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Applications of a generalized form of Gauss's second theorem to the series $_3F_2$

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Abstract. The aim of this research paper is to establish generalizations of classical Watson's theorem and Fox's theorem by employing a generalized Gauss's second summation theorem obtained earlier by Lavoie, Grondin and Rathie. Several interesting special cases are also given. The results established in this research paper are simple, interesting, easily established and may be useful.

AMS subject classifications: 33C05, 33C20

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1. Introduction

A generalized hypergeometric function with p numerator and q denominator parameters is defined by [6]

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};z\right] &= {}_pF_q\left[\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q;z\right] \\ &:= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \end{aligned} \tag{1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α by

$$(\alpha)_n = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } \ n\in\mathbb{N}=\{1,2,\cdots\}\\ 1, & \text{if } \ n=0\end{cases} \tag{2}$$

using the fundamental relation $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (n\in\mathbb{N}\cup\{0\}), \tag{3}$$

where Γ is the well-known Gamma function.

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It is well known that whenever a generalized hypergeometric function reduces to the Gamma function, the results are very important from the application point of view. We mention here some of special cases of (1).
Gauss's summation theorem [1, 2, 6]:

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4}$$

provided $\Re(c-a-b)>0$.

Gauss's second summation theorem [2]:

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1) \end{matrix};\frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})} \tag{5}$$

Watson's summation theorem [1, 2, 6]:

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix};1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} \tag{6}$$

provided $\Re(2c-a-b)>-1$.

On the other hand, Fox [3] established the following interesting result

$$\begin{aligned} &{}_pF_q\left[\begin{matrix} \alpha_1,\cdots,\alpha_p \\ \rho_1-m,\ \rho_2,\cdots,\rho_q \end{matrix};x\right] \\ &= \frac{\Gamma(\rho_1)\Gamma(\rho_1-m)\Gamma(\rho_2)\cdots\Gamma(\rho_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)} \\ &\times\sum_{r=0}^{m}\frac{x^r}{\Gamma(r+\rho_1-m)}\binom{m}{r}\frac{\Gamma(\alpha_1+r)\cdots\Gamma(\alpha_p+r)}{\Gamma(\rho_1+r)\cdots\Gamma(\rho_q+r)}\,{}_pF_q\left[\begin{matrix} \alpha_1+r,\cdots,\alpha_p+r \\ \rho_1+r,\cdots,\rho_q+r \end{matrix};x\right] \end{aligned} \tag{7}$$

where m is a positive integer.

Using (7) and (5), Fox [3] obtained the following summation theorem :

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma \\ \alpha-m,\ \frac{1}{2}(\beta+\gamma+1) \end{matrix};\frac{1}{2}\right] \\ &= \frac{\Gamma(\alpha-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\beta)\Gamma(\gamma)}2^{\beta+\gamma-2}\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)}{\Gamma(r+\alpha-m)} \end{aligned} \tag{8}$$

where m is a positive integer.

i	A_i	B_i
5	$-[(b+a+6)^2 - \frac{1}{4}(b-a+6)^2$ $-\frac{1}{2}(b-a+6)(b+a+6)$ $-11(b+a+6)$ $+\frac{13}{2}(b-a+6)+20]$	$[(b+a+6)^2 - \frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+6)$ $-17(b+a+6)$ $-\frac{1}{2}(b-a+6)+62]$
4	$\frac{1}{2}(b+a+1)(b+a-3)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a-1)$
3	$-\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
2	$\frac{1}{2}(b+a-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
-4	$\frac{1}{2}(b+a-3)(b+a+1)$ $-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$
-5	$[(b+a-4)^2 - \frac{1}{4}(b-a-4)^2)$ $-\frac{1}{2}(b+a-4)(b-a-4)$ $+4(b+a-4)$ $-\frac{7}{2}(b-a-4)]$	$[(b+a-4)^2 - \frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b-a-4)(b+a-4)$ $+8(b+a-4)$ $-\frac{1}{2}(b-a-4)+12]$

Table 1. *Table for A_i, B_i*

In 1996, Lavoie, Grondin and Rathie [5] generalized the Gauss's second summation theorem (5) in the form

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+i+1) \end{matrix};\frac{1}{2}\right] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|i|)} \\
&\times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])}+\frac{B_i}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}i-[\frac{i}{2}])}\right\}
\end{aligned} \tag{9}
$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients A_i and B_i are given in the table above [5, p. 298].

The aim of this research paper is to establish generalizations of classical Watson's theorem (6) and Fox's theorem (8).

2. Generalization of Watson's Theorem for ${}_3F_2(1)$

In this section, the result to be proved is given in the following Theorem.

Theorem 1. *For $\Re(2c-a-b) > 4$, $\Re(a+b) > 4$ and min$\{\Re(a), \Re(b)\} > 0$, we have*

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\quad b,\quad c \\ \frac{1}{2}(a+b+i+1),\ 2c \end{matrix};1\right] \\
&= \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2})} \\
&\times\sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_m(\frac{1}{2}a+\frac{1}{2})_m(\frac{1}{2}b)_m(\frac{1}{2}b+\frac{1}{2})_m}{(c+\frac{1}{2})_m m!} \\
&\times\left\{\frac{A_i^{'}}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])(\frac{1}{2}a+\frac{1}{2})_m(\frac{1}{2}b+\frac{1}{2}i-[\frac{1+i}{2}]+\frac{1}{2})_m}\right. \\
&\left.+\frac{B_i^{'}}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}i-[\frac{i}{2}])(\frac{1}{2}a)_m(\frac{1}{2}b+\frac{1}{2}i-[\frac{i}{2}])_m}\right\}
\end{aligned}
$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, and for $\Re(a) \neq \Re(b)$. The coefficients $A_i^{'}$ and $B_i^{'}$ can be obtained from the tables of A_i and B_i by changing a to $a+2m$ and b to $b+2m$, respectively.

Proof. Denoting the left-hand side by $\mathcal{L}$, we have

$$
\mathcal{L} = {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c \end{matrix};1\right].
$$

Expressing as a series,

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(\frac{1}{2}(a+b+i+1))_k (2c)_k k!}.$$

We can write this in the form

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(\frac{1}{2}(a+b+i+1))_k 2^k k!} \left\{ \frac{2^k (c)_k}{(2c)_k} \right\}.$$

Using a result

$${}_2F_1 \left[\begin{array}{c} -\frac{1}{2}k, \ -\frac{1}{2}k + \frac{1}{2} \\ c + \frac{1}{2} \end{array} ; 1 \right] = \frac{2^k (c)_k}{(2c)_k},$$

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(\frac{1}{2}(a+b+i+1))_k 2^k k!} {}_2F_1 \left[\begin{array}{c} -\frac{k}{2}, \ -\frac{k}{2} + \frac{1}{2} \\ c + \frac{1}{2} \end{array} ; 1 \right].$$

Expressing ${}_2F_1$ as a series, we have, after a little simplification

$$\begin{aligned} \mathcal{L} &= \sum_{k=0}^{\infty} \sum_{m=0}^{[\frac{k}{2}]} \frac{(a)_k (b)_k (-\frac{k}{2})_m (-\frac{k}{2} + \frac{1}{2})_m}{(\frac{1}{2}(a+b+i+1))_k (c+\frac{1}{2})_m 2^k k! m!} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{[\frac{k}{2}]} \frac{(a)_k (b)_k}{(\frac{1}{2}(a+b+i+1))_k 2^{2m+k} (c+\frac{1}{2})_m m! (k-2m)!}. \end{aligned}$$

Using the Bailey's summation method [6, p. 57, Eq. (8)]

$$\sum_{k=0}^{\infty} \sum_{m=0}^{[\frac{k}{2}]} \mathcal{B}(m,k) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{B}(m, k+2m),$$

we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{k+2m} (b)_{k+2m}}{(\frac{1}{2}(a+b+i+1))_{k+2m} 2^{k+4m} (c+\frac{1}{2})_m m! k!}.$$

Using the identity

$$(a)_{k+2m} = (a)_{2m} (a+2m)_k,$$

we have

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m} (b)_{2m}}{(\frac{1}{2}(a+b+i+1))_{2m} (c+\frac{1}{2})_m 2^{4m} m!} \sum_{k=0}^{\infty} \frac{(a+2m)_k (b+2m)_k}{(\frac{1}{2}(a+b+i+1+4m))_k 2^k k!}.$$

Summing up the inner series, we get

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m} (b)_{2m}}{(\frac{1}{2}(a+b+i+1))_{2m} (c+\frac{1}{2})_m 2^{4m} m!} {}_2F_1 \left[\begin{array}{c} a+2m, \ b+2m \\ \frac{1}{2}(a+b+i+1+4m) \end{array} ; \frac{1}{2} \right].$$

It is now easy to see that ${}_2F_1$ can be summed by the known result (9) and after a little algebra, we easily arrive at the right-hand side of Theorem 1. This completes the proof of the Theorem 1. □

Corollary 1. *The following summation formulas hold*

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \ \ c \\ \frac{1}{2}(a+b+2), \ \ 2c \end{matrix};1\right] \\
&= \frac{2^{a+b-1}\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{(a-b)\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\quad\times\left\{\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b+\frac{1}{2})}-\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b)}\right\}
\end{aligned} \tag{10}$$

provided $\Re(2c-a-b)>0$ *and* $\Re(a)\neq\Re(b)$.

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \ \ c \\ \frac{1}{2}(a+b), \ \ 2c \end{matrix};1\right] \\
&= \frac{2^{a+b-2}\Gamma(\frac{1}{2}a+\frac{1}{2}b)\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\quad\times\left\{\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b+\frac{1}{2})}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b)}\right\}
\end{aligned} \tag{11}$$

provided $\Re(2c-a-b)>0$.

The results (10) and (11) have been obtained in [3] by following a different method. Similarly, other results can also be obtained.

Remark 1. *In Theorem 1, if we put* $i=0$*, we get, after a little simplification, the classical Watson's theorem (6), while if we take* $i=1$ *and* $i=-1$*, we arrive at (10) and (11), respectively.*

3. Generalization of Fox's Theorem for ${}_3F_2(\frac{1}{2})$

In this section, generalization of Fox's theorem (8) will be established in the form of theorem.

Theorem 2. *For positive integer* $m\leq\Re(\alpha)-1$, $\Re(\beta-\gamma)>-6$ *and* $\min\{\Re(\beta),\Re(\gamma)\}>0$*, we have*

$${}_3F_2\left[\begin{matrix} \alpha, \quad \beta, \quad \gamma \\ \alpha-m, \ \frac{1}{2}(\beta+\gamma+i+1) \end{matrix};\frac{1}{2}\right]$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}\beta-\frac{1}{2}\gamma-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}\beta-\frac{1}{2}\gamma+\frac{1}{2}|i|+\frac{1}{2})}$$
$$\times \sum_{r=0}^{m} \binom{m}{r} \frac{(\beta)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)} \left\{ \frac{A_i^{'}}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])} \right.$$
$$\left. + \frac{B_i^{'}}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}i-[\frac{i}{2}])} \right\}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, *and for* $\Re(\beta) \neq \Re(\gamma)$.

The coefficients $A_i^{'}$ and $B_i^{'}$ can be obtained from the table of A_i and B_i by changing a to $\beta + r$ and b to $\gamma + r$, respectively.

Proof. If in Fox's general result (7), we put $p = 3$, $q = 2$, $x = \frac{1}{2}$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$, $\rho_1 = \alpha$ and $\rho_2 = \frac{1}{2}(\beta+\gamma+i+1)$, then for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ we have

$$\begin{aligned} {}_3F_2 &\left[\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha-m, & \frac{1}{2}(\beta+\gamma+i+1) & \end{matrix} ; \frac{1}{2} \right] \\ &= \frac{\Gamma(\alpha-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\beta)\Gamma(\gamma)} \\ &\times \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\beta+r)\Gamma(\gamma+r)}{2^r\Gamma(\alpha+r-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2}+r)} \\ &\times {}_2F_1 \left[\begin{matrix} \beta+r, \ \gamma+r \\ \frac{1}{2}(\beta+\gamma+2r+i+1) \end{matrix} ; \frac{1}{2} \right] \end{aligned} \tag{12}$$

${}_2F_1$ on the right-hand side of (12) can now be evaluated with the help of a generalized Gauss's second theorem (9) by taking $a = \beta + r$ and $b = \gamma + r$ and after a little simplification, we arrive at the right-hand side of Theorem 2. This completes the proof of Theorem 2. □

Corollary 2. *For all* $\Re(\alpha) \geq m \geq 0$ *we have the following summation fromulas*

$$\begin{aligned} {}_3F_2 &\left[\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha-m, & \frac{1}{2}(\beta+\gamma+2) & \end{matrix} ; \frac{1}{2} \right] \\ &= \frac{(\beta+\gamma)}{(\beta-\gamma)} \Gamma\left(\frac{1}{2}\right) \Gamma(\alpha-m) \Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\gamma\right) \sum_{r=0}^{m} \binom{m}{r} \frac{(\beta)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)} \\ &\times \left\{ \frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})} - \frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)} \right\} \end{aligned} \tag{13}$$

provided $\Re(\beta) \neq \Re(\gamma)$.

Note: Here we would like to mention that the right-hand side of (13) makes sense when $\beta - \gamma \longrightarrow 0^+$. The reason is given in Remark 2.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma \\ \alpha - m, & \frac{1}{2}(\beta+\gamma) & \end{matrix}; \frac{1}{2}\right] \\
&= \Gamma(\frac{1}{2})\Gamma(\alpha-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma)\sum_{r=0}^{m}\binom{m}{r}\frac{(\beta)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)} \\
&\quad\times\left\{\frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)}+\frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})}\right\}.
\end{aligned}
\tag{14}
$$

Remark 2. *The right-hand side of the result (14) is valid because of the following reason writing the right-hand side of (13) in the form*

$$
\begin{aligned}
R.H.S &= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha-m)\Gamma(\frac{1}{2}\beta+\frac{1}{2}\gamma+1)\Gamma(\frac{1}{2}\beta-\frac{1}{2}\gamma)}{\Gamma(\frac{1}{2}\beta-\frac{1}{2}\gamma+1)}\sum_{r=0}^{m}\binom{m}{r}\frac{(\beta)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)} \\
&\quad\times\left\{\frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})}-\frac{1}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}r+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)}\right\}.
\end{aligned}
\tag{15}
$$

Let us denote the positive quantity $x := \frac{(\beta-\gamma)}{2}$. *Since otherwise* $\Gamma\left(\frac{\beta-\gamma}{2}\right)$ *in (15) is not defined, we can rewrite the right-hand side of (15) for* $x \longrightarrow 0^+$ *as:*

$$
\begin{aligned}
R.H.S &\sim \Gamma(\gamma+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha-m) \\
&\quad\times\sum_{r=0}^{m}\binom{m}{r}\frac{(\gamma)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)\Gamma^2(\frac{1}{2}\gamma+\frac{1}{2}r)\Gamma^2(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})} \\
&\quad\times\left\{\frac{\Gamma(x+\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)-\Gamma(x+\frac{1}{2}\gamma+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})}{x}\right\}.
\end{aligned}
\tag{16}
$$

By employing L'Hospital's rule to (16), we have

$$
\begin{aligned}
R.H.S &\sim \Gamma(\gamma+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha-m) \\
&\quad\times\sum_{r=0}^{m}\binom{m}{r}\frac{(\gamma)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})} \qquad (17)\\
&\quad\times\left\{\frac{\Gamma'(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})}-\frac{\Gamma'(\frac{1}{2}\gamma+\frac{1}{2}r))}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)}\right\}.
\end{aligned}
$$

(18)

$$
\begin{aligned}
R.H.S \sim & \, \Gamma(\gamma+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha-m) \\
& \times \sum_{r=0}^{m} \binom{m}{r} \frac{(\gamma)_r(\gamma)_r}{2^r\Gamma(r+\alpha-m)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2})} \\
& \times\left\{\psi\left(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\gamma+\frac{1}{2}r\right)\right\},
\end{aligned}
\tag{19}
$$

which is finite. Here $\psi = (ln\Gamma)^{'}$ *denotes the digamma function. Obviously, the right-hand side of Theorem 2 and the result (13) makes sense for* β *close to* γ*, and as a by-product, we obtain a new asymptotic formula for*

$$
{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma \\ \alpha-m, & \frac{1}{2}(\beta+\gamma+2) & \end{matrix};\frac{1}{2}\right]
$$

for $\beta-\gamma \longrightarrow 0^{+}$.

Remark 3. *In Theorem 2, if we take* $i=0$*, we get, after a little simplification, Fox's theorem (8) ; for* $i=1$ *and* $i=-1$*, we get (13) and (14), respectively.*

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