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High energy asymptotics for eigenvalues of the Schrödinger operator with a matrix potential

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Abstract. We consider a Schrödinger operator with a matrix potential defined in $L_2^m(Q)$ by the differential expression $Lu = -\Delta u + Vu$ and the Neumann boundary condition, where Q is a d-dimensional parallelepiped and V a matrix potential, $d \geq 2$, $m \geq 2$. We obtain the high energy asymptotics of arbitrary order for a rich set of eigenvalues.

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We consider the Schrödinger operator with a matrix potential $V(x)$ which is defined by the differential expression

$$L = -\Delta + V \tag{1}$$

and the Neumann boundary condition

$$\frac{\partial \mathbf{\Phi}}{\partial n} \mid_{\partial Q}= 0 \tag{2}$$

in $L_2^m(Q)$, where $Q = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_d]$, ∂Q is the boundary of Q, $m \geq 2$, $d \geq 2$, Δ is a diagonal $m \times m$ matrix, its diagonal elements being the scalar Laplace operators, V is the operator of multiplication by a real valued symmetric matrix $V(x) = (v_{ij}(x))$, $i, j = 1, 2, \dots, m$, $v_{ij}(x) \in L_2(Q)$, $V^T(x) = V(x)$. We denote the operator defined by (1) and (2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of $L(V)$ by Λ_N and Ψ_N, respectively.

The eigenvalues of the operator $L(0)$ which is defined by (1) when $V(x) = 0$ and the boundary condition (2) are $\mid \gamma \mid^2$ and the corresponding eigenspaces are

$$E_\gamma = span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \dots, \Phi_{\gamma,m}(x)\},$$

where

$$\gamma \in \frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : \quad n_k \in Z^+ \bigcup \{0\}, \quad k = 1, 2, \dots, d\},$$
$$\Phi_{\gamma,j}(x) = (0, \dots, 0, u_\gamma(x), 0, \dots, 0), j = 1, 2, \dots, m,$$

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$$u_\gamma(x) = cos\tfrac{n_1\pi}{a_1}x_1 cos\tfrac{n_2\pi}{a_2}x_2 \cdots cos\tfrac{n_d\pi}{a_d}x_d,$$

$u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$. We note that the non-zero component $u_\gamma(x)$ of $\Phi_{\gamma,j}(x)$ stands in the jth component.

It can be easily calculated that the norm of $u_\gamma(x)$, $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2}$ in $L_2(Q)$ is $\sqrt{\frac{\mu(Q)}{|A_\gamma|}}$, where $\mu(Q)$ is the measure of the d-dimensional parallelepiped Q, $A_\gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} :| \alpha_k |=| \gamma^k |, k = 1, 2, \ldots, d\}$, $\frac{\Gamma}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \cdots, \frac{n_d\pi}{a_d}) : n_k \in Z, k = 1, 2, \ldots, d\}$ and $| A_\gamma |$ is the number of vectors in A_γ.

Since $\{u_\gamma(x)\}_{\gamma\in\frac{\Gamma^{+0}}{2}}$ is a complete system in $L_2(Q)$, for any $q(x)$ in $L_2(Q)$ we have

$$q(x) = \sum_{\gamma\in\frac{\Gamma^{+0}}{2}} \frac{| A_\gamma |}{\mu(Q)}(q, u_\gamma)u_\gamma(x), \tag{3}$$

where $(\cdot,\cdot)$ is the inner product in $L_2(Q)$. Using decomposition (3) and the obvious relations

$$u_\gamma(x) = u_\alpha(x), \quad (q(x), u_\gamma(x)) = (q(x), u_\alpha(x)), \quad \forall\alpha \in A_\gamma,$$
$$\frac{\Gamma}{2} = \bigcup_{\gamma\in\frac{\Gamma^{+0}}{2}} A_\gamma, \quad (q(x), u_\gamma(x)) = \frac{1}{| A_\gamma |}\sum_{\alpha\in A_\gamma}(q(x), u_\alpha(x)),$$

we have

$$\begin{aligned} q(x) &= \sum_{\gamma\in\frac{\Gamma^{+0}}{2}} \frac{| A_\gamma |}{\mu(Q)}(q(x), u_\gamma(x))u_\gamma(x) \\ &= \sum_{\gamma\in\frac{\Gamma^{+0}}{2}} \frac{| A_\gamma |}{\mu(Q)}\frac{1}{| A_\gamma |}\sum_{\alpha\in A_\gamma}(q(x), u_\alpha(x))u_\alpha(x) \\ &= \sum_{\gamma\in\frac{\Gamma}{2}} \frac{1}{\mu(Q)}(q(x), u_\gamma(x))u_\gamma(x). \end{aligned}$$

Thus one can write

$$q(x) = \sum_{\gamma\in\frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{4}$$

where $q_\gamma = \frac{1}{\mu(Q)}(q(x), u_\gamma(x))$. Since decompositions (3) and (4) are equivalent, for the sake of simplicity, we use decomposition (4).

So each matrix element $v_{ij}(x) \in L_2(Q)$ of the matrix $V(x)$ can be written in its Fourier series expansion

$$v_{ij}(x) = \sum_{\gamma\in\frac{\Gamma}{2}} v_{ij\gamma}u_\gamma(x)$$

for $i, j = 1, 2, \ldots, m$ where $v_{ij\gamma} = \frac{(v_{ij}, u_\gamma)}{\mu(Q)}$.

We assume that the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma\in\frac{\Gamma}{2}} \mid v_{ij\gamma} \mid^2 (1+ \mid \gamma \mid^{2l}) < \infty \tag{5}$$

for each $i, j = 1, 2, \ldots, m$, where $l > \frac{(d+20)(d-1)}{2} + d + 3$, which implies

$$v_{ij}(x) = \sum_{\gamma\in\Gamma^{+0}(\rho^\alpha)} v_{ij\gamma} u_\gamma(x) + O(\rho^{-p\alpha}), \tag{6}$$

where $\Gamma^{+0}(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq\mid \gamma \mid< \rho^\alpha\}$, $p = l-d$, $\alpha < \frac{1}{d+20}$, ρ is a large parameter and $O(\rho^{-p\alpha})$ is a function in $L_2(Q)$ with norm of order $\rho^{-p\alpha}$. Furthermore, a assumption (5) implies

$$M_{ij} \equiv \sum_{\gamma\in\frac{\Gamma}{2}} \mid v_{ij\gamma} \mid< \infty \tag{7}$$

for all $i, j = 1, 2, \ldots, m$.

Notice that, if a function $q(x)$ is sufficiently smooth ($q(x) \in W_2^l(Q)$) and the support of $\text{grad}q(x) = (\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d})$ is contained in the interior of the domain Q, then $q(x)$ satisfies condition (5) (see [7]). There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(Q)$,

$$q(x) = \sum_{\gamma'\in\Gamma} q_{\gamma'} u_{\gamma'}(x),$$

which is periodic with respect to a lattice $\Omega = \{(m_1a_1, m_2a_2, \ldots, m_da_d) : m_k \in Z, k = 1, 2, \ldots, d\}$ and thus it also satisfies condition (5).

In this paper and in [3], we study how the eigenvalues $\mid \gamma \mid^2$ of the unperturbed operator $L(0)$ are affected under perturbation, by using energy as a large parameter. In [3], we obtain the asymptotic formulas for the eigenvalues of the operator $L(V)$ in an arbitrary dimension. In this paper, we improve the proof of the formulas obtained in [3] so that we additionally obtain the high energy asymptotics of arbitrary order for the eigenvalues of the operator $L(V)$ in an arbitrary dimension. This is one of the essential problems related to this operator $L(V)$ that has been studied for a long time.

For the scalar case, $m = 1$, a method was first introduced by O. Veliev in [15], [16] and more recently in [17]-[19] to obtain the asymptotic formulas for the eigenvalues of the periodic Schrödinger operator with quasiperiodic boundary conditions. By some other methods, asymptotic formulas for quasiperiodic boundary conditions in two- and three-dimensional cases are obtained in [4, 5, 10, 11] and [6]. When this operator is considered with the Dirichlet boundary condition in a two-dimensional rectangle, the asymptotic formulas for the eigenvalues are obtained in [7]. The asymptotic formulas for the eigenvalues of the Schrödinger operator with Dirichlet or Neumann boundary conditions in an arbitrary dimension are obtained in [1], [8] and [9]. For the matrix case, asymptotic formulas for eigenvalues of the Schrödinger operator with quasiperiodic boundary conditions are obtained in [12].

As in [15]- [19], we divide R^d into two domains: resonance and non-resonance domains. In order to define these domains, let us introduce the following sets:

Let $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1, 2, \ldots, d-1$ and

$$\begin{aligned} V_b(\rho^{\alpha_1}) &\equiv \{x \in R^d : \quad \mid\mid x \mid^2 - \mid x+b \mid^2\mid < \rho^{\alpha_1}\}, \\ E_1(\rho^{\alpha_1}, p) &\equiv \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}), \\ U(\rho^{\alpha_1}, p) &\equiv R^d \setminus E_1(\rho^{\alpha_1}, p), \end{aligned}$$

where $\Gamma(p\rho^{\alpha}) \equiv \{b \in \frac{\Gamma}{2} : 0 <\mid b \mid < p\rho^{\alpha}\}$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $\mid \gamma \mid^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$ for all $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains, and the eigenvalue $\mid \gamma \mid^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

In this paper, we obtain the asymptotic formulas of arbitrary order for non-resonance eigenvalues, which is a rich set of eigenvalues in the following sense: The number of non-resonance eigenvalues is essentially greater than the number of resonance eigenvalues. Namely, if $N_n(\rho)$ and $N_r(\rho)$ denote the number of $\gamma \in U(\rho^{\alpha}, p) \bigcap (R(2\rho) \setminus R(\rho))$ and $\gamma \in \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha}) \bigcap (R(2\rho) \setminus R(\rho))$, respectively, then

$$\frac{N_r(\rho)}{N_n(\rho)} = O(\rho^{(d+1)\alpha - 1}) = o(1) \tag{8}$$

for $(d+1)\alpha < 1$, where $R_\rho = \{x \in R^d : \mid x \mid = \rho\}$ (see Remark 1 in [1]).

To prove the asymptotic formulas for the eigenvalues Λ_N, we use the binding formula

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > = < \Psi_N, V\Phi_{\gamma,j} > \tag{9}$$

for the eigenvalue, eigenfunction pairs Λ_N, $\Psi_N(x)$ and $\mid \gamma \mid^2$, $\Phi_{\gamma,j}(x)$ of the operators $L(V)$ and $L(0)$, respectively. Formula (9) can be obtained by multiplying the equation $L(V)\Psi_N(x) = \Lambda_N\Psi_N(x)$ by $\Phi_{\gamma,j}(x)$ and by using the facts that $L(0)$ is self-adjoint and $L(0)\Phi_{\gamma,j}(x) = \mid \gamma \mid^2 \Phi_{\gamma,j}(x)$. Here $< \cdot, \cdot >$ denotes the inner product in $L_2^m(Q)$.

We consider the eigenvalues $\mid \gamma \mid^2$ of $L(0)$ such that $\mid \gamma \mid \sim \rho$, where $\mid \gamma \mid \sim \rho$ means that $\mid \gamma \mid$ and ρ are asymptotically equal, that is, $c_1\rho \leq \mid \gamma \mid \leq c_2\rho$, c_i, $i = 1, 2, 3, \ldots$ are positive real constants which do not depend on ρ and ρ is a large parameter, $\rho \gg 1$.

Now, we decompose $V(x)\Phi_{\gamma,j}(x)$ with respect to the basis $\{\Phi_{\gamma',i}(x)\}_{\gamma' \in \frac{\Gamma}{2}, i=1,2,\ldots,m}$. By definition of $\Phi_{\gamma,j}(x)$, it is obvious that

$$V(x)\Phi_{\gamma,j}(x) = (v_{1j}(x)u_\gamma(x), \ldots, v_{mj}(x)u_\gamma(x)). \tag{10}$$

Substituting decomposition (6) of $v_{ij}(x)$ in (10), we get

$$\begin{aligned} V(x)\Phi_{\gamma,j}(x) = (&\sum_{\gamma' \in \Gamma^{+0}(\rho^{\alpha})} v_{1j\gamma'}u_{\gamma'}(x)u_\gamma(x), \ldots, \sum_{\gamma' \in \Gamma^{+0}(\rho^{\alpha})} v_{mj\gamma'}u_{\gamma'}(x)u_\gamma(x)) \\ &+O(\rho^{-p\alpha}). \end{aligned}$$

Since $\gamma \in U(\rho^{\alpha_1}, p)$, γ does not belong to the domains $V_{e_k}(\rho^{\alpha_1})$ where $e_k = (0, \ldots, 0, \frac{\pi}{a_k}, 0, \ldots, 0)$ for each $k = 1, 2, \ldots, d$, we may use the following equation

$$\sum_{\gamma\prime \in \Gamma^{+0}(\rho^{\alpha})} v_{ij\gamma\prime} u_{\gamma\prime}(x) u_{\gamma}(x) = \sum_{\gamma\prime \in \Gamma^{+0}(\rho^{\alpha})} v_{ij\gamma\prime} u_{\gamma+\gamma\prime}(x)$$

which is proved in [8] (see equation (18) in [8]), and obtain

$$\begin{aligned} V(x)\Phi_{\gamma,j}(x) &= (\sum_{\gamma\prime \in \Gamma^{+0}(\rho^{\alpha})} v_{1j\gamma\prime} u_{\gamma+\gamma\prime}(x), \ldots, \sum_{\gamma\prime \in \Gamma^{+0}(\rho^{\alpha})} v_{mj\gamma\prime} u_{\gamma+\gamma\prime}(x)) + O(\rho^{-p\alpha}) \\ &= \sum_{i=1}^{m} \sum_{\gamma\prime \in \Gamma^{+0}(\rho^{\alpha})} v_{ij\gamma\prime} \Phi_{\gamma+\gamma\prime,i}(x) + O(\rho^{-p\alpha}). \end{aligned} \tag{11}$$

Expressions (9) and (11) together imply that

$$\begin{aligned} < \Psi_N, \Phi_{\gamma\prime,j} > &= \frac{< \Psi_N, V\Phi_{\gamma\prime,j} >}{(\Lambda_N - \mid \gamma\prime \mid^2)} \\ &= \sum_{i=1}^{m} \sum_{\gamma_1 \in \Gamma^{+0}(\rho^{\alpha})} v_{ij\gamma_1} \frac{< \Psi_N, \Phi_{\gamma\prime+\gamma_1,i} >}{(\Lambda_N - \mid \gamma\prime \mid^2)} + O(\rho^{-p\alpha}) \end{aligned} \tag{12}$$

for every vector $\gamma\prime \in \frac{\Gamma}{2}$, satisfying the condition

$$\mid \Lambda_N - \mid \gamma\prime \mid^2 \mid > \frac{1}{2}\rho^{\alpha_1}.$$

If $\gamma \in U(\rho^{\alpha_1}, p)$ and Λ_N satisfies

$$\mid \Lambda_N - \mid \gamma \mid^2 \mid < \frac{1}{2}\rho^{\alpha_1}, \tag{13}$$

which is called the iterability condition, then

$$\mid \Lambda_N - \mid \gamma + b \mid^2 \mid \geq \mid\mid \Lambda_N - \mid \gamma \mid^2 \mid - \mid\mid \gamma + b \mid^2 - \mid \gamma \mid^2 \mid\mid > \frac{1}{2}\rho^{\alpha_1}, \tag{14}$$

for all $b \in \Gamma^{+0}(p\rho^{\alpha})$ with $b \neq 0$.

Let $\gamma \in U(\rho^{\alpha_1}, p)$ with $\mid \gamma \mid \sim \rho$. Now, we start the iteration by substituting (11) into the binding formula (9) and obtain

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > = \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma^{+0}(\rho^{\alpha})} v_{i_1 j\gamma_1} < \Psi_N, \Phi_{\gamma+\gamma_1,i_1} > + O(\rho^{-p\alpha}).$$

Isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$, that is, $\gamma_1 = 0$, for each $i = 1, 2, \ldots, m$, we get

$$\begin{aligned} (\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > &= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > \\ &+ \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma^{+0}(\rho^{\alpha})} v_{i_1 j\gamma_1} < \Psi_N, \Phi_{\gamma+\gamma_1,i_1} > + O(\rho^{-p\alpha}). \end{aligned}$$

In the second summation of the above equation, if Λ_N satisfies (13), then since $\gamma \in U(\rho^{\alpha_1}, p)$ and $\gamma_1 \in \Gamma^{+0}(\rho^{\alpha})$ with $\gamma_1 \neq 0$, by (14), we can use (12) replacing $\gamma\prime$ by $\gamma + \gamma_1$ and obtain

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > = \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > + \sum_{i_1,i_2=1}^{m} \sum_{\gamma_1,\gamma_2 \in \Gamma^{+0}(\rho^{\alpha})} v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2} \frac{< \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2,i_2} >}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)} + O(\rho^{-p\alpha}).$$

Again, in the second summation of the above equation, isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$, that is, $\gamma_1 + \gamma_2 = 0$, $\gamma_1 \neq 0$ for each $i = 1, 2, \ldots, m$, we get

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > \tag{15}$$

$$= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > + \sum_{i_1,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma^{+0}(\rho^{\alpha}) \\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j \gamma_1} v_{i i_1 \gamma_2}}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma,i} >$$

$$+ \sum_{i_1,i_2=1}^{m} \sum_{\gamma_1,\gamma_2 \in \Gamma^{+0}(\rho^{\alpha})} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2}}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2,i_2} > + O(\rho^{-p\alpha}). \tag{16}$$

Writing this equation for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, m$, after the first step of the iteration we obtain the following system:

$$[(\Lambda_N - \mid \gamma \mid^2) I - V_0] A(N, \gamma) = S^1 A(N, \gamma) + R^1 + O(\rho^{-p\alpha}),$$

where I is an $m \times m$ identity matrix, $V_0 = \int_Q V(x) dx$, which is again an $m \times m$ matrix, $A(N, \gamma)$ is the $m \times 1$ vector

$$A(N, \gamma) = (< \Psi_N, \Phi_{\gamma,1} >, < \Psi_N, \Phi_{\gamma,2} >, \ldots, < \Psi_N, \Phi_{\gamma,m} >),$$

$S^1 = (s^1_{ji})$ is an $m \times m$ matrix whose entries are

$$s^1_{ji} = \sum_{i_1=1}^{m} \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma^{+0}(\rho^{\alpha}) \\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j \gamma_1} v_{i i_1 \gamma_2}}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)}, \quad j, i = 1, 2, \ldots, m,$$

and $R^1 = (r^1_j)$ is the vector whose components are

$$r^1_j = \sum_{i_1,i_2=1}^{m} \sum_{\gamma_1,\gamma_2 \in \Gamma^{+0}(\rho^{\alpha})} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2}}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2,i_2} >, \quad j = 1, 2, \ldots, m.$$

Now, we continue to iterate equation (15). In the third summation of equation (15), if Λ_N satisfies (13), then since $\gamma \in U(\rho^{\alpha_1}, p)$ and $\gamma_1 + \gamma_2 \in \Gamma^{+0}(2\rho^{\alpha})$ with

$\gamma_1+\gamma_2\neq 0$, by (14) we can use (12) replacing $\gamma\prime$, for this time, by $\gamma+\gamma_1+\gamma_2$ and obtain

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > \\ = \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > + \sum_{i_1,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j\gamma_1} v_{ii_1\gamma_2}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma,i} > \\ + \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^\alpha)}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3, i_3} > \\ + O(\rho^{-p\alpha}).$$

Isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$ for each $i = 1, 2, \ldots, m$, we get

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} > \\ = \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > + \sum_{i_1,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j\gamma_1} v_{ii_1\gamma_2}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma,i} > \\ + \sum_{i_1,i_2,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3=0}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{ii_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} < \Psi_N, \Phi_{\gamma,i} > \\ + \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^\alpha)}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3, i_3} > \\ + O(\rho^{-p\alpha}).$$

Again, if we write this equation for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, m$, after the second step of the iteration we obtain the following system:

$$[(\Lambda_N - \mid \gamma \mid^2) I - V_0] A(N,\gamma) = (S^1 + S^2) A(N,\gamma) + R^2 + O(\rho^{-p\alpha}),$$

where this time $S^2 = (s^2_{ji})$ is an $m\times m$ matrix whose entries are

$$s^2_{ji} = \sum_{i_1,i_2=1}^{m} \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3=0}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{ii_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)},$$

$j, i = 1, 2, \ldots, m$ and $R^2 = (r^2_j)$ is an $m\times 1$ vector whose components are

$$r^2_j = \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^\alpha)}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3, i_3} >,$$

$j = 1, 2, \ldots, m$.

If we continue to iterate in this manner after the p_1st step where $p_1 = [\frac{p+1}{2}]$ and $[\cdot]$ is the integer function, we obtain the following system:

$$[(\Lambda_N - \mid \gamma \mid^2) I - V_0] A(N,\gamma) = (\sum_{k=1}^{p_1} S^k) A(N,\gamma) + R^{p_1} + O(\rho^{-p\alpha}), \tag{17}$$

where

$$S^k(\Lambda_N) = (s^k_{ji}(\Lambda_N)), \quad k = 1, 2, \ldots, p_1, \quad j, i = 1, 2, \ldots, m, \tag{18}$$

$$s^k_{ji}(\Lambda_N) = \sum_{\substack{i_1, i_2, \ldots, \\ i_k = 1}}^{m} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_{k+1} \in \Gamma^{+0}(\rho^\alpha) \\ \gamma_1 + \gamma_2 + \cdots + \gamma_{k+1} = 0}} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2} \ldots v_{i i_k \gamma_{k+1}}}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2) \ldots (\Lambda_N - \mid \gamma + \gamma_1 + \cdots + \gamma_k \mid^2)},$$

$$R^{p_1} = (r^{p_1}_j), \quad j = 1, 2, \ldots, m,$$

and

$$r^{p_1}_j = \sum_{\substack{i_1, i_2, \ldots, \\ i_{p_1+1} = 1}}^{m} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \\ \gamma_{p_1+1} \in \Gamma^{+0}(\rho^\alpha)}} \frac{v_{i_1 j \gamma_1} \ldots v_{i_{p_1+1} i_{p_1} \gamma_{p_1+1}} < \Psi_N, \Phi_{\gamma + \gamma_1 + \cdots + \gamma_{p_1+1}, i_{p_1+1}} >}{(\Lambda_N - \mid \gamma + \gamma_1 \mid^2) \ldots (\Lambda_N - \mid \gamma + \gamma_1 + \cdots + \gamma_{p_1} \mid^2)}. \tag{19}$$

If Λ_N satisfies (13), then since $\gamma \in U(\rho^{\alpha_1}, p)$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_k \in \Gamma^{+0}(k\rho^\alpha)$ with $\gamma_1 + \gamma_2 + \cdots + \gamma_k \neq 0$, by (14) and (7),

$$\begin{aligned} &|s^k_{ji}(\Lambda_N)| \\ &\quad \leq \sum_{i_1, i_2, \ldots, i_k = 1}^{m} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_{k+1} \in \Gamma^{+0}(\rho^\alpha) \\ \gamma_1 + \gamma_2 + \cdots + \gamma_{k+1} = 0}} \frac{|v_{i_1 j \gamma_1}||v_{i_2 i_1 \gamma_2}||v_{i_3 i_2 \gamma_3}| \ldots |v_{i i_k \gamma_{k+1}}|}{|(\Lambda_N - |\gamma + \gamma_1|^2)| \ldots |(\Lambda_N - |\gamma + \gamma_1 + \cdots + \gamma_k|^2)|} \\ &\quad \leq \frac{1}{(2\rho^{\alpha_1})^k} \sum_{i_1, i_2, \ldots, i_k = 1}^{m} M_{i_1 j} M_{i_2 i_1} \ldots M_{i i_k}, \end{aligned}$$

for each $k = 1, 2, \ldots, p_1$, $i, j = 1, 2, \ldots, m$. Thus

$$S^k(\Lambda_N) = O(\rho^{-k\alpha_1}), \quad \forall k = 1, 2, \ldots, p_1 \quad \Rightarrow \quad \sum_{k=1}^{p_1} S^k = O(\rho^{-\alpha_1}). \tag{20}$$

Similarly,

$$\begin{aligned} |r^{p_1}_j| &\leq \sum_{\substack{i_1, i_2, \ldots, \\ i_{p_1+1} = 1}}^{m} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \\ \gamma_{p_1+1} \in \Gamma^{+0}(\rho^\alpha)}} \frac{|v_{i_1 j \gamma_1}| \ldots |v_{i_{p_1+1} i_{p_1} \gamma_{p_1+1}}|| < \Psi_N, \Phi_{\gamma + \gamma_1 + \cdots + \gamma_{p_1+1}, i_{p_1+1}} > |}{|(\Lambda_N - \mid \gamma + \gamma_1 \mid^2)| \ldots |(\Lambda_N - \mid \gamma + \gamma_1 + \cdots + \gamma_{p_1} \mid^2)|} \\ &\leq \frac{1}{(2\rho^{\alpha_1})^{p_1}} \sum_{i_1, i_2, \ldots, i_{p_1+1} = 1}^{m} M_{i_1 j} M_{i_2 i_1} \ldots M_{i_{p_1+1} i_{p_1}}, \end{aligned}$$

that is,

$$R^{p_1} = O(\rho^{-p_1 \alpha_1}). \tag{21}$$

Note that, in order to obtain (20), we have only used the assumption that Λ_N satisfies (13), that is, $\Lambda_N \in J$ where $J = [|\gamma|^2 - \frac{1}{2}\rho^{\alpha_1}, |\gamma|^2 + \frac{1}{2}\rho^{\alpha_1}]$. Hence we may write

$$\sum_{k=1}^{p_1} S^k(a) = O(\rho^{-\alpha_1}), \quad \forall a \in J. \tag{22}$$

Similarly, (17) holds for $\Lambda_N \in J$.

Note that, since we have chosen $p_1 = [\frac{p+1}{2}]$, we have the obvious inequalities

$$p_1 \geq \frac{p}{2}, \quad p_1\alpha_1 > p\alpha, \quad p > \frac{(d+20)(d-1)}{2} \tag{23}$$

by definitions of α, α_1, l and p.

For any Λ_N and $a \in J$, using (21) and inequalities (23) in (17), we have

$$[D(\Lambda_N, \gamma) - S(a, p_1)]A(N, \gamma) = O(\rho^{-p\alpha}), \tag{24}$$

where $D(\Lambda_N, \gamma) \equiv (\Lambda_N - |\gamma|^2)I - V_0$, $S(a, p_1) \equiv \sum_{k=1}^{p_1} S^k(a)$. We note that since V is symmetric, V_0 and $S(a, p_1)$ are symmetric real valued matrices, hence $D(\Lambda_N, \gamma) - S(a, p_1)$ is a symmetric real valued matrix.

We denote the eigenvalues of V_0, counted with multiplicity, and the corresponding orthonormal eigenvectors by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ and $\omega_1, \omega_2, \ldots, \omega_m$, respectively. Thus

$$V_0\omega_i = \lambda_i\omega_i, \quad \omega_i \cdot \omega_j = \delta_{ij},$$

where " $\cdot$ " denotes the inner product in R^m.

We let $\beta_i \equiv \beta_i(\Lambda_N, \gamma, a)$ denote an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S(a, p_1)$ and $f_i \equiv f_i(\Lambda_N, \gamma, a)$ its corresponding normalized eigenvector. That is,

$$[D(\Lambda_N, \gamma) - S(a, p_1)]f_i = \beta_i f_i, \tag{25}$$

where $f_i \cdot f_j = \delta_{ij}$, $i, j = 1, 2, \ldots, m$.

Lemma 1. *Let* $| \gamma |^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $| \gamma | \sim \rho$.

(a) Let β_i *be an eigenvalue of the matrix* $D(\Lambda_N, \gamma) - S(a, p_1)$ *and* $f_i = (f_{i_1}, \ldots, f_{i_m})$ *its corresponding normalized eigenvector. Then there exists an integer* $N \equiv N_i$ *such that* Λ_N *satisfies (13) and*

$$| A(N, \gamma) \cdot f_i | > c_3\rho^{\frac{-(d-1)}{2}}. \tag{26}$$

(b) Let Λ_N *be an eigenvalue of the operator* $L(V)$ *satisfying inequality (13). Then there exists an eigenfunction* $\Phi_{\gamma,i}(x)$ *of the operator* $L(0)$ *such that*

$$|< \Phi_{\gamma,i}, \Psi_N >| > c_4\rho^{\frac{-(d-1)}{2}} \tag{27}$$

holds.

Proof. (a): We use a result from perturbation theory which states that the Nth eigenvalue of the operator $L(V)$ lies in the M-neighborhood of the Nth eigenvalue of the operator $L(0)$. Let the Nth eigenvalues of $L(V)$ and $L(0)$ be Λ_N and $| \gamma |^2$, respectively. Then there is an integer N such that $| \Lambda_N - | \gamma |^2 | < \frac{1}{2}\rho^{\alpha_1}$.

On the other hand, since $L(V)$ is a self adjoint operator, the eigenfunctions $\{\Psi_N(x)\}_{N=1}^{\infty}$ of $L(V)$ form an orthonormal basis for $L_2^m(Q)$. By Parseval's relation, we have

$$\| \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j} \|^2 = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} |< \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j}, \Psi_N >|^2 + \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} |< \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j}, \Psi_N >|^2 . \tag{28}$$

Now, we estimate the last expression in (28). By using the Cauchy-Schwarz inequality and (9), we get

$$\begin{aligned} &\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} |< \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j}, \Psi_N >|^2 \\ &= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} | \sum_{j=1}^{m} f_{ij} < \Phi_{\gamma,j}, \Psi_N >|^2 \\ &\leq \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} [\sum_{j=1}^{m} | f_{ij} |^2 \sum_{j=1}^{m} |< \Psi_N, \Phi_{\gamma,j} >|^2] \\ &\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} \sum_{j=1}^{m} \frac{|< \Psi_N, V\Phi_{\gamma,j} >|^2}{| \Lambda_N - | \gamma |^2|^2} \\ &\leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} \sum_{j=1}^{m} |< \Psi_N, V\Phi_{\gamma,j} >|^2 \\ &\leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \sum_{j=1}^{m} \| V\Phi_{\gamma,j} \|^2 \end{aligned}$$

from which together with (7) we obtain

$$\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} |< \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j}, \Psi_N >|^2 = O(\rho^{-2\alpha_1}).$$

It follows from the last equation and (28) that

$$\sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} |< \sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j}, \Psi_N >|^2 = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} | A(N,\gamma) \cdot f_i |^2 = 1 - O(\rho^{-2\alpha_1}). \tag{29}$$

On the other hand, if $a \sim \rho$, then the number of $\gamma \in \frac{\Gamma}{2}$ satisfying $|| \gamma |^2 - a^2 |< 1$ is less than $c_5\rho^{d-1}$. Therefore, the number of eigenvalues of $L(0)$ lying in (a^2-1, a^2+1) is less than $c_6\rho^{d-1}$. By this result and the result of perturbation theory, the number of eigenvalues Λ_N of $L(V)$ in the interval $[\mid \gamma \mid^2 - \frac{1}{2}\rho^{\alpha_1}, \mid \gamma \mid^2 + \frac{1}{2}\rho^{\alpha_1}]$ is less than $c_7\rho^{d-1}$. Thus

$$1 - O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} \mid A(N,\gamma) \cdot f_i \mid^2 < c_7\rho^{d-1} \mid A(N,\gamma) \cdot f_i \mid^2 \quad (30)$$

from which we get (26).

(b): Since $L(0)$ is a self adjoint operator, the set of eigenfunctions

$$\{\Phi_{\gamma,i}(x)\}_{\gamma \in \frac{\Gamma}{2}, i=1,2,\ldots,m}$$

of $L(0)$ forms an orthonormal basis for $L_2^m(Q)$. By Parseval's relation, we have

$$\begin{aligned} \| \Psi_N \|^2 = & \sum_{\gamma:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 \\ & + \sum_{\gamma:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 . \end{aligned} \quad (31)$$

We estimate the last expression in (31). Hence for a fixed $i = 1, 2, \ldots, m$, using (9) together with (7) we get

$$\begin{aligned} & \sum_{\gamma:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 \\ & = \sum_{\gamma:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{|< \Psi_N, V\Phi_{\gamma,i} >|^2}{\mid \Lambda_N - \mid \gamma \mid^2 \mid^2} \\ & \leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \sum_{\gamma:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< V\Psi_N, \Phi_{\gamma,i} >|^2 \\ & \leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \parallel V\Psi_N \parallel^2, \end{aligned} \quad (32)$$

that is,

$$\sum_{\gamma:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 = O(\rho^{-2\alpha_1}).$$

From the last equality and (31) we obtain

$$\sum_{\gamma:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 = 1 - O(\rho^{-2\alpha_1}).$$

Arguing as in the proof of part(a) we get

$$1 - O(\rho^{-2\alpha_1}) = \sum_{\gamma:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |< \Psi_N, \Phi_{\gamma,i} >|^2 \leq c_8\rho^{d-1} \, |< \Psi_N, \Phi_{\gamma,i} >|^2$$

from which (27) follows. □

Theorem 1. *Let* $\mid \gamma \mid^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $\mid \gamma \mid \sim \rho$.

(a) For each eigenvalue λ_i *of the matrix* V_0*, there exists an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying*

$$\Lambda_N = \mid \gamma \mid^2 + \lambda_i + O(\rho^{-\alpha_1}). \tag{33}$$

(b) For each eigenvalue Λ_N *of the operator* $L(V)$ *satisfying* (13)*, there exists an eigenvalue* λ_i *of the matrix* V_0 *satisfying* (33).

Proof. (a): By Lemma(1a), there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying (13), that is, $\Lambda_N \in J$ and (26) hold. Thus we consider equation (24) for $a = \Lambda_N$, that is,

$$[D(\Lambda_N, \gamma) - S(\Lambda_N, p_1)]A(N, \gamma) = O(\rho^{-p\alpha}).$$

Let β_i be an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S(\Lambda_N, p_1)$ and f_i its corresponding normalized eigenvector. Multiplying both sides of the above equation by f_i, we obtain

$$\beta_i[A(N, \gamma) \cdot f_i] = O(\rho^{-p\alpha}).$$

Using inequality (26) in the above equation, we get

$$\beta_i = O(\rho^{-(p - \frac{d-1}{2\alpha})\alpha}). \tag{34}$$

Since $D(\Lambda_N, \gamma)$ and $S(\Lambda_N, p_1)$ are symmetric real valued matrices, by a well known result in matrix theory (see [13]), $|\beta_i - (\Lambda_N - \mid \gamma \mid^2 - \lambda_i)| \leq \|S(\Lambda_N, p_1)\|$, which together with (22) implies that

$$\beta_i = \Lambda_N - \mid \gamma \mid^2 - \lambda_i + O(\rho^{-\alpha_1}). \tag{35}$$

Hence, choosing $p > \frac{d-1}{2\alpha} + 1$ and using (35) and (34), we get the result.

(b): By Lemma(1b), there exists $\Phi_{\gamma,i}(x)$ satisfying (27) from which we have

$$\|A(N, \gamma)\| > c_9 \rho^{\frac{-(d-1)}{2}}. \tag{36}$$

Now, we consider equation (24) for these (N, γ) pairs:

$$[(\Lambda_N - \mid \gamma \mid^2)I - V_0]A(N, \gamma) = S(\Lambda_N, p_1)A(N, \gamma) + O(\rho^{-p\alpha}).$$

Applying $\frac{1}{\|A(N,\gamma)\|}[(\Lambda_N - \mid \gamma \mid^2)I - V_0]^{-1}$ to both sides of the above equation, taking the norm of both sides, and using (36), we obtain

$$1 \leq \|[(\Lambda_N - \mid \gamma \mid^2)I - V_0]^{-1}\| \| \sum_{k=1}^{p_1} S^k \| + \|[(\Lambda_N - \mid \gamma \mid^2)I - V_0]^{-1}\| [O(\rho^{-(p\alpha - \frac{(d-1)}{2})}].$$

By estimation (20), we get

$$1 \leq \max_{i=1,2,\ldots,m} \frac{1}{|\Lambda_N - \mid \gamma \mid^2 - \lambda_i|}[O(\rho^{-\alpha_1}) + O(\rho^{-(p\alpha - \frac{d-1}{2})})].$$

Choosing $p > \frac{d-1}{2\alpha} + 1$, we obtain

$$\min_{i=1,2,\ldots,m} |\Lambda_N - \mid \gamma \mid^2 - \lambda_i| \leq c_{10}\rho^{-\alpha_1},$$

where the minimum is taken over all eigenvalues of the matrix V_0 from which we obtain the result. □

Now, we define the following $m \times m$ matrices:

$$F_0 = 0, \quad F_1 = S^1(\mu_{\gamma,s}), \quad F_j = S(\mu_{\gamma,s} + \|F_{j-1}\|, j), \quad j \geq 2, \tag{37}$$

where $\mu_{\gamma,s} \equiv \mid \gamma \mid^2 + \lambda_s$. Then we have

$$\|F_j\| = O(\rho^{-\alpha_1}) \tag{38}$$

for all $j = 1, 2, \ldots, p - c$, $c = [\frac{d-1}{2\alpha}] + 1$. Indeed, since $F_0 = 0$, $\|F_0\| = 0$ and if we assume that $\|F_{j-1}\| = O(\rho^{-\alpha_1})$, then since $\mu_{\gamma,s} + \|F_{j-1}\| \in J$, by (22), we have $\|F_j\| = O(\rho^{-\alpha_1})$.

By (38), we have $\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}) \in J$. Thus substituting $a \equiv \mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1})$ into $S(a, p_1)$ in (24), we get

$$[D(\Lambda_N, \gamma) - S(\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1)]A(N, \gamma) = O(\rho^{-p\alpha}). \tag{39}$$

Adding and subtracting the term $F_j A(N, \gamma) = S(\mu_{\gamma,s} + \|F_{j-1}\|, j)A(N, \gamma)$ into the left-hand side of equation (39), we obtain

$$[D(\Lambda_N, \gamma) - F_j]A(N, \gamma) - E_j A(N, \gamma) = O(\rho^{-p\alpha}), \tag{40}$$

where

$$\begin{aligned} E_j &= [S(\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), j) - S(\mu_{\gamma,s} + \|F_{j-1}\|, j)] \\ &\quad + (\sum_{k=j+1}^{p_1} S^k(\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}))). \end{aligned}$$

By (20), we have

$$\sum_{k=j+1}^{p_1} S^k(\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1})) = O(\rho^{-(j+1)\alpha_1}). \tag{41}$$

If we prove that

$$\|S(\mu_{\gamma,s} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), j) - S(\mu_{\gamma,s} + \|F_{j-1}\|, j)\| = O(\rho^{-(j+1)\alpha_1}), \tag{42}$$

then it follows from (41) and (42) that

$$\|E_j\| = O(\rho^{-(j+1)\alpha_1}). \tag{43}$$

Now, we prove (42). Since $\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1})\in J$ and $\mu_{\gamma,s}+\|F_{j-1}\|\in J$ satisfy (13), by (14), we have

$$
\begin{aligned}
|\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1+\cdots+\gamma_t\mid^2\mid &> \frac{1}{2}\rho^{\alpha_1},\\
|\mu_{\gamma,s}+\|F_{j-1}\|-\mid\gamma+\gamma_1+\cdots+\gamma_t\mid^2\mid &> \frac{1}{2}\rho^{\alpha_1},
\end{aligned}
\tag{44}
$$

for all $\gamma_t\in\Gamma(\rho^\alpha)$ and $t=1,2,\ldots,p_1$. By its definition, $S(a,j)\equiv\sum_{k=1}^{j}S^k(a)$. Thus we first calculate the order of the first term of the summation in (42). To do this, we consider each entry of this term, and use (44) and (7):

$$
\begin{aligned}
&|s^1_{li}(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-s^1_{li}(\mu_{\gamma,s}+\|F_{j-1}\|)|\\
&\quad\le\sum_{i_1=1}^{m}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2=0}}|v_{i_1l\gamma_1}||v_{ii_1\gamma_2}|O(\rho^{-j\alpha_1})\\
&\quad\times\frac{1}{|(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1\mid^2)||(\mu_{\gamma,s}+\|F_{j-1}\|-\mid\gamma+\gamma_1\mid^2)|}\\
&\quad\le c_{11}\rho^{-(j+2)\alpha_1},
\end{aligned}
$$

for each $l,i=1,2,\ldots,m$ which implies

$$
\|S^1(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^1(\mu_{\gamma,s}+\|F_{j-1}\|)\|=O(\rho^{-(j+2)\alpha_1}).
$$

If we consider each entry of the second term of the summation in (42), then again by (44) and (7) we see

$$
\begin{aligned}
&|s^2_{li}(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-s^2_{li}(\mu_{\gamma,s}+\|F_{j-1}\|)|\\
&\quad\le\sum_{i_1,i_2=1}^{m}\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3=0}}|v_{i_1l\gamma_1}||v_{i_2i_1\gamma_2}||v_{ii_2\gamma_3}|O(\rho^{-j\alpha_1})\\
&\quad\times\Big\{\frac{1}{|(a'+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1\mid^2)(a'+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1+\gamma_2\mid^2)(a'-\mid\gamma+\gamma_1+\gamma_2\mid^2)|}\\
&\quad+\frac{1}{|(a'+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1\mid^2)(a'-\mid\gamma+\gamma_1\mid^2)(a'+O(\rho^{-j\alpha_1})-\mid\gamma+\gamma_1+\gamma_2\mid^2)|}\Big\}\\
&\quad\le c_{12}\rho^{-(j+3)\alpha_1},
\end{aligned}
$$

for each $l,i=1,2,\ldots,m$, where we use the notation $a'\equiv\mu_{\gamma,s}+\|F_{j-1}\|$ for the sake of simplicity, which implies

$$
\|S^2(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^2(\mu_{\gamma,s}+\|F_{j-1}\|)\|=O(\rho^{-(j+3)\alpha_1}).
$$

Therefore, by direct calculations, it can be easily seen that

$$
\|S^k(\mu_{\gamma,s}+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^k(\mu_{\gamma,s}+\|F_{j-1}\|)\|=O(\rho^{-(j+k+1)\alpha_1})
$$

from which we obtain (42).

Theorem 2. *Let* $|\gamma|^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $|\gamma| \sim \rho$.

(a) For any eigenvalue λ_i, $i = 1, 2, \ldots, m$ *of the matrix* V_0, *there exits an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying the following formula:*

$$\Lambda_N = \mu_{\gamma,i} + \|F_{k-1}\| + O(\rho^{-k\alpha_1}), \tag{45}$$

where $\mu_{\gamma,i} = |\gamma|^2 + \lambda_i$, F_{k-1} *is given by* (37), $k = 1, 2, \ldots, p - c$.

(b) For any eigenvalue Λ_N *of the operator* $L(V)$ *satisfying* (13), *there is an eigenvalue* λ_i *of the matrix* V_0 *satisfying* (45).

Proof. (a): By Lemma(1a), there exist Λ_N and $\Psi_N(x)$ satisfying (13) and (26), respectively. We prove the theorem by induction. For $k = 1$, we obtain the result by Theorem(1a).

Now, assume that for $k = j - 1$ formula (45) is true, that is,

$$\Lambda_N = \mu_{\gamma,i} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}). \tag{46}$$

Let β_i be an eigenvalue of the matrix $D(\Lambda_N, \gamma) - S(\mu_{\gamma,i} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1)$. If we multiply both sides of equation (39) by its corresponding normalized eigenvector f_i, and use (26), then we obtain

$$\beta_i = O(\rho^{-(p-c)\alpha}). \tag{47}$$

On the other hand, the matrix $D(\Lambda_N, \gamma) - S(\mu_{\gamma,i} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1)$ in (39) is decomposed as follows

$$D(\Lambda_N, \gamma) - S(\mu_{\gamma,i} + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1) = D(\Lambda_N, \gamma) - F_j - E_j.$$

Thus, by (43), (47) and a well known result in matrix theory,

$$|\beta_i - (\Lambda_N - \mu_{\gamma,i})| \leq \|F_j\| + O(\rho^{-(j+1)\alpha_1}),$$

where $1 \leq j + 1 \leq p - c$, we get the proof of (45).

(b): Again we prove this part of the theorem by induction. For $j = 1$, we obtain the result by Theorem (1b).

Now, assume that for $k = j - 1$ formula (45) is true. To prove (45) for $k = j$, we use equation (40). By using the definition of the matrix $D(\Lambda_N, \gamma)$ and (40), we have

$$[(\Lambda_N - |\gamma|^2)I - D_j]A(N, \gamma) = E_j A(N, \gamma) + O(\rho^{-p\alpha}),$$

where $D_j = V_0 + F_j$. Applying $\dfrac{1}{\|A(N,\gamma)\|}[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}$ to both sides of the above equation, taking the norm of both sides, and using estimations (36) and (43), we obtain

$$1 \leq \|[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}\|[O(\rho^{-(j+1)\alpha_1}] + \|[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}\|[O(\rho^{-(p-c)\alpha})]$$
$$\leq \max_{i=1,2,\ldots,m} \frac{1}{|\Lambda_N - |\gamma|^2 - \widetilde{\lambda}_i(j)|}[O(\rho^{-(j+1)\alpha_1})],$$

or

$$\min_{i=1,2,\ldots,m} \mid \Lambda_N - \mid \gamma \mid^2 - \widetilde{\lambda}_i(j) \mid \leq c_{13}\rho^{-(j+1)\alpha_1},$$

where the minimum is taken over all eigenvalues $\widetilde{\lambda}_i(j)$ of the matrix D_j, $1 \leq j + 1 \leq p - c$. By the last inequality and the well known result in matrix theory, $|\widetilde{\lambda}_i(j) - \lambda_i| \leq \|F_j\|$, we obtain the result. □

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