (\widetilde{D}_h\varphi, \widetilde{D}_h\varphi) = (\partial^{*+}_h\widetilde{D}_h\varphi, \varphi) = \sqrt{-1}([\Lambda^h\partial_{\overline{h}} - \partial_{\overline{h}}\Lambda^h]\widetilde{D}_h\varphi, \varphi) \\ &= \sqrt{-1}(\Lambda^h[\partial_{\overline{h}}\widetilde{D}_h + \widetilde{D}_h\partial_{\overline{h}}]\varphi, \varphi) - \sqrt{-1}(\Lambda^h\widetilde{D}_h\varphi, \partial^+_{\overline{h}}\varphi) \\ &\leq \sqrt{-1}(\Lambda^h e(R)\varphi, \varphi). \end{aligned}$$

Then, using also the adjointness of Λ^h and L^h we have

Theorem 1. *For any $\partial_{\overline{h}}$-harmonic $V^{1,0}\widetilde{M}$-valued form φ with compact support of the mixed type $(p, q+r+s)$ one has*

$$\sqrt{-1}(e(R)\varphi, L^h\varphi) \geq 0 \tag{39}$$

equality holding if and only if $\widetilde{D}_h\varphi = 0$.

Remark 1. *We notice that in [8] the first horizontal Chern form of a complex Finsler bundle is defined in terms of the horizontal curvature R by $\mathcal{C}_1^h(\widetilde{M}) = \frac{\sqrt{-1}}{2\pi}R_k^k$. Thus, our Nakano type inequality from the above theorem becomes*

$$(e(\mathcal{C}_1^h(\widetilde{M}))\varphi, L^h\varphi) \geq 0 \tag{40}$$

for any $\partial_{\overline{h}}$-harmonic $V^{1,0}\widetilde{M}$-valued form φ with compact support of the mixed type $(p, q+r+s)$.

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