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Exact solutions of the mKdV equation with time-dependent coefficients

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Abstract. In this paper, we study the time variable coefficient modified Korteweg-de Vries (mKdV) equation from group-theoretic point of view. We obtain Lie point symmetries admitted by the mKdV equation for various forms for the time variable coefficients. We use the symmetries to construct the group-invariant solutions for each of the cases of the arbitrary variable coefficients. Finally, the solitary wave ansatz will be used to carry out the integration of the mKdV equation that will be supported by a concrete example.

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1. Introduction

There are various nonlinear evolution equations (NLEEs) that are being presently studied in the literature. The issues that are addressed in these studies are the integrability aspects, conservation laws, symmetry analysis, stochasticity aspects and others. One of the main interesting and most widely studied aspects is the integrability issue. Sometimes these NLEEs have time-dependent coefficients that are more closely related to real life situations. Therefore it is more practical that these time-dependencies be taken into account.

There are various methods to address the integrability issues. Some of these common methods of integrability issues are Adomian decomposition method, Inverse Scattering Transform, G'/G method, exponential function method, F-expansion method and many others (see e.g. [1, 2, 9, 11, 8, 7, 12, 13, 14, 15] and the references therein). In this paper, the method of Lie symmetry will be used to carry out the integration of an important NLEE that is the mKdV equation with time-dependent coefficients. For the theory and application of the Lie symmetry groups the reader is referred to [3, 6, 10]. This equation is used as a mathematical model to study physical

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phenomena arising in several areas of interest. For example, in the study of coastal waves in ocean and liquid drops and bubbles, in the issues of atmospheric blocking phenomenon and dipole blocking (see [2, 9, 15]). The dimensionless form of the mKdV equation that will be studied in this paper with time-dependent coefficients is given by

$$q_t + q^2 q_x + a(t)q + b(t)q_{xxx} = 0. \tag{1}$$

In (1), the first term represents the evolution term while the second term represents the nonlinear term. The third term represents linear damping while the fourth term is the dispersion term. The time-dependent coefficients of damping and dispersion are $a(t)$ and $b(t)$, respectively. Equation (1) will be studied by Lie symmetry methods and also 1-soliton solution will be obtained by the solitary wave ansatz method.

2. Lie point symmetries and group-invariant solutions

In this section, we present the Lie point symmetry generators obtained for various cases of the time variable coefficients $a(t)$ and $b(t)$. Moreover, we obtain the exact solutions for these special cases using the symmetry generators or combination of symmetry generators of the equations.

Case 2.1. $a(t) = \dfrac{1}{t}$, $b(t) = \dfrac{K}{t^2}$, K is a constant.

In this case, equation (1) takes the form

$$q_t + q^2 q_x + \frac{1}{t}q + \frac{K}{t^2} q_{xxx} = 0. \tag{2}$$

A vector field

$$X = \xi^1(t,x,q)\frac{\partial}{\partial t} + \xi^2(t,x,q)\frac{\partial}{\partial x} + \eta(t,x,q)\frac{\partial}{\partial q}, \tag{3}$$

is a generator of point symmetry of equation (2) if

$$X^{[3]}\left(q_t + q^2 q_x + \frac{1}{t}q + \frac{K}{t^2} q_{xxx}\right) = 0, \tag{4}$$

whenever (2) is satisfied. Here the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t \frac{\partial}{\partial q_t} + \zeta_x \frac{\partial}{\partial q_x} + \zeta_{xxx} \frac{\partial}{\partial q_{xxx}}$$

and the coefficients ζ_t, ζ_x and ζ_{xxx} are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - q_t D_t(\xi^1) - q_x D_t(\xi^2),\\
\zeta_x &= D_x(\eta) - q_t D_x(\xi^1) - q_x D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x) - q_{xt} D_x(\xi^1) - q_{xx} D_x(\xi^2),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - q_{xxt} D_x(\xi^1) - q_{xxx} D_x(\xi^2).
\end{aligned}$$

The operator D_i denotes the total derivative operator and it is defined by

$$D_i = \frac{\partial}{\partial x^i} + q_i \frac{\partial}{\partial q} + q_{ij} \frac{\partial}{\partial q_j} + \ldots, \quad i = 1, 2,$$

and $(x^1, x^2) = (t, x)$. Moreover, we use the notations $q_1 = q_t, q_2 = q_x, q_{11} = q_{tt}, q_{21} = q_{xt}$, and so on, to denote a different order of partial derivatives of the dependent variable q with respect to the independent variables t and x.

The functions ξ^1, ξ^2 and η are calculated by solving the determining equation (4). The coefficients ξ^1, ξ^2 and η are independent of the derivatives of q. Hence the coefficients of like derivatives of q in (4) can be equated to yield an overdetermined system of linear partial differential equations (PDEs). Therefore, the determining equation for symmetries after some tedious calculations yield

$$\xi^1 = \xi^1(t),\, \xi^2_q = 0,\, \eta_{qq} = 0, \tag{5}$$

$$-\frac{2K}{t^3}\,\xi^1 + \frac{K}{t^2}\,\xi^1_t - \frac{3K}{t^2}\,\xi^2_x = 0, \tag{6}$$

$$\eta_{xq} - \xi^2_{xx} = 0, \tag{7}$$

$$2q\eta + q^2\,\xi^1_t - \xi^2_t - q^2\,\xi^2_x + \frac{K}{t^2}\,(3\eta_{xxq} - \xi^2_{xxx}) = 0, \tag{8}$$

$$-\frac{1}{t^2}q\,\xi^1 + \frac{1}{t}\eta + \eta_t - \frac{1}{t}q\,\eta_q + \frac{1}{t}q\,\xi^1_t + q^2\,\eta_x + \frac{K}{t^2}\eta_{xxx} = 0. \tag{9}$$

Solving equations (5-9) we obtain

$$\xi^1 = c_1\,t^2 - c_3 t,\ \xi^2 = \frac{c_3}{3}x + c_2,\ \eta = \left(\frac{2c_3}{3} - c_1\,t\right)q,$$

where c_1, c_2, c_3 are constants. Thus, equation (2) admits a three-dimensional Lie algebra of symmetries given by

$$X_1 = \frac{\partial}{\partial x},\ X_2 = t^2\frac{\partial}{\partial t} - tq\frac{\partial}{\partial q},\ X_3 = -t\frac{\partial}{\partial t} - \frac{x}{3}\frac{\partial}{\partial x} + \frac{2q}{3}\frac{\partial}{\partial q}.$$

First we consider the symmetry generator X_2 for the group-invariant solution. The Lagrange system of equations corresponding to X_2 is given by

$$\frac{dt}{t^2} = \frac{dx}{0} = \frac{dq}{-tq}. \tag{10}$$

Solving the equations in (10) yields the following group invariants, namely, $\gamma = x$, $\beta = tq$, so that a group-invariant solution to (2) is given by $q = t^{-1}h(x)$, where $h(x)$ is an arbitrary function of its argument. Substituting the derivatives of q with respect to t and x into equation (2) gives a third-order nonlinear ordinary differential equation (ODE) with a dependent variable h and an independent variable γ, viz.,

$$Kh^{'''} + h^2 h^{'} = 0. \tag{11}$$

We integrate equation (11) with respect to γ once to obtain

$$h^{''} = -\frac{h^3}{3K} + \tilde{A}, \tag{12}$$

where $\tilde{A}$ is a constant. Further integration of (12) yields the following nonlinear first-order ODE

$$\left(\frac{dh}{d\gamma}\right)^2 = A + Bh - \frac{h^4}{6K}, \tag{13}$$

where A and B are constants of integration.
If $A = B = 0$ and $K = -1$, then solving equation (13), we find that

$$h(\gamma) = \frac{\sqrt{6}}{C \pm \gamma},$$

where C is a constant. Hence a group-invariant solution to (2) is

$$q(x,t) = \frac{\sqrt{6}}{t(C \pm x)}.$$

Now we consider a linear combination of the symmetry generators X_1 and X_2, i.e., $X_1 + X_2$. The group invariants are

$$\gamma = x + \frac{1}{t}, \quad \beta = tq.$$

Thus the group-invariant solution of (2) corresponding to $X_1 + X_2$ is given by $q(x,t) = t^{-1}h(x+1/t)$, where $h(\gamma)$ satisfies the following third-order nonlinear ODE

$$Kh^{'''} + h^2h^{'} - h^{'} = 0. \tag{14}$$

We integrate equation (14) with respect to γ once to obtain

$$h^{''} = -\frac{h^3}{3K} + \frac{h}{K} + \tilde{A}, \tag{15}$$

where $\tilde{A}$ is a constant. Further integration of (15) yields the following nonlinear first-order ODE

$$\left(\frac{dh}{d\gamma}\right)^2 = A + Bh + \frac{h^2}{K} - \frac{h^4}{6K}, \tag{16}$$

where A and B are constants.
If we take $A = B = 0$ and $K = 1$, then solving equation (16) yields

$$h(\gamma) = \sqrt{6}\ \mathrm{sech}(C \pm \gamma),$$

where C is a constant. Hence a solitary wave solution to (2) is

$$q(x,t) = \frac{\sqrt{6}}{t}\ \mathrm{sech}\left[C \pm \left(x + \frac{1}{t}\right)\right].$$

Now if we choose $A = B = 0$ and $K = -1$ we obtain another group-invariant solution to (2), namely

$$q(x,t) = \frac{\sqrt{6}}{t}\ \sec\left[C \pm \left(x + \frac{1}{t}\right)\right],$$

where C is a constant.

Remark 1. *Here we restrict ourselves to obtaining some interesting particular solutions of equation (2) by letting the constants, namely A and B, to be zero in (13) and (16), respectively. However, in general equations (13) and (16) with $A \neq 0$ and $B \neq 0$ have solutions expressed in terms of Jacobi elliptic functions (see e.g., [4, 5]).*

Case 2.2. $a(t) = K_0,\ b(t) = K_1 e^{-2K_0 t},\ K_0, K_1$ are constants.
In this case, equation (1) becomes

$$q_t + q^2 q_x + K_0 q + K_1 e^{-2K_0 t} q_{xxx} = 0. \tag{17}$$

Lie point symmetries admitted by equation (17) are given by

$$X_1 = \frac{\partial}{\partial x},\ X_2 = e^{2K_0 t}\frac{\partial}{\partial t} - K_0 e^{2K_0 t} q \frac{\partial}{\partial q},\quad X_3 = -\frac{\partial}{\partial t} + \frac{2K_0 x}{3}\frac{\partial}{\partial x} + \frac{K_0 q}{3}\frac{\partial}{\partial q}.$$

The symmetry generator X_2 gives rise to the group-invariant solution

$$q(x,t) = e^{-K_0 t} h(x),$$

where $\gamma = x$ and $h(\gamma)$ satisfies the third-order nonlinear ODE

$$K_1 h^{'''} + h^2 h^{'} = 0. \tag{18}$$

This ODE is similar to the ODE (11) and hence it reduces to

$$\left(\frac{dh}{d\gamma}\right)^2 = A + Bh - \frac{h^4}{6K_1}, \tag{19}$$

where A and B are constants. In (19) by taking $A = B = 0$ and $K_1 = -1$ and solving the equation one obtains the exact group-invariant solution to (17) given by

$$q(x,t) = \frac{\sqrt{6}\ e^{-K_0 t}}{(C \pm x)},$$

where C is a constant.

A linear combination of the symmetry generators X_1 and X_2, i.e., $X_1 + X_2$ gives the following group-invariant solution of (17)

$$q(x,t) = e^{-K_0 t} h\left(x + \frac{e^{-2K_0 t}}{2K_0}\right),$$

where $\gamma = x + e^{-2K_0 t}/2K_0$ and $h(\gamma)$ satisfies the following third-order nonlinear ODE

$$K_0 h^{'''} + h^2 h^{'} - h^{'} = 0. \tag{20}$$

Equation (20) can be simplified as before to the following nonlinear first-order ODE

$$\left(\frac{dh}{d\gamma}\right)^2 = A + Bh + \frac{h^2}{K_1} - \frac{h^4}{6K_1}, \tag{21}$$

where A and B are constants.
By choosing $A = B = 0$ and $K_1 = 1$ and then solving equation (21) we find that

$$h(\gamma) = \sqrt{6}\ \mathrm{sech}(C \pm \gamma),$$

where C is a constant. Hence a group-invariant solution to (17) is

$$q(x,t) = \sqrt{6}\ e^{-K_0 t}\ \mathrm{sech}\left[C \pm \left(x + \frac{e^{-2K_0 t}}{2K_0}\right)\right].$$

Likewise, one can obtain another exact group-invariant solution

$$q(x,t) = \sqrt{6}\ e^{-K_0 t}\ \sec\left[C \pm \left(x + \frac{e^{-2K_0 t}}{2K_0}\right)\right],$$

where C is a constant, to (17) by taking $A = B = 0$ and $K_1 = -1$ in (21).

3. Solitary wave solutions

In order to obtain a solitary wave solution to (1), the starting hypothesis is

$$q(x,t) = \frac{A(t)}{\cosh^p\left[B(t)\,(x - v(t))\right]}, \tag{22}$$

where A represents the amplitude of the soliton, while B is the inverse width of the soliton and v represents the velocity of the soliton. It needs to be noted that since damping and dispersion have time-dependent coefficients, one needs to have, in general,

$$A = A(t), \tag{23}$$
$$B = B(t) \tag{24}$$

and

$$v = v(t). \tag{25}$$

Thus from (22) one gets

$$q_t = \frac{dA}{dt}\frac{1}{\cosh^p\tau} + pAB\left(v + t\frac{dv}{dt}\right)\frac{\tanh\tau}{\cosh^p\tau} - \frac{pA}{B}\frac{dB}{dt}\frac{\tau\tanh\tau}{\cosh^p\tau}, \tag{26}$$
$$q^2 q_x = -pA^3B\frac{\tanh\tau}{\cosh^{3p}\tau}, \tag{27}$$
$$a(t)q = \frac{a(t)A}{\cosh^p\tau}, \tag{28}$$
$$b(t)q_{xxx} = p(p+1)(p+2)b(t)AB^3\frac{\tanh\tau}{\cosh^{p+2}\tau} - p^3b(t)AB^3\frac{\tanh\tau}{\cosh^p\tau}. \tag{29}$$

Substituting (26) to (29) into (1) yields

$$\begin{aligned}&\left(\frac{dA}{dt}+a(t)A\right)\frac{1}{\cosh^p\tau}+pAB\left(v+t\frac{dv}{dt}-b(t)p^2B^2\right)\frac{\tanh\tau}{\cosh^p\tau}\\&-\frac{pA}{B}\frac{dB}{dt}\frac{\tau\tanh\tau}{\cosh^p\tau}-pA^3B\frac{\tanh\tau}{\cosh^{3p}\tau}+p(p+1)(p+2)b(t)AB^3\frac{\tanh\tau}{\cosh^{p+2}\tau}=0.\end{aligned} \tag{30}$$

From (30), one can say that the last two terms match up, provided the exponent of the cosh functions are the same. This gives

$$3p = p + 2, \tag{31}$$

which yields

$$p = 1. \tag{32}$$

Also, from (30), one can see that the functions $1/\cosh^p\tau$, $\tanh\tau/\cosh^p\tau$, $\tau\tanh\tau/\cosh^p\tau$ are linearly independent and therefore their coefficients must vanish, respectively. This leads to the following relations

$$\frac{dA}{dt}+a(t)A = 0, \tag{33}$$

$$v + t\frac{dv}{dt} = bp^2B^2 \tag{34}$$

and

$$\frac{dB}{dt} = 0. \tag{35}$$

From (33), (34) and (35), one can respectively conclude

$$A(t) = A_0 e^{-\int a(t)dt}, \tag{36}$$

$$v(t) = \frac{p^2B^2}{t}\int b(t)dt \tag{37}$$

and

$$B(t) = \text{constant}, \tag{38}$$

where A_0 is the initial amplitude of the soliton. In order to determine the constant width B, one needs to set the sum of the coefficients of the last two terms in (30) to zero. This gives

$$A^2 = b(p+1)(p+2)B^2, \tag{39}$$

which yields

$$B(t) = 2\left[\frac{A_0^2 e^{-2\int a(t)dt}}{24b(t)}\right]^{\frac{1}{2}} \tag{40}$$

so that from (40), one needs to have $b(t) > 0$. In order to conform to the fact that $B(t)$ must be a constant, one needs to have the constraint of the time-dependent coefficients $a(t)$ and $b(t)$ from (40) related as

$$b(t) = ke^{-2\int a(t)dt} \tag{41}$$

for some positive constant k. Thus, the solitary wave solution to (1) is finally given by

$$q(x,t) = \frac{A(t)}{\cosh\left[B\left(x - v(t)\right)\right]} \tag{42}$$

where the soliton parameters $A(t)$, $B(t)$ and $v(t)$ are given by (36), (40) and (37), respectively, while the coefficients $a(t)$ and $b(t)$ are related as given in (41). The only necessary condition for the solitons to exist is that the time-dependent coefficients $a(t)$ and $b(t)$ must be Riemenn integrable, as evident from (37) and (41).

3.1. Example

In this section, the same example, as considered before [11] will be studied, to illustrate the above technique. The gKdV equation considered here is

$$q_t + q^2 q_x + aq + ce^{-2at} q_{xxx} = 0, \tag{43}$$

where a and c are constants. Starting with the same ansatze as in (22), equation (30) modifies to

$$\begin{aligned}
&\left(\frac{dA}{dt} + aA\right)\frac{1}{\cosh^p \tau} - \frac{pA}{B}\frac{dB}{dt}\frac{\tau \tanh\tau}{\cosh^p\tau} \\
&\quad -pA^3B\frac{\tanh\tau}{\cosh^{3p}\tau} + pAB\left(v + t\frac{dv}{dt} - cp^2B^2e^{-2at}\right)\frac{\tanh\tau}{\cosh^p\tau} \\
&\quad +p(p+1)(p+2)cAB^3e^{-2at}\frac{\tanh\tau}{\cosh^{p+2}\tau} = 0,
\end{aligned} \tag{44}$$

which leads to the solitary wave solution

$$q(x,t) = \frac{A}{\cosh\left[B\left(x - vt\right)\right]}, \tag{45}$$

where

$$A(t) = A_0 e^{-at}, \tag{46}$$

$$B = \frac{1}{\sqrt{6c}} \tag{47}$$

and

$$v(t) = -\frac{A_0^2 e^{-2at}}{12at}. \tag{48}$$

The constraint relation (40) is meaningful here as c is a constant as seen from the hypothesis. From (46), it can be seen that

$$\lim_{t\to\infty} A(t) = \begin{cases} 0, & a > 0 \\ \infty, & a < 0, \end{cases} \tag{49}$$

while from (48), one can conclude that

$$\lim_{t\to\infty} v(t) = \begin{cases} 0, & a > 0 \\ \infty, & a < 0, \end{cases} \tag{50}$$

which shows that the amplitude and velocity of the soliton die down gradually, provided $a > 0$. Thus, the solitary wave solution of (43) is given by (45) with the respective parameters defined in (46), (47) and (48).

4. Concluding remarks

We have studied the mKdV equation with time variable coefficients using the Lie symmetry method. This approach has enabled us to obtain various possible forms for time variable coefficients of the underlying equation. For each of the possible forms of the coefficients of the equation, we derived the Lie point symmetries admitted by the equation and then used them to construct group-invariant solutions. Finally, the 1-soliton solution is obtained by using the solitary wave ansatz. In this context, it was established that time-dependent coefficients must be Riemann integrable for the solitons to exist.

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