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Symmetry-type graphs of Platonic and Archimedean solids

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Abstract. A recently developed theory of flag-graphs and k-orbit maps classifies maps according to their symmetry-type graphs. We propose a similar classification for polyhedra showing that Platonic and Archimedean solids with the same vertex pattern have isomorphic symmetry-type graphs and introducing some tools for the determination of symmetry-type graphs of any polyhedron.

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1. Introduction

The goal of this paper is to propose a classification of polyhedra based on their symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of two kinds: the first ones are defined by all the isometries of Euclidean space preserving a given polyhedron $\mathcal{P}$, and the others only by orientation preserving isometries (rotations) of $\mathbb{R}^3$.
Showing that symmetry-type graphs of Archimedean solids depend only on the type of their vertex pattern (p^q), $(p.q.p.q)$, $(p.q.q)$, $(p.q.q.q)$, $(p.q.r.q)$, $(p.q.r)$ or $(p.p.p.p.q)$ (Theorem 1, Section 3) we make a first step towards such classification. The notation by vertex pattern is explained in [4]. The tools from Section 5 (unique face, position vector, domino rule) and the algorithm from Section 6 can be applied for the determination of symmetry-type graphs of any uniform polyhedron or tiling.

2. Basic notions

In this section we define or mention some basic concepts, notations and facts about polyhedra and their symmetries and list some classes of polyhedra with most symmetries. We also introduce the concepts of a flag graph and a symmetry-type graph and explain how they are represented with drawings.

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2.1. Polyhedra: structure and symmetry

POLYHEDRON. There are many definitions of the concept of a polyhedron. Questions such as: *Are polyhedra surfaces or solids?, What are the milestones in the history of polyhedra?* are discussed for instance in [15]. To us a *polyhedron* $\mathcal{P}$ is a solid in Euclidean space $\mathbb{R}^3$ with given sets of *vertices* $V(\mathcal{P})$, *edges* $E(\mathcal{P})$ and (polygonal or star) *faces* $F(\mathcal{P})$.
TYPE OF A FACE. Two faces $f, g \in F(\mathcal{P})$ are of the same *type*, if they are congruent. The type of a regular polygonal face is denoted by the number of its edges $(3, 4, 5, 6, 8, 10, 12$, etc.), the type of a star face is expressed with two numbers $(5/2, 10/3$, etc.), while the type of a rhomb can be denoted by its acute interior angle α.
ISOMETRIES OF EUCLIDEAN SPACE $\mathbb{R}^3$. The group of isometries $I(\mathbb{R}^3)$ of Euclidean space $\mathbb{R}^3$ consists of translations, reflections (over a plane, over a point or central reflection, and glide-reflections) and rotations.
SYMMETRIES OF A POLYHEDRON. The *symmetry group* $I(\mathcal{P})$ of polyhedron $\mathcal{P}$, defined as the group of isometries $h \in I(\mathbb{R}^3)$ preserving $\mathcal{P}$, consists of the sets of rotations $Rot(\mathcal{P})$ and reflections (over a plane or a point) $Ref(\mathcal{P})$. The elements of $I(\mathcal{P})$ are called *symmetries* of $\mathcal{P}$.
GEOMETRIC DESCRIPTION OF SOME SYMMETRIES OF A POLYHEDRON. Let $R_{f,\alpha}$, $R_{e,\alpha}$, $R_{v,\alpha} \in Rot(\mathcal{P})$ denote the rotation symmetries for the angle α with symmetry axes going through the center of $\mathcal{P}$ and the center of a face f, the center of an edge e and a vertex v, respectively. Let $S_{f,v}, S_{f,e} \in Ref(\mathcal{P})$ denote the reflection symmetries with the reflection planes $\Pi(f, v)$ and $\Pi(f, e)$ orthogonal to f, passing through the center of f and meeting the vertex v or the center of e, respectively.

2.2. Some classes of polyhedra with most symmetries

Polyhedra are fascinating objects, interesting to people because of their beautiful symmetries. Therefore the most studied classes of polyhedra are those with many symmetries, for example:
EQUILATERAL POLYHEDRA. Most symmetries have polyhedra with equilateral faces (regular polygons, regular stars, rhombs).
VERTEX-TRANSITIVE POLYHEDRA. A polyhedron $\mathcal{P}$ is *vertex-transitive* if for any $u, v \in V(\mathcal{P})$ there is a symmetry $h \in I(\mathcal{P})$ such that $h(u) = v$.
UNIFORM POLYHEDRA. A polyhedron $\mathcal{P}$ is *uniform* if it is vertex-transitive and if all its faces are regular polygons or regular stars.
POLYHEDRA WITH THE SAME VERTEX PATTERN. Uniform polyhedra (and uniform tilings, defined likewise, too) can be described by their *vertex pattern* – the cycle of faces around any of their vertices. Some of them, like a snub cube (3.3.3.3.4) and a tiling $(3^4.6)$, have two different forms, being mirror images of each other. A regular-faced polyhedron with only one type of vertex is not necessarily uniform: J.C.P. Miller discovered a non-uniform polyhedron with one vertex type, the same as that of a rhombicuboctahedron (3.4.4.4) ([4], p.172).
REGULAR POLYHEDRA. A polyhedron $\mathcal{P}$ is called *regular* if it has the same number of faces of the same type around each vertex. There are five convex regular polyhedra with regular polygonal faces, called the *Platonic solids*: (3.3.3), (3.3.3.3), (4.4.4),

(3.3.3.3.3), (5.5.5). And there are four non-convex regular polyhedra with regular polygonal or regular star faces, called the *Kepler-Poinsot polyhedra*. Both Platonic and Kepler-Poinsot polyhedra belong to the class of uniform polyhedra.

SEMI-REGULAR POLYHEDRA. A polyhedron $\mathcal{P}$ is called *semi-regular* if it has two or more different types of faces and the same vertex pattern around each vertex. There are thirteen *Archimedean polyhedra*, defined as semi-regular convex uniform polyhedra with regular polygonal faces: (3.4.3.4), (3.5.3.5),(3.6.6), (3.8.8), (4.6.6), (3.10.10), (5.6.6), (3.4.4.4), (3.4.5.4), (4.6.8), (4,6,10),(3.3.3.3.4), (3.3.3.3.5).

2.3. Flags, flag graphs and symmetry-type graphs

FLAGS AND THEIR VERTICES. If all the faces f of a polyhedron $\mathcal{P}$ are regular polygons, we can make a baricentric subdivision of its faces into triangles, called *flags*. The vertices of any such flag Φ, denoted by Φ_2 (the center of the face f incident with Φ), Φ_1 (the center of the edge e incident with Φ) and Φ_0 (the vertex of the edge e incident with Φ), are called the *face*, the *edge* and the *vertex* of Φ, respectively.

ADJACENT FLAGS. Each flag Φ has three *adjacent flags*, sharing an edge with Φ: the 0-*adjacent flag* Φ^0 lies in the same face f as Φ and along the same edge of f; the 1-*adjacent flag* Φ^1 lies in the same face f as Φ, but not along the same edge; the *2-adjacent flag* Φ^2 lies along the same edge of f, but not in the same face as Φ [9].

FLAG GRAPH. The *flag graph* $G_{\mathcal{P}}$ of a polyhedron $\mathcal{P}$ is a graph whose vertex set consists of all the triangles (flags) obtained from the baricentric subdivision of its faces. The edges connecting pairs of adjacent flags (Φ, Φ^0), (Φ, Φ^1), (Φ, Φ^2) are labeled $0, 1$ and 2, respectively.

MONODROMY GROUP. Involutions s_0, s_1 and s_2 of the flag graph, carrying flags $\Phi \in G_{\mathcal{P}}$ into their adjacent flags: $s_0(\Phi) = \Phi^0$, $s_1(\Phi) = \Phi^1$, $s_2(\Phi) = \Phi^2$, satisfy the relations $(s_0 s_2)^2 = id = s_0^2 = s_1^2 = s_2^2$. The group $Mon(\mathcal{P})$, generated by s_0, s_1, s_2 is called the *monodromy group* of $\mathcal{P}$.

AUTOMORPHISMS OF THE FLAG GRAPH. Let $Aut(G_{\mathcal{P}})$ denote the group of automorphisms of the flag graph $G_{\mathcal{P}}$, preserving not only adjacency of vertices of $G_{\mathcal{P}}$ but also the labels $0, 1, 2$ of edges.

COMBINATORIAL DESCRIPTION OF SYMMETRIES OF A POLYHEDRON. Given any two flags Φ and Ψ of $G_{\mathcal{P}}$, there is at most one automorphism $\widetilde{h} \in Aut(G_{\mathcal{P}})$ carrying a flag Φ into a flag Ψ [9]. Thus $\widetilde{h}$ can be denoted by an ordered pair (Φ, Ψ). Consequently, any symmetry $h \in I(\mathcal{P})$ can be described by an ordered pair $(\Phi, \widetilde{h}(\Phi))$. $I(\mathcal{P})$ is isomorphic to a subgroup $\widetilde{I}(\mathcal{P})$ of $Mon(\mathcal{P})$, since to different isometries $h_1, h_2 \in I(\mathcal{P})$ correspond different automorphisms $\widetilde{h_1}, \widetilde{h_2} \in Aut(G_{\mathcal{P}})$. Since $Mon(\mathcal{P})$ is a subgroup of $Aut(G_{\mathcal{P}})$ and $\widetilde{I}(\mathcal{P})$ is a subgroup of $Mon(\mathcal{P})$, any symmetry $h \in I(\mathcal{P})$ can be described by an ordered pair $(\Phi, \widetilde{h}(\Phi))$, where $\widetilde{h} \in Mon(\mathcal{P})$, or denoted by $h(\Phi, \widetilde{h}(\Phi))$.

ORBIT OF A FLAG. The *orbit* $T(\Phi)$ of a flag $\Phi \in V(G_{\mathcal{P}})$ is a set of all flags into which Φ is carried by all the isometries $h \in I(\mathcal{P})$ preserving the polyhedron: $T(\Phi) = \{\widetilde{h}(\Phi), h \in I(\mathcal{P})\}$.

Each member of the orbit $T(\Phi)$ is called a *representative* of that orbit. Now we can describe any symmetry of a given polyhedron $\mathcal{P}$ just by telling which $s \in Mon(\mathcal{P})$ preserve the orbit of a chosen flag Φ.

symmetry-type graph $T(\mathcal{P})$. The quotient graph of $G_{\mathcal{P}}$ under the action of $\widetilde{I}(\mathcal{P})$ (whose vertices are orbits of flags of $G_{\mathcal{P}}$ and whose edges labeled 0, 1 and 2 correspond to labeled edges of their representatives) is called the *symmetry-type graph* of the polyhedron $\mathcal{P}$ and is denoted by $T(\mathcal{P})$. From this definition immediately follows:

$$T(s_i(\Phi)) = s_i(T(\Phi))$$

for all three involutions s_0, s_1, s_2 of the flag graph. Hence $T(s(\Phi) = s(T(\Phi))$ for any $s \in Mon(\mathcal{P})$ and any $\Phi \in G_{\mathcal{P}}$.

symmetry-type graph $T_R(\mathcal{P})$. For the classification of polyhedra we will use also another type of quotient graph of $G_{\mathcal{P}}$, denoted by $T_R(\mathcal{P})$. Here the orbit $T_R(\Phi)$ of Φ consists only of those flags Ψ, for which there is a rotation of polyhedron, carrying Φ into Ψ.

symmetry-type graphs of tilings. For a tiling $\mathcal{T}$ the flag graph $G_{\mathcal{T}}$ and the symmetry-type graphs $T(\mathcal{T})$ and $T_R(\mathcal{T})$ are defined likewise. Since there are four types of plane isometry (reflection over a line, rotation, translation and glide reflection [3], p.26), the symmetry-type graph $T(\mathcal{T})$ is defined by all of them and $T_R(\mathcal{T})$ only by rotations and translations.

2.4. Drawings of symmetry-type graphs

labeling the edges. When we represent flag graphs and symmetry-type graphs with drawings, we usually label the edges with numbers $0, 1, 2$ or color them blue, yellow and red. We can also simply mark the 1-edges with a small line (–) and the 2-edges with two parallel small lines (=).

pre-graphs and half-edges. Flag-graphs are 3-regular, while in the symmetry-type graphs there may be loops (edges connecting an orbit with itself). Introducing the concept of *pre-graphs* and using *half-edges* instead of loops [10] we can represent them as 3-regular pregraphs without loops. Half-edges may be also of three types: $0, 1$ and 2.

3. Classification

Figure 1 shows representations of symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of Platonic and Archimedean solids. Some of these graphs are well known. The notation we use here just indicates the number of their orbits and distinguishes between graphs with the same number of orbits.

Theorem 1. *There are 11 different symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of the 5 Platonic and 13 Archimedean solids and they have have 1, 2, 3, 4, 6, 8, 10 or 12 orbits.*

These symmetry-type graphs can be described by the permutations of orbits $a, b, c, \ldots$, induced by the involutions s_0, s_1, s_2 of flags, as follows:

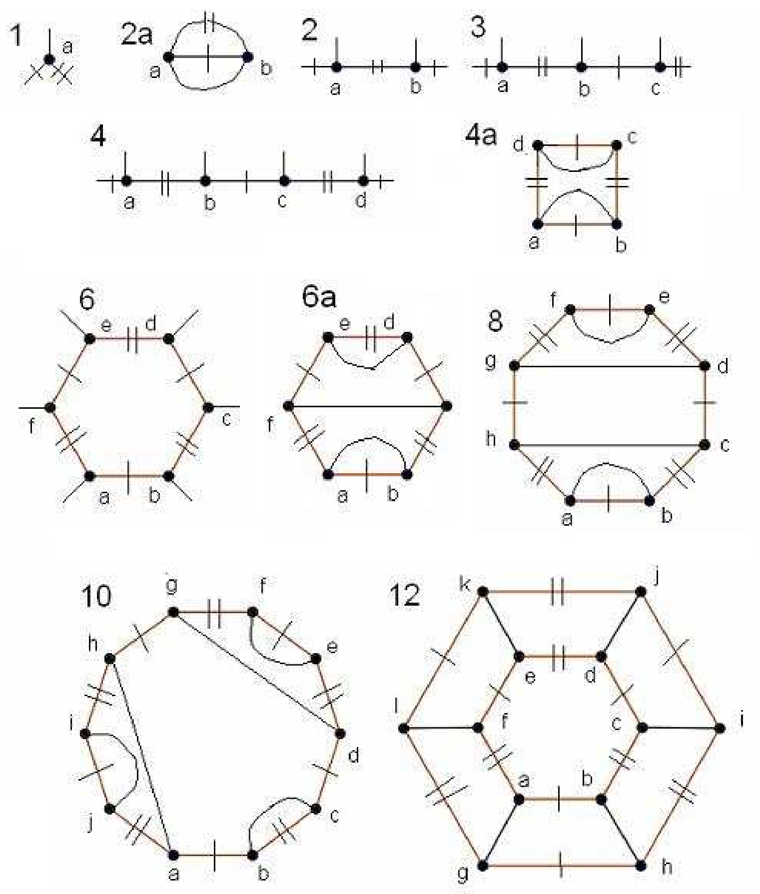

Figure 1: *Symmetry-type graphs of Platonic and Archimedean solids*

graph	s_0	s_1	s_2
1	*id*	*id*	*id*
2a	*(ab)*	*(ab)*	*(ab)*
2	*id*	*id*	*(ab)*
3	*id*	*(a)(bc)*	*(ab)(c)*
4	*id*	*(a)(bc)(d)*	*(ab)(cd)*
4a	*(ab)(cd)*	*(ab)(cd)*	*(bc)(da)*
6	*id*	*(ab)(cd)(ef)*	*(af)(bc)(de)*
6a	*(ab)(cf)(de)*	*(ab)(cd)(ef)*	*(af)(bc)(de)*
8	*(ab)(ch)(dg)(ef)*	*(ab)(cd)(ef)(gh)*	*(ah)(bc)(de)(fg)*
10	*(ah)(bc)(dg)(ef)(ij)*	*(ab)(cd)(ef)(gh)(ij)*	*(aj)(bc)(de)(fg)(hi)*
12	*(ag)(bh)(ci)(dj)(ek)(fl)*	*(ab)(cd)(ef)(gh)(ij)(kl)*	*(af)(bc)(de)(hi)(jk)(lg)*

The symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of Platonic and Archimedean solids $\mathcal{P}$ depend only on the type of their vertex pattern (p^r), $(p.q.p.q)$, $(p.q.q)$, $(p.q.r.q)$, $(p.q.q.q)$, $(p.q.r)$, $(p.p.p.p.q)$. They define the following 6 classes:

class of $\mathcal{P}$	*vertex type of $\mathcal{P}$*	$T(\mathcal{P})$	$T_R(\mathcal{P})$
Regular (Platonic)	(p^q)	*1*	$2a$
Quasi-regular	$(p.q.p.q)$	*2*	$4a$
Truncated regular	$(p.q.q)$	*3*	$6a$
Versi-quasi regular	$(p.q.r.q)$ *and* $(p.q.q.q)$	*4*	*8*
Truncated quasi-regular	$(p.q.r)$	*6*	*12*
Snub quasi-regular	$(p.p.p.p.q)$	*10*	*10*

class	*Platonic or Archimedean solid $\mathcal{P}$*	*vertex pattern*	$T(\mathcal{P})$	$T_R(\mathcal{P})$
I.	*tetrahedron*	*(3.3.3)*	*1*	*2a*
I.	*octahedron*	*(3.3.3.3)*	*1*	*2a*
I.	*cube*	*(4.4.4)*	*1*	*2a*
I.	*icosahedron*	*(3.3.3.3.3)*	*1*	*2a*
I.	*dodecahedron*	*(5.5.5)*	*1*	*2a*
II.	*cuboctahedron*	*(3.4.3.4)*	*2*	*4a*
II.	*icosidodecahedron*	*(3.5.3.5)*	*2*	*4a*
III.	*truncated tetrahedron*	*(3.6.6)*	*3*	*6a*
III.	*truncated cube*	*(3.8.8)*	*3*	*6a*
III.	*truncated octahedron*	*(4.6.6)*	*3*	*6a*
III.	*truncated dodecahedron*	*(3.10.10)*	*3*	*6a*
III.	*truncated icosahedron*	*(5.6.6)*	*3*	*6a*
IV.	*rhombicuboctahedron*	*(3.4.4.4)*	*4*	*8*
IV.	*rhombicosidodecahedron*	*(3.4.5.4)*	*4*	*8*
V.	*truncated cuboctahedron*	*(4.6.8)*	*6*	*12*
V.	*truncated icosidodecahedron*	*(4.6.10)*	*6*	*12*
VI.	*snub cube*	*(3.3.3.3.4)*	*10*	*10*
VI.	*snub dodecahedron*	*(3.3.3.3.5)*	*10*	*10*

The proof of this theorem (the main result of this paper) is given in Section 6. The comparison of this table with the one given in ([13], Figure 13), presenting the derivation of the Platonic and Archimedean polyhedra from the tetrahedron using the operations on polyhedra called truncation Tr, medial Me, snub Sn and dual Du, shows that our classes are related as follows: $Tr(I) = III$, $Tr(II) = VI$, $Me(II) = IV$. Besides this, the two solids from class II are the medials of the two solids from class I.

4. A method: tools for finding $T(\mathcal{P})$ and $T_R(\mathcal{P})$

In this section we introduce some tools used in the proof of Theorem 1 in Section 6. They are also useful for the determination of symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of other polyhedra.

Definition 1 (Reflections S_{ϕ_0} and S_{ϕ_1} over two sides of a flag)**.** *For any flag Φ and for any $i \in \{0, 1\}$ let $S_{\phi_i} : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection over a plane $\prod_{\Phi_i}$ orthogonal to the face containing flags Φ and Φ^i.*

To express a fact that $\mathcal{P}$ is preserved by a S_{Φ_i} we can write: $S_{\Phi_i} \in I(\mathcal{P})$.

Proposition 1. *Let $\mathcal{P}$ be a polyhedron. If there is a reflection symmetry $S_{\Phi_i}(\mathcal{P}) = \mathcal{P}$, it implies $T(\Phi) = T(s_i(\Phi))$. In that case there are half-edges (representing loops) labeled i in the symmetry-type graph $T(G)$.*

Proof. If $S_{\Phi_i}(\mathcal{P}) = \mathcal{P}$, then S_{Φ_i} induces an automorphism of $G_{\mathcal{P}}$, and flags Φ and $s_i(\Phi) = S_i(\Phi)$ belong to the same orbit. □

Definition 2 (Unique face)**.** *A face x is called unique around a given vertex u, if it is the only face of its type incident with u.*

For example, in a polyhedron (3.4.5.4), faces 3 and 5 are unique (around each vertex), while the two identical faces 4 are not. Likewise, in a polyhedron (10.10.5/2), a pentagram star 5/2 is a unique face, while the two regular 10-gons are not.

Proposition 2. *If a uniform polyhedron $\mathcal{P}$ has a unique face x, then $T(\mathcal{P})$ has either n or $2n$ vertices (representing orbits of flags).*

Proof. Uniform polyhedra are vertex-transitive. If x is a unique face of a uniform polyhedron $\mathcal{P}$, then the pairs of flags incident to any of the vertices of x belong to at most two orbits. Since all the orbits contain the same number of flags, there must be at least n orbits of flags in $T(\mathcal{P})$. And since the number of orbits $\#o$ divides $2n$, it is either $\#o = n$ (if there is a reflection symmetry $S_{X_1}(\mathcal{P}) = \mathcal{P}$) or $\#o = 2n$ (if there is no such symmetry). □

If two flags X and Y lie in the same orbit, we can write $X \approx Y$ instead of $T(X) = T(Y)$.

Proposition 3 (The domino rule)**.** *If two flags X and Y lie in the same orbit: $X \approx Y$, the same is true for their adjacent flags: $X^0 \approx Y^0$, $X^1 \approx Y^1$, $X^2 \approx Y^2$. Consequently, $X \approx Y$ if and only if $h(X) \approx h(Y)$ for any h, composed of involutions s_0, s_1 and s_2 of the flag graph. Thus, if $h(X)$ and $h(Y)$ lie on faces of different types, then X and Y cannot lie in the same orbit.*

Proof. This rule follows directly from the definition of a symmetry-type graph. □

It is named after dominoes standing in a line: if one falls, all the others fall as well.

Definition 3 (Odd face). *A face of a polyhedron or a tiling with an odd number of edges is called an odd face.*

Proposition 4. *If a uniform polyhedron $\mathcal{P}$ contains an odd unique face x, then along each edge of x there must be flags X and X^1, hence $X^0 \approx X^1$ and $(T(X))^0 = T(X^1)$. In that case we can find another 0-edge between orbits: $T(X^2)^0 = T(X^0)^2$, since the 0-2 cycles of flags have length 4.*

Proof. An odd unique face contains only two types of flags: X and X^1, which must alternate along the edges. □

Definition 4 (Position vector). *Let $f(X)$ denote the type $3, 4, 5$ or 6 of the face x containing the flag X. The position vector $v(X)$ of a flag X is defined as $v(X) = (f(X), f(s_2(X)))$.*

Proposition 5. *Any two flags X and Y belonging to the same orbit have the same position vectors: if $T(X) = T(Y)$, then $v(X) = v(Y)$. Any two flags X and Y belonging to a pair of 0-adjacent orbits have the same position vectors: if $T(X) = s_0(T(Y))$, then $v(X) = v(Y)$. Thus if two flags have different position vectors, then they cannot lie in the same orbit, and they cannot even lie in the 0-adjacent orbits.*

Proof. This is also a direct consequence of the domino rule. □

Bilinski introduced cyclical sequences $A = [a, b, c, \ldots, \mathrm{k}, l, m]$ with elements $a, b, c, \ldots, k, l, m$ in any set S and used them for the classification of homogeneous planar nets [1]. Likewise, it is very useful to study cycles of flags around a vertex, an edge or a face in a flag graph G (of a polyhedron $\mathcal{P}$ or of a tiling $\mathcal{T}$), or the cycles of orbits of flags (represented by the vertices of the corresponding symmetry-type graphs $T(G)$ or $T_R(G)$). For example, they are useful in the study of local flag arrangements [9], where the numbers of consecutive flags in a cycle are usually written into the triangles representing flags. We represent such cycles as graphs with an even number of vertices (labeled $1, 2, \ldots, 2n$ or $a, b, c, \ldots, k, l, m$), since such representation is already the first step towards drawing the symmetry-type graph of a polyhedron.

Definition 5 (1-2 cycle, 2-0 cycle, 0-1 cycle). *Let G be any 3-regular graph with edges labeled 0,1 and 2.*

A 1-2 cycle is a cyclical sequence of vertices $(X_1, X_2 \ldots X_{2n})$, such that $s_1(X_{2i-1}) = X_{2i}$ and $s_2(X_{2i}) = X_{2i+1 (mod\, 2n)}$ for any $i \in \{1, 2, \ldots, n\}$.

A 2-0 cycle is a cyclical sequence of vertices (X_1, X_2, X_3, X_4), such that $s_2(X_{2i-1}) = X_{2i}$ and $s_0(X_{2i}) = X_{2i+1 (mod\, 4)}$ for any $i \in \{1, 2\}$.

A 0-1 cycle is a cyclical sequence of vertices $(X_1, X_2, \ldots X_{2m})$, such that $s_0(X_{2i-1}) = X_{2i}$ and $s_1(X_{2i}) = X_{2i+1 (mod\, 2n)}$ for any $i \in \{1, 2, \ldots, m\}$.

Proposition 6. *Let P be a flag of the unique face p of an uniform polyhedron $\mathcal{P}$. If there is a reflection symmetry $S_{P_1} \in I(\mathcal{P})$, then $\mathcal{P}$ remains vertex-transitive even if we forbid reflections. The same conclusion holds in a more general case when all the flags of any face p (not necessarily unique) of a uniform polyhedron $\mathcal{P}$ lie in the same orbit.*

Proof. The reflection symmetry S_{P_1} implies all the flags from the faces p are in the same orbit. Since $\mathcal{P}$ is vertex-transitive, we can always find symmetries $h_1, h_2 \in I(\mathcal{P})$ carrying a flag P into a pair of 1-adjacent flags incident with any vertex $v \in V(\mathcal{P})$. One of the symmetries h_1, h_2 is reflection, the other is rotation. Hence $\mathcal{P}$ remains vertex-transitive even if we forbid reflections. $\square$

This proposition is very useful when we try to determine symmetry-type graphs $T_R(\mathcal{P})$ of Platonic and Archimedean solids (and other uniform polyhedra as well), since many of them have both a unique face p and the symmetry S_{P_1}, or all the flags of some face p in the same orbit.

5. The algorithm for finding $T(\mathcal{P})$ and $T_R(\mathcal{P})$

In this section we present an algorithm for finding symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ for any uniform polyhedron $\mathcal{P}$, based on tools devised in Section 4. In Section 6 we apply it to Platonic and Archimedean solids.
To determine symmetry-type graphs $T(\mathcal{P})$ of any uniform solid $\mathcal{P}$ (or any uniform tiling) it suffices to execute the following general procedure:

Algorithm 1 (symmetry-type graphs of uniform solids)**.**

(1) Label the 1-2 cycle of flags around a chosen vertex $1, 2, \ldots, 2n$.

(2) Identify flags of this cycle belonging to the same orbit.

(3) Find the 0-edges between orbits.

This algorithm suggests only *what* to do, not specifying *how* to do it. One possible way is the following:
(1) DRAW A BASIC 1-2 CYCLE. Draw a regular $2n$-gon, label its vertices with numbers $1, 2, \ldots, 2n$ and for each $i \in \{1, \ldots, n\}$ label the edges $(2i-1, 2i)$ with 1 and edges $(2i, 2i+1 (mod\, 2n))$ with 2.
(2) IDENTIFY ORBITS. Use position vectors or other similar tools to obtain a lower bound m for the number $\#o$ of orbits of flags, and use the symmetries of a polyhedron to obtain an upper bound M for $\#o$. Deduce from the vertex pattern which symmetries are possible at all (and then check their existence by looking at a polyhedron net, picture or 3D-model). Thus in the case $(p.q.q)$ there is at most one reflection symmetry and in the case $(p.q.r)$ there are no symmetries, while $(p.q.p.q)$ may have a reflection symmetry and a rotation symmetry around the axis going through the vertex of a polyhedron.
(3) FIND ORIENTATIONS OF 1-2 CYCLES AND LABEL THEM. Find out how the 1-2 cycles of flags labeled $1, 2, \ldots, 2n$ must be oriented in all the adjacent vertices of a chosen vertex.

a) AN ODD UNIQUE FACE. For example, if $\mathcal{P}$ has an odd unique face p, there is an alternating 0-1 cycle of just two types of flags of p (belonging to at most 2 different orbits. Consequently all the 1-2 cycles of flags must have the same orientation, and starting at unique face there is only one way to label the flags around each a vertex with numbers $1, 2, \ldots, 2n$. Now we can determine which orbits are connected with edges labeled 0 simply by looking at the part of the polyhedron net containing labeled 1-2 cycles around one chosen vertex and around all of its adjacent vertices. This works for six Archimedean polyhedra (3.6.6), (3.8.8), (3.10.10), (5.6.6), (3.4.4.4), (3.4.5.4), (3.3.3.3.5) and also for one Archimedean tiling (3.12.12).

b) AN ODD FACE AND A ROTATION SYMMETRY AROUND ITS CENTER. Likewise, if there is an odd face p surrounded by the faces of the same type q and if there is a rotation symmetry around the center of this face for the angle $2\pi/p$, then we have an alternating 0-1 cycle of just two types of flags in p. Consequently, all the 1-2 cycles of flags have the same orientation and the odd flags (labeled with odd numbers) are always 0-adjacent to even flags (labeled with even numbers). This works for all five Platonic polyhedra (3^3), (3^4), (4^3), (3^5) and (5^3), all three regular tilings (3^6), (4^4) and (6^3), eleven Archimedean polyhedra (different from (4.6.8) and (4.6.10)) and six additional Archimedean tilings: (3^2.4.3.4), (3.4.6.4), (3.6.3.6), (3.12^2), (4.8^2) and two enantiomorphic forms of (3^4.6).

c)VERTEX TYPE $(p.q.r)$. This vertex type implies opposite orientations of 1-2 cycles in adjacent vertices. A unique face p uniquely defines labels $1, 2, ..., 2n$ of 1-2 cycles around each vertex. This works for the remaining two Archimedean solids (4.6.8) and (4.6.10) and for the Archimedean tiling (4.6.12).

To determine the symmetry-type graphs $T_R(\mathcal{P})$ of any uniform solid $\mathcal{P}$ (and any uniform tiling) we have to be a little more careful. For if we forbid reflections, the polyhedron may not be vertex-transitive any more.

For example, in the case $(p.q.r)$ we have two different 1-2 cycles of labeled flags: $1, 2, 3, 4, 5, 6$ and $7, 8, 9, 10, 11, 12$ around two types of vertices (say black and white) and the orbits of flags $1, 2, 3, 4, 5, 6$ are different from the orbits of flags $7, 8, 9, 10, 11$, 12. We will see that in the case $(p.q.r)$ both symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ can be simply calculated.

Sometimes (especially when we can prove that the polyhedron remains vertex-transitive even if we forbid reflections) it is better first to determine $T_R(\mathcal{P})$ and then, if there are any reflection symmetries $h(\Phi, \Phi^1)$, we simply identify pairs of vertices in $T_R(\mathcal{P})$ to get $T(\mathcal{P})$.

6. Classification by symmetry-type graphs

In this section we determine symmetry-type graphs of Archimedean and Platonic solids. For each of these solids we also give its uniform notation number Uxy and Wenninger ([17]) notation number Wxy.

Johnson classified uniform polyhedra into ten classes according to the types of their vertex figure. Since our classification of Archimedean and Platonic polyhedra by their symmetry-type graphs coincides with his, we use his names (regular, quasi-regular, truncated regular, versi quasi-regular, truncated quasi-regular and snub quasi-regular polyhedra) for our six different classes of Archimedean and Platonic polyhedra.

6.1. The proof of Theorem 1

We will consider the following six classes of Platonic and Archimedean solids:
Class I (regular polyhedra) consists of the five Platonic solids with vertex pattern (p^q):

U01 W01 Tetrahedron (3.3.3)
U05 W02 Octahedron (3.3.3.3)
U06 W06 Cube (4.4.4)
U22 W04 Icosahedron (3.3.3.3.3)
U23 W05 Dodecahedron (5.5.5).

It is easy to see that for any of these solids $\mathcal{P}$ there are reflection symmetries carrying any chosen flag Φ into its adjacent flags Φ^0, Φ^1, Φ^2; consequently, there is only one orbit of flags in $T(\mathcal{P})$ and all the edges must be half-edges. Hence (by Proposition 6) $\mathcal{P}$ remains vertex-transitive even if we forbid reflections. Rotations $R_{p,2\pi/p}$ around the center of each face p for the angle $2\pi/p$ ensure that there are only two orbits of flags in $T_R(\mathcal{P})$. All the flags at odd distances from any chosen initial flag Φ (if we define the distance between two flags as the smallest number of involutions needed to come from one to the other) lie in the same orbit a and all flags at even distances from Φ lie in the same orbit b of $T_R(\mathcal{P})$. Since adjacent pairs of flags lie in different orbits a and b, the orbits a and b are connected with edges labeled 0, 1 and 2.
Class II (quasi-regular Archimedean polyhedra) consists of two solids with vertex pattern ($p.q.p.q$):

U07 W11 Cuboctahedron (3.4.3.4) and
U24 W12 Icosidodecahedron (3.5.3.5).

Here let us first find $T_R(\mathcal{P})$. These two polyhedra are the only two Archimedean polyhedra without unique faces. But they both have an odd face p with 3 edges and a rotation symmetry $R_{3,2\pi/3}$ around the center of any triangle. We already know that this implies that the 1-2 cycles of flags $1, 2, 3, 4, 5, 6, 7, 8$ around each of the vertices of p have the same orientation. Four of them $(1, 2, 5, 6)$ belong to the triangles and the other four $(3, 4, 7, 8)$ to squares or pentagons. But since these two polyhedra are both symmetrical by the rotation symmetries R_π with axes going through their vertices, they both have at most four orbits of flags: $1 \approx 5, 2 \approx 6$ in the triangles and $3 \approx 7, 4 \approx 8$ in the squares or pentagons. Thus the flags 1 and 2 form an alternating 0-1 cycle $1, 2, 1, 2, 1, 2$ in any triangle, hence $1^0 \approx 2, 5^0 \approx 6$. The fact that 0-2 cycles have the length 4 implies $3^0 = 3^{202} \approx 8$ and likewise $4^0 = 4^{202} \approx 7$. So there are half-edges labeled 0 in each of the orbits $1, 2, 3, 4$. (This could also be seen directly by looking at 3D-models of polyhedra, since there are obviously reflection symmetries $S_{f,e}$ in any face f). Thus $T_R(\mathcal{P})$ of both polyhedra has four orbits $a(1, 5), b(2, 6), c(3, 7), d(4, 8)$ forming a 1-2 cycle, and these orbits are permuted by involutions s_0, s_1, s_2 like this: $s_0 = (a)(b)(c)(d)$, $s_1 = (a)(bc)(d)$, $s_2 = (ab)(cd)$.
If we also allow reflection symmetries, the reflection symmetry $h(1, 2)$ (it can also be written as S_{P_1}) implies $1 \approx 2, 3 \approx 8$ and the symmetry-type graph $T(\mathcal{P})$ is described

with the permutations of 2 orbits $a(1,2,5,6), b(3,8,4,7)$ like this: $s_0 = (a)(b)$, $s_1 = (a)(b)$, $s_2 = (ab)$.

Class III (TRUNCATED REGULAR ARCHIMEDEAN POLYHEDRA) consists of five solids with the vertex pattern $(p.q.q)$:

U02 W6 Truncated tetrahedron (3.6.6)
U08 W7 Truncated octahedron (4.6.6)
U09 W8 Truncated cube (3.8.8)
U25 W9 Truncated icosahedron (5.6.6)
U26 W10 Truncated dodecahedron (3.10.10)

Here again let us first determine $T_R(\mathcal{P})$. We have three faces around each vertex with flags $1,2,3,4,5,6$. So the number of orbits in the symmetry-type graph $T(\mathcal{P})$ is at most 6. The unique face p is surrounded only by faces q. It is easy to see that all the above polyhedra $(p.q.q)$ are symmetrical by rotations for the angle $2\pi/p$ around the axes going through the center of any of the unique faces and orthogonal to it. Since the vertex-transitivity of each solid is ensured by these rotations, we can label flags 1, 2, 3, 4, 5, 6 around each vertex also in our computations of $T_R(\mathcal{P})$. These rotations also imply that we have an alternating 0-1 cycle of flags of type 1 and 2 in p. Hence all the 1-2 cycles around each vertex have the same orientation, thus odd flags are always 0-neighbours of some even flags. So the 0-edges must be: $1^0 \approx 2$, $3^0 = 3^{202} \approx 6$, $5^0 \approx 4$. The flags $1,3$ and 5 must lie in different orbits since they have different position vectors: 1 lies in a p-gon and 1^2 lies in a q-gon, 3 lies in a q-gon and 3^2 lies in a p-gon, 5 lies in a q-gon and 5^2 lies in a q-gon. So we have at least three orbits of flags. But since we have no rotations in axes going through vertices of polyhedra $(p.q.q)$, the symmetry-type graph $T_R(\mathcal{P})$ has 6 orbits $a(1)$, $b(2)$, $c(3)$, $d(4)$, $e(5)$, $f(6)$, and can be described with the permutations $s_0 = (a)(b)(c)(d)(e)(f)$, $s_1 = (ab)(cd)(ef)$, $s_2 = (bc)(de)(fa)$. Since all the above polyhedra are symmetrical with respect to the reflection S_{P_1}, the orbits of the following pairs of flags are the same: $1 \approx 2$, $3 \approx 6$, $5 \approx 4$, hence $T(\mathcal{P})$ has 3 orbits $a(1,2), b(3,6), c(5,4)$ and can be described with the permutations: $s_0 = (a)(b)(c)$, $s_1 = (a)(bc)$, $s_2 = (ab)(c)$.

Class IV (VERSI QUASI-REGULAR ARCHIMEDEAN POLYHEDRA) consists of two solids with the vertex pattern $(p.q.q.q)$ and $(p.q.r.q)$:

U10 W13 Rhombicuboctahedron (3.4.4.4) and
U27 W14 Rhombicosidodecahedron (3.4.5.4).

Let us first find $T_R(\mathcal{P})$. In both of these polyhedra we have a unique face $p = 3$ and a rotation symmetry around its center for the angle $2\pi/p$; hence all the 0-1 cycles of labeled flags around each vertex are oriented in the same direction. In the unique face $p = 3$ there is an alternating 0-1 cycle of flags 1 and 2. Hence $1^0 \approx 2$ and therefore $3^0 = 3^{202} \approx 8$.

In (3.4.5.4) we have another unique face r, hence $5^0 \approx 6$ and therefore $4^0 = 4^{202} \approx 7$ and the symmetry type $T_R(\mathcal{P})$ of (3.4.5.4) is known.

Likewise we have $5^0 \approx 6$ and consequently $4^0 \approx 7$ also in (3.4.4.4). For all other options lead into contradiction: in $T_R(\mathcal{P})$ there are no half-edges, and the only remaining cases $5^0 \approx 7$, $4^0 \approx 8$ or $5^0 \approx 4$, $7^0 \approx 8$ would imply a 1-0 cycle of period 6 in each of the squares having only 8 flags.

The reflection symmetry $h(1,2)$ identifies pairs of orbits: $1 \approx 2$, $3 \approx 8$, $5 \approx 6$ and $7 \approx 4$. Since there are at least three different position vectors of flags $((p,q),(q,p)$

and (q, q)), there are at least four orbits in $T(\mathcal{P})$, and we already know all the 0-edges between them, so $T(\mathcal{P})$ is also known.

Class V (truncated quasi-regular Archimedean polyhedra) consists of two solids with the vertex pattern $(p.q.r)$:

U11 W15 Truncated cuboctahedron (4.6.8) and

U28 W16 Great truncated icosidodecahedron (4.6.10)

In that case we can find the symmetry-type graph $T(\mathcal{P})$ as follows: The flags $1, 2, 3, 4, 5, 6$ have different position vectors: $v(1) = (p, r)$, $v(2) = (p, q)$, $v(3) = (q, p)$, $v(4) = (q, r)$, $v(5) = (r, q)$, $v(6) = (r, p)$, so they must lie in different orbits. Consequently, since the position vectors of 0-adjacent flags are the same, the 0-adjacent flags belong to the same orbit. In other words, all the 0-edges in $T(\mathcal{P})$ must be half-edges. We could see this directly by looking at 3D-models of polyhedra. But the value of the above argument lies in the fact that it applies to all uniform polyhedra with the vertex pattern $(p.q.r)$.

We have already mentioned in Section 5 that in the calculation of $T_R(\mathcal{P})$ in the case of the vertex type $(p.q.r)$ we have two types of vertices (say white and black) and two 1-2 cycles of labeled flags $1, 2, 3, 4, 5, 6$ around the white vertices and 7, 8, 9, 10, 11, 12 around the black ones. These two types of cycles have opposite orientations, thus 0-neighbours of odd flags are odd flags and 0-neighbours of even flags are even flags. Since the position vectors of 0-adjacent flags must be the same, the 0-edges between 12 different orbits must be the following: $1^0 \approx 7$, $2^0 \approx 8$, $3^0 \approx 9$, $4^0 \approx 10$, $5^0 \approx 11$, $6^0 \approx 12$.

Class VI (Snub quasi-regular Archimedean polyhedra) consists of two solids with the vertex pattern $(p.p.p.p.q)$:

U12 W17 Snub cube (3.3.3.3.4) and

U29 W18 Snub dodecahedron (3.3.3.3.5)

It is easy to see that they have no reflection symmetries, therefore for them $T_R(\mathcal{P}) = T(\mathcal{P})$. (We can also prove algebraically that all flags around one vertex lie in different orbits, for if we count the distances of these flags from the two closest unique faces in the polyhedron net, we always get a flag with a unique pair at such distances. And since all orbits contain the same number of flags, each of these ten flags must lie in a different orbit.) Since they both have a unique face q and rotation symmetry around its center for the angle $2\pi/q$, they must have the same orientation of 1-2 cycles of flags $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ around each vertex, and we can determine 0-edges in their symmetry-type graphs by labeling the 1-2 cycles of ten flags around each vertex in the polyhedron net, starting at the unique face. Each of these two polyhedra has two enantiomorphic forms, which are mirror images of each other, and the same holds for their nets. But their symmetry-type graphs are isomorphic. This completes the proof of Theorem 1.

Using the techniques described in Sections 3, 4 and 5 it is easy to see that there is an Archimedean tiling $\mathcal{T}$ with a vertex type $(3^3.4^2)$ having symmetry-type graphs different from all the aforementioned ones: $T(\mathcal{T})$ has five orbits and $T_R(\mathcal{T})$ has 10 orbits.

Likewise, for the uniform solid (3.5/3.3.5/2.3.3) we get a new symmetry-type graph $T(\mathcal{P}) = T_R(\mathcal{P})$ with 12 orbits. These symmetry-type graphs can be described by permutations s_0, s_1, s_2 of orbits like this:

	$T(3^3.4^2)$	$T_R(3^3.4^2)$	$T(3.5/3.3.5/2.3.3)$
s_0	$(ab)(c)(d)(e)$	$(ab)(cj)(dg)(ef)(hi)$	$(aj)(be)(cd)(fi)(gh)(kl)$
s_1	$(a)(bc)(de)$	$(ab)(cd)(ef)(gh)(ij)$	$(ab)(cd)(ef)(gh)(ij)(kl)$
s_2	$(ab)(cd)(e)$	$(bc)(de)(fg)(hi)(ja)$	$(bc)(de)(fg)(hi)(jk)(la)$

The first two symmetry-type graphs have no symmetries. The symmetry-type graph of the polyhedron (3.5/3.3.5/2.3.3) can be represented by a regular 12-gonal drawing possessing rotation symmetry for the angle $2\pi/3$.

7. Symmetry-type graphs and operations on polyhedra

Flag graphs and symmetry-type graphs can be defined purely algebraically and for any map [8],[9]. A *k-orbit map* [8] is a map M with k distinct orbits of flags. By recent discoveries in the theory of k-orbit maps the medial $Me(M)$ of a k-orbit map is either a k-orbit map (if and only if M is self dual) or a $2k$-orbit map. Likewise, for any map M its truncation $Tr(M)$ is either a k-orbit map, a $3k/2$-orbit map or a $3k$-map([8], p. 420).

Proposition 7. *For any polyhedron $\mathcal{P}$ obtained from a tetrahedron by a sequence of operations medial Me, dual Du and truncation Tr, the number of orbits of $T(\mathcal{P})$ and $T_R(\mathcal{P})$ is of the form $2^a 3^b, a, b \in \mathbb{N}$.*

Proof. This is an obvious corollary of the above mentioned result on k-orbit maps and the fact that the symmetry-type graph of the dual has the same vertices and edges, and interchanged labels 0 and 2 of edges. □

Now we can ask: Is it possible to determine symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of any polyhedron $\mathcal{P} = F(\mathcal{Q})$, obtained from another polyhedron $\mathcal{Q}$ by a transformation F, composed of operations on polyhedra like Tr, Me, Du, Sn, etc., directly from the following two data:
a) the sequence of these operations transforming $\mathcal{Q}$ to $\mathcal{P}$,
b) the symmetry-type graphs $T_R(\mathcal{Q})$ and $T(\mathcal{Q})$ of polyhedron $\mathcal{Q}$?
If this could be done, the classification of polyhedra by their symmetry-types would be much easier.
Most recent research in this area [6] shows that the medials of two self-dual maps with the same symmetry-type graph can be different! The reason for this lies in different kinds of transformations of maps establishing self-duality of a given map. But since most of the symmetry-type graphs obtained from Archimedean and Platonic polyhedra are not self-dual (only types 1 and 2a from our Theorem 1 are self-dual), we can still hope to calculate the symmetry-types of most polyhedra, derived from this family of polyhedra by operations like medial, truncation, etc.
Thus we come to the following conjecture: (OPERATIONS DETERMINE SYMMETRY TYPES) For any non self-dual polyhedron $\mathcal{P}$ the symmetry-type graphs $T(Tr(\mathcal{P}))$, $T(Me(\mathcal{P}))$, $T_R(Tr(\mathcal{P}))$ and $T_R(Me\ (\mathcal{P}))$ depend only on the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$.
An analogous conjecture can be made for any map M and also for other operations on polyhedra (Snub Sn, Dual Du, Leapfrog $Le = Tr(Du)$ [13], etc.). If this conjecture turns out to be true, it would make the classification of polyhedra by

their symmetry-type graphs much easier, at least in theory. But this would really work only if we could solve another problem: (FIND INDUCED OPERATIONS ON SYMMETRY-TYPE GRAPHS). For any basic operation $O \in \{Tr, Me, Du, Sn\}$ defined on any non self-dual polyhedron $\mathcal{P}$ find an algorithm for deducing the symmetry-type graphs $T(O((\mathcal{P}))$ and $T_R(O(\mathcal{P}))$ directly from the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$.
So the question is: can we find operations Tr^*, Me^*, Du^*, Sn^* on symmetry-type graphs such that $T(O(\mathcal{P})) = O^*(T(\mathcal{P}))$ holds for any operation $O \in \{Tr, Me, Du, Sn\}$ and for any non self-dual polyhedron $\mathcal{P}$?
One way to verify or falsify our conjecture would be by using computer programs like Vega ([12]). If our conjecture is true, then the problem of finding operations Tr^*, Me^* on symmetry-type graphs should not be too difficult to solve (at least for reasonably small k), since by the already mentioned results of the theory of k-orbit maps ([8], p. 420) there are only two or three possible numbers of orbits of $Tr(\mathcal{P})$ and $Me(\mathcal{P})$. And maybe these operations Tr^*, Me^* on symmetry-type classes, if they exist at all, could be even described by some simple operations on the corresponding symmetry-type graphs. Then we could draw a graph of symmetry-classes whose vertices would correspond to symmetry-type graphs and its directed edges to operations on them. Indeed, such graphs have already been made for the medials of maps with at most four orbits in [6].
Such a transformation Du^* on symmetry-type graphs exists at least for the operation dual Du: it leaves the vertices and edges, and interchanges the labels 0 and 2 of edges.

8. Summary

We have seen (Theorem 1) that the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of any Archimedean and Platonic solid $\mathcal{P}$ depend only on its vertex type:

class of $\mathcal{P}$	vertex type of $\mathcal{P}$	$T(\mathcal{P})$	$T_R(\mathcal{P})$
Regular (Platonic)	(p^q)	1	$2a$
Quasi-regular	$(p.q.p.q)$	2	$4a$
Truncated regular	$(p.q.q)$	3	$6a$
Versi-quasi regular	$(p.q.r.q)$ and $(p.q.q.q)$	4	8
Truncated quasi-regular	$(p.q.r)$	6	12
Snub quasi-regular	$(p.p.p.p.q)$	10	10

Our classification of Archimedean and Platonic solids by their symmetry-type graphs is thus in perfect accordance with Johnson's classification for uniform polyhedra (see the names in the left column of the table above). But since it is based on symmetry-type graphs, it can be extended to the classification of any family of polyhedra and tilings, too.

The comparison of this table with the one given in [13] together with some results of the theory of k-orbit maps [8] suggested our conjecture that the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of a given polyhedron can be simply calculated if we know how a given polyhedron $\mathcal{P} = F(\mathcal{Q})$ can be obtained by the sequence F of operations like truncation, medial, dual, snub, etc. from some other polyhedron $\mathcal{Q}$, at least if $\mathcal{Q}$ is not a self-dual polyhedron.

This paper builds on the ideas and results from many papers from this area: A classification of edge-transitive maps has been made in [7]. An enumeration of edge-transitive types is given in [2]. The classification of all edge-transitive maps in the torus according to their automorphism group type is given in [16]. Flag graphs first appeared in [5] (there the term gems, an acronym for graph-encoded maps, was used). Flag-graphs and transformations on maps are discussed in [11]. The question of enumeration of uniform polyhedra (also with skew faces) is discussed in Problem 26 of [14].

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