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ARH – quasigroups

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Abstract. In this paper, the concept of an ARH–quasigroup is introduced and identities valid in that quasigroup are studied. The geometrical concept of an affine–regular heptagon is defined in a general ARH–quasigroup and geometrical representation in the quasigroup $\mathbb{C}(2\cos\frac{\pi}{7})$ is given. Some statements about new points obtained from the vertices of an affine–regular heptagon are also studied.
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1. Definition and examples

A quasigroup $(Q,\cdot)$ will be called an *ARH–quasigroup* if it satisfies the identities of *idempotency* and *mediality*, i.e. we have the identities

$$aa = a, \tag{1}$$
$$ab\cdot cd = ac\cdot bd, \tag{2}$$

and besides, if the identity

$$(a\cdot ab)b = ba \tag{3}$$

also holds.

Example 1. *Let $(G,+)$ be a commutative group in which there is an automorphism φ which satisfies the identity*

$$(\varphi\circ\varphi\circ\varphi)(a) - (\varphi\circ\varphi)(a) - \varphi(a) - \varphi(a) + a = 0, \tag{4}$$

where $\circ$ is the composition of functions. Let us define multiplication $\cdot$ on the set G by the formula

$$ab = a + \varphi(b-a). \tag{5}$$

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Now we shall prove that $(G, \cdot)$ is an ARH–quasigroup.

For each $a, b \in G$, because of (5), the equations $ax = b$ and $ya = b$ are equivalent to the equations

$$a + \varphi(x - a) = b \quad \text{and} \quad y + \varphi(a) - \varphi(y) = b. \tag{6}$$

The first equation has a unique solution $x = a + \varphi^{-1}(b - a)$, and the second one can be written in the form

$$(\varphi \circ \varphi)(y) - (\varphi \circ \varphi \circ \varphi)(y) = (\varphi \circ \varphi)(b) - (\varphi \circ \varphi \circ \varphi)(a),$$

i.e. according to (4) in the form

$$y - \varphi(y) - \varphi(y) = (\varphi \circ \varphi)(b) - (\varphi \circ \varphi \circ \varphi)(a)$$

or owing to (6) it gets the form

$$b - \varphi(a) - \varphi(y) = (\varphi \circ \varphi)(b) - (\varphi \circ \varphi \circ \varphi)(a).$$

The last equation has a unique solution

$$y = \varphi^{-1}[b - (\varphi \circ \varphi)(b) - \varphi(a) + (\varphi \circ \varphi \circ \varphi)(a)],$$

which also satisfies equation (6) as due to (4) we get

$$\begin{aligned} y - \varphi(y) &= \varphi^{-1}[b - (\varphi \circ \varphi)(b) - \varphi(a) + (\varphi \circ \varphi \circ \varphi)(a)] \\ &\quad -[b - (\varphi \circ \varphi)(b) - \varphi(a) + (\varphi \circ \varphi \circ \varphi)(a)] \\ &= \varphi^{-1}[b - \varphi(b) - (\varphi \circ \varphi)(b) + (\varphi \circ \varphi \circ \varphi)(b)] \\ &\quad -[a - \varphi(a) - (\varphi \circ \varphi)(a) + (\varphi \circ \varphi \circ \varphi)(a)] \\ &= \varphi^{-1}[\varphi(b)] - \varphi(a) = b - \varphi(a). \end{aligned}$$

We have just proved that $(G, \cdot)$ is a quasigroup. Its idempotency is obvious from (5). According to (5), it follows

$$\begin{aligned} ab \cdot cd &= ab + \varphi(cd - ab) = a + \varphi(b - a) + \varphi[c + \varphi(d - c) - a - \varphi(b - a)] \\ &= a - \varphi(a) - \varphi(a) + (\varphi \circ \varphi)(a) + \varphi(b) - (\varphi \circ \varphi)(b) + \varphi(c) - (\varphi \circ \varphi)(c) \\ &\quad +(\varphi \circ \varphi)(d) \end{aligned}$$

and the symmetry of the obtained expression by b and c proves mediality (2). From (5) firstly follows

$$a \cdot ab = a + \varphi[a + \varphi(b - a) - a] = a - (\varphi \circ \varphi)(a) + (\varphi \circ \varphi)(b)$$

and by means of (4) we get

$$\begin{aligned} (a \cdot ab)b &= a \cdot ab + \varphi(b - a \cdot ab) \\ &= a - (\varphi \circ \varphi)(a) + (\varphi \circ \varphi)(b) + \varphi[b - a + (\varphi \circ \varphi)(a) - (\varphi \circ \varphi)(b)] \\ &= a - \varphi(a) - (\varphi \circ \varphi)(a) + (\varphi \circ \varphi \circ \varphi)(a) + \varphi(b) + (\varphi \circ \varphi)(b) - (\varphi \circ \varphi \circ \varphi)(b) \\ &= \varphi(a) + b - \varphi(b) = b + \varphi(a - b) = ba; \end{aligned}$$

therefore, identity (3) also holds.

Example 2. *Let $(F,+,\cdot)$ be a field in which the equation*

$$q^3 - q^2 - 2q + 1 = 0 \tag{7}$$

has the solution q and the operation $*$ on the set F is defined by

$$a * b = (1-q)a + qb. \tag{8}$$

Then it is obvious that the identity $\varphi(a) = qa$ defines an automorphism of the commutative group $(F,+)$, and since equality (7) holds, then for each $a \in F$ equality (4) holds. However, equation (8) can be written in the form

$$a * b = a + \varphi(b-a),$$

and based on Example 1 it follows that $(F,*)$ is a ARH–quasigroup.

Example 3. *Let $(\mathbb{C},+,\cdot)$ be a field of complex numbers and $*$ a binary operation on the set $\mathbb{C}$ defined by (8), where q is the solution of (7). Example 2 implies that $(\mathbb{C},*)$ is an ARH–quasigroup.*

Namely, besides the trivial solution $\varphi = 0$, the equation $\sin 3\varphi = \sin 4\varphi$ has also the solutions $\varphi \in \{\frac{\pi}{7}, \frac{3\pi}{7}, \frac{9\pi}{7}\}$ due to $\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}$, $\sin\frac{9\pi}{7} = \sin\frac{12\pi}{7}$, $\sin\frac{27\pi}{7} = \sin\frac{36\pi}{7}$. That equation gets the form

$$8\sin\varphi\cos^3\varphi - 4\sin\varphi\cos^2\varphi - 4\sin\varphi\cos\varphi + \sin\varphi = 0,$$

and without the factor $\sin\varphi$ it can be written in the form (7) where $q = 2\cos\varphi$. Therefore, the solutions of the obtained equation are

$$q_1 = 2\cos\frac{\pi}{7} \sim 1,8019377,\ q_2 = 2\cos\frac{3\pi}{7} \sim 0,4450419,\ q_3 = 2\cos\frac{9\pi}{7} \sim -1,2469796. \tag{9}$$

For each of these three solutions we get a nice geometrical interpretation which justifies studying ARH–quasigroups and defining geometrical concepts in them. Let us consider the set of complex numbers as a set of the points of Euclidean plane. For two different points a and b equation (8) can be written in the form

$$\frac{a*b-a}{b-a} = q,$$

meaning that points a, b, $a*b$ form the cross ratio q. Let symbol $*_i$ represent the operation $*$ defined by formula (8) with value $q = q_i$, where, with $i \in \{1,2,3\}$, that value q_i is given by (9). Then, Figure 1 shows all three operations $*$, for which $(\mathbb{C},*)$ is an ARH–quasigroup.

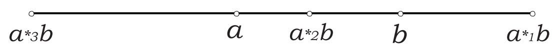

Figure 1:

These three quasigroups will be denoted by $\mathbb{C}(q_1)$, $\mathbb{C}(q_2)$ and $\mathbb{C}(q_3)$ because for $a = 0$ and $b = 1$ we get $a *_i b = q_i$. Each identity in an ARH–quasigroup can be interpreted as a geometrical theorem. So, in Figure 2 the illustration of the identity (3) in the quasigroup $\mathbb{C}(q_1)$ is given, where instead of e.g. $a *_1 b$ it will be written ab, and such notation will be used in all figures.

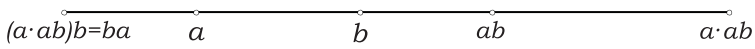

Figure 2:

2. Basic properties of ARH–quasigroups

Direct consequences of identities (1) and (2) are the following identities of *elasticity* and *left* and *right distributivity*, i.e.

$$ab \cdot a = a \cdot ba, \tag{10}$$

$$a \cdot bc = ab \cdot ac, \tag{11}$$

$$ab \cdot c = ac \cdot bc. \tag{12}$$

Because of (12), identity (3) can also be written in the form

$$ab \cdot (ab \cdot b) = ba, \tag{13}$$

which will be very useful later. Let us prove the following theorem now.

Theorem 1. *In an ARH–quasigroup* $(Q, \cdot)$ *the following identities*

$$a(ab \cdot b) = ba \cdot a, \tag{14}$$

$$(a \cdot ab)c = (c \cdot cb)a, \tag{15}$$

$$(a \cdot bc)c = c(ba \cdot a) \tag{16}$$

hold.

Proof. (see [2], Th. 27) We get successively

$$a(ab \cdot b) \overset{(11)}{=} (a \cdot ab) \cdot ab \overset{(11)}{=} (a \cdot ab)a \cdot (a \cdot ab)b \overset{(3)}{=} (a \cdot ab)a \cdot ba \overset{(12)}{=} (a \cdot ab)b \cdot a \overset{(3)}{=} ba \cdot a.$$

For each $a, b, c \in Q$ there is $d \in Q$ such that

$$cd = b. \tag{17}$$

Now we get

$$\begin{aligned}(a \cdot ab)c \cdot (a \cdot ab)c &\overset{(1)}{=} (a \cdot ab)c \overset{(17)}{=} a(a \cdot cd) \cdot c \overset{(11)}{=} a(ac \cdot ad) \cdot c \overset{(11)}{=} (a \cdot ac)(a \cdot ad) \cdot c \\ &\overset{(12)}{=} (a \cdot ac)c \cdot (a \cdot ad)c \overset{(3)}{=} ca \cdot (a \cdot ad)c \overset{(2)}{=} c(a \cdot ad) \cdot ac \\ &\overset{(11)}{=} (ca)(ca \cdot cd) \cdot ac \overset{(17)}{=} (ca)(ca \cdot b) \cdot ac \overset{(12)}{=} (ca)(cb \cdot ab) \cdot ac \\ &\overset{(2)}{=} (c \cdot cb)(a \cdot ab) \cdot ac \overset{(2)}{=} (c \cdot cb)a \cdot (a \cdot ab)c,\end{aligned}$$

so identity (15) follows. For each $a, b, c \in Q$ there is an element $e \in Q$ such that

$$ae = b, \tag{18}$$

and then there follows

$$\begin{aligned}(a\cdot bc)c &\overset{(18)}{=} a(ae\cdot c)\cdot c \overset{(12)}{=} a(ac\cdot ec)\cdot c \overset{(11)}{=} (a\cdot ac)(a\cdot ec)\cdot c \overset{(12)}{=} (a\cdot ac)c\cdot (a\cdot ec)c\\ &\overset{(3)}{=} ca\cdot (a\cdot ec)c \overset{(11)}{=} ca\cdot (ae\cdot ac)c \overset{(18)}{=} ca\cdot (b\cdot ac)c \overset{(12)}{=} ca\cdot (bc)(ac\cdot c)\\ &\overset{(2)}{=} (c\cdot bc)\cdot a(ac\cdot c) \overset{(14)}{=} (c\cdot bc)(ca\cdot a) \overset{(2)}{=} (c\cdot ca)(bc\cdot a) \overset{(12)}{=} (c\cdot ca)(ba\cdot ca)\\ &\overset{(12)}{=} (c\cdot ba)\cdot ca \overset{(11)}{=} c(ba\cdot a),\end{aligned}$$

thus identity (16) holds. □

3. Affine–regular heptagon

From now on let $(Q,\cdot)$ be any ARH–quasigroup. The elements of the set Q are said to be *points*.

Theorem 2. *In the cyclical order of seven equalities* $a_i a_{i+1} = a_{i+3}a_{i+2}$ $(i = 1,2,3,4,5,6,7)$*, where the indexes are taken modulo 7 from the set* $\{1,2,3,4,5,6,7\}$*, each four consecutive equalities imply the remaining three equalities.*

Proof. It is sufficient to prove that from equalities

$$a_1a_2 = a_4a_3, \tag{19}$$

$$a_2a_3 = a_5a_4, \tag{20}$$

$$a_3a_4 = a_6a_5, \tag{21}$$

$$a_4a_5 = a_7a_6 \tag{22}$$

follows the equality

$$a_5a_6 = a_1a_7. \tag{23}$$

Firstly, let us prove that equalities (19) and (20) imply the equality

$$a_3\cdot a_3a_4 = a_1a_5, \tag{24}$$

and then (with the substitution $i\to i+2$) in the same way from (21) and (22) follows the equality

$$a_5\cdot a_5a_6 = a_3a_7. \tag{25}$$

Indeed, we get successively

$$\begin{aligned}(a_3\cdot a_3a_4)a_2 &\overset{(15)}{=} (a_2\cdot a_2a_4)a_3 \overset{(12)}{=} a_2a_3\cdot(a_2a_3\cdot a_4a_3)\\ &\overset{(1)}{=} (a_2a_3\cdot a_2a_3)\cdot(a_2a_3\cdot a_2a_3)(a_4a_3)\\ &\overset{(20)}{=} (a_5a_4\cdot a_2a_3)\cdot(a_5a_4\cdot a_2a_3)(a_4a_3)\\ &\overset{(2)}{=} (a_5a_2\cdot a_4a_3)\cdot(a_5a_2\cdot a_4a_3)(a_4a_3)\\ &\overset{(19)}{=} (a_5a_2\cdot a_1a_2)\cdot(a_5a_2\cdot a_1a_2)(a_1a_2) \overset{(12)}{=} (a_5a_1\cdot a_2)\cdot(a_5a_1\cdot a_1)a_2\\ &\overset{(12)}{=} (a_5a_1)(a_5a_1\cdot a_1)\cdot a_2 \overset{(12)}{=} (a_5\cdot a_5a_1)a_1\cdot a_2 \overset{(3)}{=} a_1a_5\cdot a_2,\end{aligned}$$

wherefrom follows equality (24). Then we also prove equality (23), which follows from

$$a_5a_6 \cdot a_5 \overset{(1)}{=} (a_5a_6 \cdot a_5)(a_5a_6 \cdot a_5) \overset{(10)}{=} (a_5 \cdot a_6a_5)(a_5a_6 \cdot a_5) \overset{(21)}{=} (a_5 \cdot a_3a_4) \cdot (a_5a_6 \cdot a_5)$$
$$\overset{(2)}{=} (a_5 \cdot a_5a_6)(a_3a_4 \cdot a_5) \overset{(25)}{=} a_3a_7 \cdot (a_3a_4 \cdot a_5) \overset{(2)}{=} (a_3 \cdot a_3a_4) \cdot a_7a_5$$
$$\overset{(24)}{=} a_1a_5 \cdot a_7a_5 \overset{(12)}{=} a_1a_7 \cdot a_5.$$

□

We shall say that $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is an *affine–regular heptagon* with the vertices a_1, a_2, a_3, a_4, a_5, a_6, a_7 and it is denoted by ARH$(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ if any four adjacent equalities, and then also all seven, out of seven equalities $a_ia_{i+1} = a_{i+3}a_{i+2}$ $(i = 1, 2, 3, 4, 5, 6, 7)$ hold (Figure 3).

The possibility of introducing the concept of the affine–regular heptagon into this quasigroup justifies naming it an ARH–quasigroup.

Corollary 1. *If* $(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *is any cyclic permutation of* $(1, 2, 3, 4, 5, 6, 7)$ *or of* $(7, 6, 5, 4, 3, 2, 1)$*, then from ARH* $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ *follows ARH* $(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}, a_{i_7})$.

Corollary 2. *An affine–regular heptagon is uniquely determined by any three adjacent vertices of its vertices.*

Proof. If e.g. a_1, a_2, a_3 are the given vertices, then the vertices a_4, a_5, a_6, a_7 are given by the identities (19) – (22), respectively. □

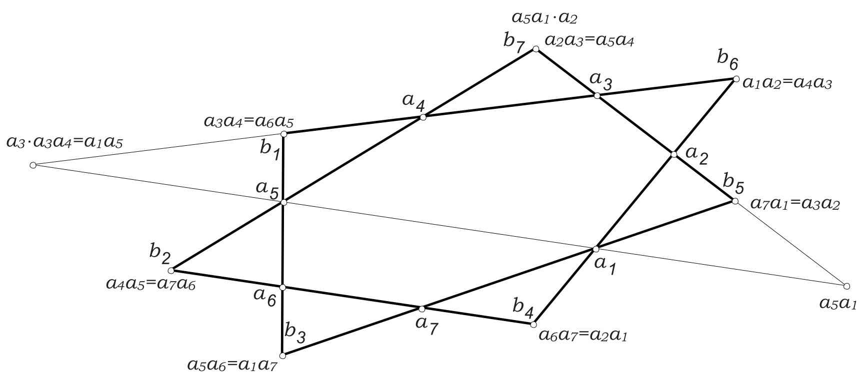

Figure 3:

Theorem 3. *An affine–regular heptagon is uniquely determined by any three of its vertices.*

Proof. By means of the statement of Corollary 2 it is necessary to prove three more statements which, by means of cyclical permutation, can lead to the proofs of statements (i), (ii) and (iii) out of the following proofs:

(i) The vertices a_1, a_2, a_4 determine the vertex a_3 uniquely. This statement is obvious from (19).

(ii) The vertices a_1, a_2, a_5 determine the vertex a_3 uniquely. In fact, let a_3 be the point such that $a_5a_1 \cdot a_2 = a_2a_3$, and then let a_4 be the point such that $a_1a_2 = a_4a_3$. It is necessary to prove that equality $a_2a_3 = a_5a_4$ holds. Here is the proof of this fact:

$$a_2a_3 \cdot a_2a_3 \overset{(1)}{=} a_2a_3 = a_5a_1 \cdot a_2 \overset{(12)}{=} a_5a_2 \cdot a_1a_2 = a_5a_2 \cdot a_4a_3 \overset{(2)}{=} a_5a_4 \cdot a_2a_3.$$

(iii) The vertices a_1, a_3, a_5 determine the vertex a_4 uniquely. In fact, let a_4 be the point such that $a_3 \cdot a_3a_4 = a_1a_5$, and then let a_2 be the point so that $a_2a_3 = a_5a_4$. It is necessary to prove the equality $a_1a_2 = a_4a_3$. This equality follows from this conclusion:

$$\begin{aligned} a_1a_2 \cdot a_5a_3 &\overset{(2)}{=} a_1a_5 \cdot a_2a_3 = (a_3 \cdot a_3a_4) \cdot a_5a_4 \overset{(11)}{=} (a_3 \cdot a_3a_4)a_5 \cdot (a_3 \cdot a_3a_4)a_4 \\ &\overset{(3)}{=} (a_3 \cdot a_3a_4)a_5 \cdot a_4a_3 \overset{(2)}{=} (a_3 \cdot a_3a_4)a_4 \cdot a_5a_3 \overset{(3)}{=} a_4a_3 \cdot a_5a_3. \end{aligned}$$

□

Let us examine more precisely those points, which can be explicitly expressed by means of the vertices of the affine–regular heptagon $\mathrm{ARH}(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. Let us always take the indexes modulo 7 from the set $\{1, 2, 3, 4, 5, 6, 7\}$. The same products of the adjacent vertices from the definition of an affine–regular heptagon will be labelled so that

$$a_{i+2}\, a_{i+3} = b_i = a_{i-2}\, a_{i-3} \tag{26}$$

(Figures 3 and 4). Besides that, let us state

$$b_{i,i+3} = a_i\, a_{i+3}, \qquad b_{i,i-3} = a_i\, a_{i-3} \tag{27}$$

(Figure 4).

Equality (24) can be written in the form $a_3b_1 = b_{15}$, and then according to Corollary 1, general equalities

$$b_{i,i-3} = a_{i+2}\, b_i = a_{i+2} \cdot a_{i+2}\, a_{i+3}, \tag{28}$$

$$b_{i,i+3} = a_{i-2}\, b_i = a_{i-2} \cdot a_{i-2}\, a_{i-3} \tag{29}$$

are valid. Since by substituting $i \to -i$ equality (24) implies the equality $a_4 \cdot a_4a_3 = a_6a_2$, i.e. the equality $b_{62} = a_4b_6$, then we get

$$b_{63}b_7 \overset{(27),(26)}{=} a_6a_3 \cdot a_2a_3 \overset{(12)}{=} a_6a_2 \cdot a_3 = b_{62}a_3 = a_4b_6 \cdot a_3 \overset{(26)}{=} (a_4 \cdot a_4a_3)a_3 \overset{(3)}{=} a_3a_4 \overset{(26)}{=} b_1,$$

what implies general equalities

$$b_{i-2,i+1}\, a_{i+2} = b_i, \qquad b_{i+2,i-1}\, a_{i-2} = b_i, \tag{30}$$

$$b_{i-2,i+2}\, b_{i-1} = b_i, \qquad b_{i+2,i-2}\, b_{i+1} = b_i. \tag{31}$$

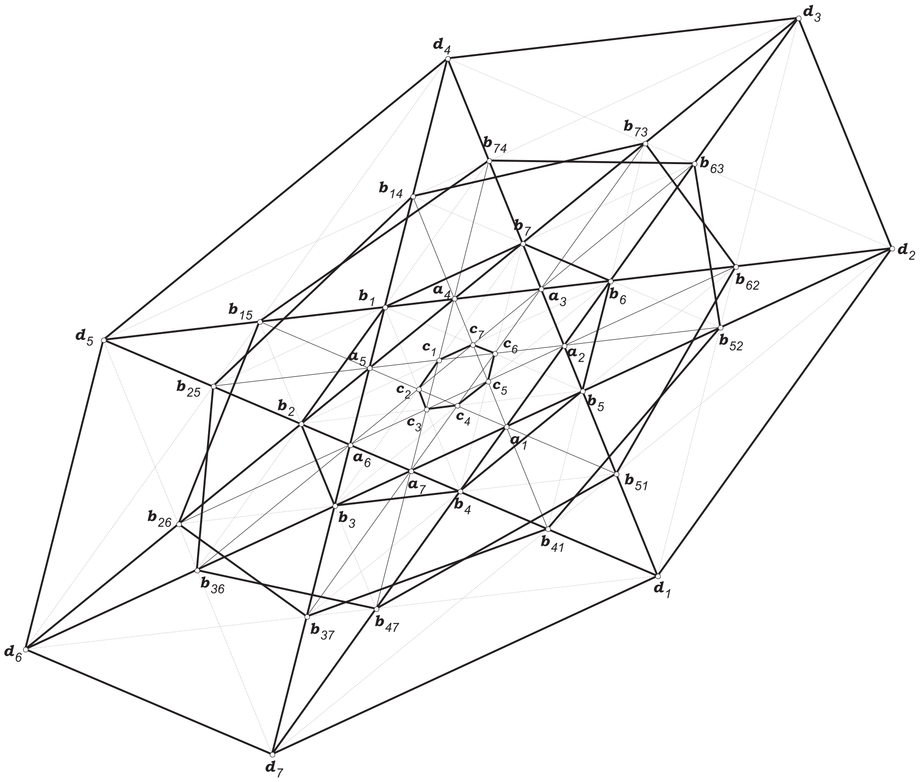

Figure 4:

Further, we get

$$\begin{aligned} b_{52}a_2 \cdot b_6 &\overset{(26)}{=} b_{52}a_2 \cdot a_1a_2 \overset{(12)}{=} b_{52}a_1 \cdot a_2 \overset{(30)}{=} b_3a_2 \overset{(26)}{=} a_1a_7 \cdot a_2 \overset{(12)}{=} a_1a_2 \cdot a_7a_2 \\ &\overset{(26)}{=} a_4a_3 \cdot a_7a_2 \overset{(2)}{=} a_4a_7 \cdot a_3a_2 \overset{(27),(26)}{=} b_{47} \cdot a_7a_1 \overset{(11)}{=} b_{47}a_7 \cdot b_{47}a_1 \\ &\overset{(30)}{=} b_{47}a_7 \cdot b_6, \end{aligned}$$

so the equality $b_{52}a_2 = b_{47}a_7$ follows. The obtained equal products are labelled by c_1 in Figure 4. In general, we get equalities

$$b_{i-3,i+1}\, a_{i+1} = c_i = b_{i+3,i-1}\, a_{i-1}, \tag{32}$$

which define the points c_i in Figure 4. Because of (27), equalities (32) can be written in the form

$$a_{i-3}\, a_{i+1} \cdot a_{i+1} = c_i = a_{i+3}\, a_{i-1} \cdot a_{i-1}. \tag{33}$$

Now we get

$$a_5c_1 \overset{(33)}{=} a_5(a_5a_2 \cdot a_2) \overset{(14)}{=} a_2a_5 \cdot a_5 \overset{(33)}{=} c_6,$$

and generally the equalities

$$a_{i-3}\, c_i = c_{i-2}, \qquad a_{i+3}\, c_i = c_{i+2} \tag{34}$$

are valid. Besides that, we get

$$b_{25}c_6 \overset{(27),(33)}{=} a_2a_5 \cdot (a_2a_5 \cdot a_5) \overset{(12)}{=} (a_2 \cdot a_2a_5)a_5 \overset{(3)}{=} a_5a_2 \overset{(27)}{=} b_{52},$$

i.e. the equalities

$$b_{i,i+3}\, c_{i-3} = b_{i+3,i}, \quad b_{i,i-3}\, c_{i+3} = b_{i-3,i} \tag{35}$$

hold. As it follows

$$\begin{aligned} b_6c_1 &\overset{(26),(33)}{=} a_4a_3 \cdot (a_4a_7 \cdot a_7) \overset{(2)}{=} (a_4 \cdot a_4a_7) \cdot a_3a_7 \overset{(12)}{=} (a_4 \cdot a_3a_7)(a_4a_7 \cdot a_3a_7) \\ &\overset{(12)}{=} (a_4 \cdot a_3a_7)(a_4a_3 \cdot a_7) \overset{(26)}{=} (a_4 \cdot a_3a_7)(a_1a_2 \cdot a_7) \overset{(12)}{=} (a_4 \cdot a_3a_7)(a_1a_7 \cdot a_2a_7) \\ &\overset{(2)}{=} (a_4 \cdot a_1a_7)(a_3a_7 \cdot a_2a_7) \overset{(12)}{=} (a_4 \cdot a_1a_7)(a_3a_2 \cdot a_7) \overset{(26)}{=} (a_4 \cdot a_1a_7)(a_7a_1 \cdot a_7) \\ &\overset{(10)}{=} (a_4 \cdot a_1a_7)(a_7 \cdot a_1a_7) \overset{(12)}{=} a_4a_7 \cdot a_1a_7 \overset{(26)}{=} a_4a_7 \cdot a_5a_6 \overset{(2)}{=} a_4a_5 \cdot a_7a_6 \\ &\overset{(26)}{=} b_2b_2 \overset{(1)}{=} b_2, \end{aligned}$$

so the general equalities hold

$$b_{i-2}\, c_i = b_{i+1}, \qquad b_{i+2}\, c_i = b_{i-1}. \tag{36}$$

Further, we get

$$\begin{aligned} a_2 \cdot a_5a_1 &\overset{(27)}{=} a_2b_{51} \overset{(29)}{=} a_2(a_3 \cdot a_3a_2) \overset{(11)}{=} a_2a_3 \cdot (a_2 \cdot a_3a_2) \\ &\overset{(10)}{=} a_2a_3 \cdot (a_2a_3 \cdot a_2) \overset{(2)}{=} (a_2 \cdot a_2a_3) \cdot a_3a_2 \overset{(28)}{=} b_{74} \cdot a_3a_2 \\ &\overset{(27),(26)}{=} a_7a_4 \cdot a_7a_1 \overset{(11)}{=} a_7 \cdot a_4a_1 \overset{(27)}{=} a_7b_{41}, \end{aligned} \tag{37}$$

and, as in Figure 4, the obtained equal products will be designated by d_1. Generally, we have equalities

$$a_{i+1} \cdot a_{i-3}\, a_i = a_{i+1}\, b_{i-3,i} = d_i = a_{i-1}\, b_{i+3,i} = a_{i-1} \cdot a_{i+3}a_i, \tag{38}$$

which define the points d_i in Figure 4. For these points some interesting relations also hold. So, we get

$$\begin{aligned} d_1b_{51} &= a_2b_{51}\cdot b_{51} \overset{(29)}{=} a_2(a_3\cdot a_3a_2)\cdot(a_3\cdot a_3a_2) \overset{(2)}{=} a_2a_3\cdot(a_3\cdot a_3a_2)(a_3a_2)\\ &\overset{(11)}{=} a_2a_3\cdot a_3(a_3a_2\cdot a_2) \overset{(14)}{=} a_2a_3\cdot(a_2a_3\cdot a_3) \overset{(12)}{=} (a_2\cdot a_2a_3)a_3 \overset{(3)}{=} a_3a_2 \overset{(26)}{=} b_5,\\ d_1b_5 &= a_2(a_3\cdot a_3a_2)\cdot a_3a_2 \overset{(2)}{=} a_2a_3\cdot(a_3\cdot a_3a_2)a_2 \overset{(3)}{=} a_2a_3\cdot a_2a_3 \overset{(1)}{=} a_2a_3 \overset{(26)}{=} b_7,\\ d_1a_2 &= a_2(a_3\cdot a_3a_2)\cdot a_2 \overset{(10)}{=} a_2\cdot(a_3\cdot a_3a_2)a_2 \overset{(3)}{=} a_2\cdot a_2a_3 \overset{(28)}{=} b_{74},\\ d_1a_3 &= a_2(a_3\cdot a_3a_2)\cdot a_3 \overset{(12)}{=} a_2a_3\cdot(a_3\cdot a_3a_2)a_3 \overset{(3)}{=} (a_3\cdot a_3a_2)a_2\cdot(a_3\cdot a_3a_2)a_3\\ &\overset{(11)}{=} (a_3\cdot a_3a_2)\cdot a_2a_3 \overset{(2)}{=} a_3a_2\cdot(a_3a_2\cdot a_3) \overset{(10)}{=} a_3a_2\cdot(a_3\cdot a_2a_3)\\ &\overset{(11)}{=} a_3(a_2\cdot a_2a_3) \overset{(28)}{=} a_3b_{74} \overset{(38)}{=} d_4,\\ d_1b_{62} &= a_2b_{51}\cdot b_{62} \overset{(27)}{=} a_2b_{51}\cdot a_6a_2 \overset{(2)}{=} a_2a_6\cdot b_{51}a_2 \overset{(30)}{=} a_2a_6\cdot b_7 \overset{(26)}{=} a_2a_6\cdot a_2a_3\\ &\overset{(11)}{=} a_2\cdot a_6a_3 \overset{(27)}{=} a_2b_{63} \overset{(38)}{=} d_3, \end{aligned}$$

and the following general equalities

$$d_i\,b_{i-3,i} = b_{i-3}, \qquad d_i\,b_{i+3,i} = b_{i+3}, \tag{39}$$
$$d_i\,b_{i-3} = b_{i-1}, \qquad d_i\,b_{i+3} = b_{i+1}, \tag{40}$$
$$d_i\,a_{i+1} = b_{i-1,i+3}, \qquad d_i\,a_{i-1} = b_{i+1,i-3}, \tag{41}$$
$$d_i\,a_{i+2} = d_{i+3}, \qquad d_i\,a_{i-2} = d_{i-3}, \tag{42}$$
$$d_i\,b_{i-2,i+1} = d_{i+2}, \qquad d_i\,b_{i+2,i-1} = d_{i-2} \tag{43}$$

hold. In (37), the equality $d_1 = b_{74}\cdot a_3a_2$ holds, which, because of (26), gets the form $b_{74}b_5 = d_1$, thus the general equalities

$$b_{i-1,i+3}\,b_{i-3} = d_i, \qquad b_{i+1,i-3}\,b_{i+3} = d_i \tag{44}$$

hold. For the points a_i, b_i, c_i, d_i the following equalities

$$a_i\,c_i = b_i, \qquad d_i\,a_i = b_i \tag{45}$$

also hold because, e.g., we get

$$\begin{aligned} a_1c_1 &\overset{(33)}{=} a_1(a_5a_2\cdot a_2) \overset{(11)}{=} (a_1a_5\cdot a_1a_2)\cdot a_1a_2 \overset{(27),(26)}{=} b_{15}b_6\cdot b_6 \overset{(44)}{=} d_2b_6 \overset{(40)}{=} b_1,\\ d_1a_1 &\overset{(37)}{=} a_2(a_3\cdot a_3a_2)\cdot a_1 \overset{(12)}{=} a_2a_1\cdot(a_3\cdot a_3a_2)a_1 \overset{(3),(15)}{=} (a_1\cdot a_1a_2)a_2\cdot(a_1\cdot a_1a_2)a_3\\ &\overset{(11)}{=} (a_1\cdot a_1a_2)\cdot a_2a_3 \overset{(28),(26)}{=} b_{63}b_7 \overset{(31)}{=} b_1. \end{aligned}$$

We have proved the following theorem.

Theorem 4. *If the statement $ARH(a_1,a_2,a_3,a_4,a_5,a_6,a_7)$ holds, then there are the points b_i, $b_{i,i+3}$, $b_{i,i-3}$, c_i, d_i so that equalities (26)–(36) and (38)–(45) hold, where the indexes are taken modulo 7 from the set $\{1,2,3,4,5,6,7\}$.*

Numerous proofs in this article use the properties of cancellation and solvability in a quasigroup $(Q,\cdot)$, so it is interesting to observe one direct proof, in which only properties (1)–(3) and (10)–(16) of a groupoid $(Q,\cdot)$ are used explicitly. We shall prove that equality $a_6a_7 = a_2a_1$ follows from equalities

$$a_1a_2 = a_4a_3,\quad a_2a_3 = a_5a_4,\quad a_3a_4 = a_6a_5,\quad a_4a_5 = a_7a_6.$$

We get successively

$$\begin{aligned}
a_6a_7 &\overset{(13)}{=} a_7a_6\cdot(a_7a_6\cdot a_6) = a_4a_5\cdot(a_4a_5\cdot a_6) \overset{(2)}{=} (a_4\cdot a_4a_5)\cdot a_5a_6\\
&\overset{(13)}{=} (a_4\cdot a_4a_5)\cdot(a_6a_5)(a_6a_5\cdot a_5) = (a_4\cdot a_4a_5)\cdot(a_3a_4)(a_3a_4\cdot a_5)\\
&\overset{(2)}{=} (a_4\cdot a_4a_5)\cdot(a_3\cdot a_3a_4)(a_4a_5) \overset{(12)}{=} a_4(a_3\cdot a_3a_4)\cdot a_4a_5\\
&\overset{(13)}{=} a_4(a_3\cdot a_3a_4)\cdot(a_5a_4)(a_5a_4\cdot a_4) = a_4(a_3\cdot a_3a_4)\cdot(a_2a_3)(a_2a_3\cdot a_4)\\
&\overset{(11)}{=} (a_4a_3)(a_4\cdot a_3a_4)\cdot(a_2a_3)(a_2a_3\cdot a_4) \overset{(2)}{=} (a_4a_3\cdot a_2a_3)\cdot(a_4\cdot a_3a_4)(a_2a_3\cdot a_4)\\
&\overset{(10)}{=} (a_4a_3\cdot a_2a_3)\cdot(a_4a_3\cdot a_4)(a_2a_3\cdot a_4) = (a_1a_2\cdot a_2a_3)\cdot(a_1a_2\cdot a_4)(a_2a_3\cdot a_4)\\
&\overset{(12)}{=} (a_1a_2\cdot a_2a_3)\cdot(a_1a_2\cdot a_2a_3)a_4 \overset{(2)}{=} (a_1a_2)(a_1a_2\cdot a_2a_3)\cdot(a_2a_3\cdot a_4)\\
&\overset{(12)}{=} (a_1a_2)(a_1a_2\cdot a_2a_3)\cdot(a_2a_4\cdot a_3a_4)\\
&\overset{(13)}{=} (a_1a_2)(a_1a_2\cdot a_2a_3)\cdot[a_2a_4\cdot(a_4a_3)(a_4a_3\cdot a_3)]\\
&= (a_1a_2)(a_1a_2\cdot a_2a_3)\cdot[a_2a_4\cdot(a_1a_2)(a_1a_2\cdot a_3)]\\
&\overset{(2)}{=} (a_1a_2\cdot a_2a_4)[(a_1a_2\cdot a_2a_3)\cdot(a_1a_2)(a_1a_2\cdot a_3)]\\
&\overset{(11)}{=} (a_1a_2\cdot a_2a_4)[a_1a_2\cdot(a_2a_3)(a_1a_2\cdot a_3)] \overset{(11)}{=} a_1a_2\cdot[a_2a_4\cdot(a_2a_3)(a_1a_2\cdot a_3)]\\
&\overset{(12)}{=} a_1a_2\cdot[a_2a_4\cdot(a_2\cdot a_1a_2)a_3] \overset{(2)}{=} a_1a_2\cdot[a_2(a_2\cdot a_1a_2)\cdot a_4a_3]\\
&= a_1a_2\cdot[a_2(a_2\cdot a_1a_2)\cdot a_1a_2] \overset{(10)}{=} a_1a_2\cdot[a_2(a_2a_1\cdot a_2)\cdot a_1a_2]\\
&\overset{(10)}{=} a_1a_2\cdot[(a_2\cdot a_2a_1)a_2\cdot a_1a_2] \overset{(12)}{=} a_1a_2\cdot[(a_2\cdot a_2a_1)a_1\cdot a_2]\\
&\overset{(3)}{=} a_1a_2\cdot(a_1a_2\cdot a_2) \overset{(13)}{=} a_2a_1.
\end{aligned}$$

By means of labels from Theorem 4 the following theorem can be proved.

Theorem 5. *If the statement $ARH(a_1,a_2,a_3,a_4,a_5,a_6,a_7)$ holds, then the statements $ARH(b_1,b_2,b_3,b_4,b_5,b_6,b_7)$, $ARH(c_1,c_2,c_3,c_4,c_5,c_6,c_7)$, $ARH(d_1,d_2,d_3,d_4,d_5,d_6,d_7)$, $ARH(b_{14},b_{25},b_{36},b_{47},b_{51},b_{62},b_{73})$, $ARH(b_{15},b_{26},b_{37},b_{41},b_{52},b_{63},b_{74})$ hold (Figure 4).*

Proof. It can be proved that for each i from the set $\{1,2,3,4,5,6,7\}$ the following equalities hold

$$b_ib_{i+1} \overset{(26)}{=} a_{i+5}a_{i+4} \cdot a_{i+6}a_{i+5} \overset{(2)}{=} a_{i+5}a_{i+6} \cdot a_{i+4}a_{i+5} \overset{(26)}{=} b_{i+3}b_{i+2}$$

$$\begin{aligned} c_ic_{i+1} &\overset{(33)}{=} (a_{i+3}a_{i-1} \cdot a_{i-1})(a_{i+4}a_i \cdot a_i) \overset{(2)}{=} (a_{i+3}a_{i-1} \cdot a_{i+4}a_i) \cdot a_{i-1}a_i \\ &\overset{(2)}{=} (a_{i+3}a_{i+4} \cdot a_{i-1}a_i) \cdot a_{i-1}a_i \overset{(26)}{=} (a_{i+6}a_{i+5} \cdot a_{i+2}a_{i+1}) \cdot a_{i+2}a_{i+1} \\ &\overset{(2)}{=} (a_{i+6}a_{i+2} \cdot a_{i+5}a_{i+1}) \cdot a_{i+2}a_{i+1} \\ &\overset{(2)}{=} (a_{i+6}a_{i+2} \cdot a_{i+2})(a_{i+5}a_{i+1} \cdot a_{i+1}) \\ &\overset{(33)}{=} c_{i+3}c_{i+2} \end{aligned}$$

$$\begin{aligned} d_id_{i+1} &\overset{(38)}{=} (a_{i+1} \cdot a_{i+4}a_i)(a_{i+2} \cdot a_{i+5}a_{i+1}) \overset{(2)}{=} a_{i+1}a_{i+2} \cdot (a_{i+4}a_i \cdot a_{i+5}a_{i+1}) \\ &\overset{(2)}{=} a_{i+1}a_{i+2} \cdot (a_{i+4}a_{i+5} \cdot a_ia_{i+1}) \overset{(26)}{=} a_{i+4}a_{i+3} \cdot (a_ia_{i+6} \cdot a_{i+3}a_{i+2}) \\ &\overset{(2)}{=} a_{i+4}a_{i+3} \cdot (a_ia_{i+3} \cdot a_{i+6}a_{i+2}) \overset{(2)}{=} (a_{i+4} \cdot a_ia_{i+3})(a_{i+3} \cdot a_{i+6}a_{i+2}) \\ &\overset{(38)}{=} d_{i+3}d_{i+2} \end{aligned}$$

$$\begin{aligned} b_{i,i+3}b_{i+1,i+4} &\overset{(27)}{=} a_ia_{i+3} \cdot a_{i+1}a_{i+4} \overset{(2)}{=} a_ia_{i+1} \cdot a_{i+3}a_{i+4} \overset{(26)}{=} a_{i+3}a_{i+2} \cdot a_{i+6}a_{i+5} \\ &\overset{(2)}{=} a_{i+3}a_{i+6} \cdot a_{i+2}a_{i+5} \overset{(27)}{=} b_{i+3,i+6}b_{i+2,i+5} \end{aligned}$$

$$\begin{aligned} b_{i,i-3}b_{i+1,i-2} &\overset{(27)}{=} a_ia_{i-3} \cdot a_{i+1}a_{i-2} \overset{(2)}{=} a_ia_{i+1} \cdot a_{i-3}a_{i-2} \overset{(26)}{=} a_{i+3}a_{i+2} \cdot a_ia_{i-1} \\ &\overset{(2)}{=} a_{i+3}a_i \cdot a_{i+2}a_{i-1} \overset{(27)}{=} b_{i+3,i}b_{i+2,i-1}, \end{aligned}$$

where the indexes are taken modulo 7 from the set $\{1, 2, 3, 4, 5, 6, 7\}$, so the assertions of the theorem are proved. □

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References

[1] V. Volenec, *Admissible identitities in complex IM–quasigroups I.*, Rad Jugoslav. Akad. Znan. Umjetn. **413**(1985), 61–85.

[2] V. Volenec, *Admissible identitities in complex IM–quasigroups II.*, Rad Jugoslav. Akad. Znan. Umjetn. **421**(1986), 45–78.