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Lamarle formula in 3-dimensional Lorentz space*

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Abstract. The Lamarle Formula is known as a relationship between the Gaussian curvature and the distribution parameter (Drall) of a ruled surface in the surface theory, [12]. In $\mathbb{R}^3_1$, the non-null ruled surfaces were investigated in five different classes with respect to the character of base curves and rulings, [8]–[10]. In this paper, by excluding the null tangent of ruling, three different kinds of non-null ruled are taken into consideration and the relationships between the Gaussian curvatures and distribution parameters of spacelike ruled surface, timelike ruled surface with spacelike ruling and timelike ruled surface with timelike ruling are obtained, respectively. These relationships are called as Lorentzian Lamarle formulae. Finally, some examples concerning these relations are given.

AMS subject classifications: 53B30, 53C50

Key words: Ruled surface, distribution parameter (Drall), Gaussian curvature, Lamarle formula

1. Introduction

The study of a ruled surface in $\mathbb{R}^3$ is a classical subject in the differential geometry. It has again been studied in some areas (i.e. Projective geometry, [17], Computer-aided design, [15], etc.) Also, it is well known that the geometry of ruled surface is very important in kinematics or spatial mechanisms in $\mathbb{R}^3$, [5], [11]. A ruled surface is one which can be generated by sweeping a line through space. Developable surfaces are special cases of ruled surfaces, [14]. Cylindrical surfaces are examples of developable surfaces. On a developable surface at least one of the two principal curvatures is zero at all points. Consequently, the Gaussian curvature is zero everywhere, too. So, it is meaningful for us to study non-cylindrical ruled surfaces. The relationship between the Gaussian curvature and the distribution parameter (Drall) of a non-cylindrical ruled surface is $K = -\frac{D^2}{(D^2+v^2)^2}$ in $\mathbb{R}^3$, which is called Lamarle formula, [12].

The Lorentz metrics in a $3-$dimensional Lorentz space $\mathbb{R}^3_1$ is indefinite. In the theory of relativity, geometry of indefinite metric is very crucial. Hence, the theory of a ruled surface in Lorentz space $\mathbb{R}^3_1$, which has the metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$, attracted much attention.

*For interpretation of color in all figures, the reader is referred to the web version of this article available at www.mathos.hr/mc.

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The situation is much more complicated than the Euclidean case, since ruled surfaces may have a definite metric (spacelike surfaces), Lorentz metric (timelike surfaces) or mixed metric. Timelike and spacelike ruled surfaces are defined and the characterizations of these non-null ruled surfaces are found in [2, 3] [6]–[10], [13, 18] and [19].

2. Preliminaries

Let $\mathbb{R}^3_1$ denote the 3−dimensional Lorentz space, i.e. the usual vector space $\mathbb{R}^3$ with standard flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3_1$. Since $\langle\ ,\ \rangle$ is indefinite metric, recall that a vector $\vec{v}$ in $\mathbb{R}^3_1$ can have one of three casual characters: it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s) \subset \mathbb{R}^3_1$ can be locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike or null (lightlike), respectively. The norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|\langle \vec{v}, \vec{v} \rangle|}$. Therefore, $\vec{v}$ is a unit vector if $\langle \vec{v}, \vec{v} \rangle = \mp 1$. Furthermore, vectors $\vec{v}$ and $\vec{w}$ are said to be orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$, [14].
Let the set of all timelike vectors in $\mathbb{R}^3_1$ be Γ. For $\vec{u} \in \Gamma$, we call

$$C(\vec{u}) = \{\, \vec{v} \in \Gamma |\, \langle \vec{v}, \vec{u} \rangle < 0\}$$

time-conic of Lorentz space $\mathbb{R}^3_1$ including vector $\vec{u}$, [14].
Let $\vec{v}$ and $\vec{w}$ be two timelike vectors in Lorentz space $\mathbb{R}^3_1$. Then there exists the following inequality

$$|\langle \vec{v}, \vec{w} \rangle| \geq \|\vec{v}\|\, \|\vec{w}\|$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent.
If timelike vectors $\vec{v}$ and $\vec{w}$ stay inside the same time-conic, then there is a unique non-negative real number of $\theta \geq 0$ such that

$$\langle \vec{v}, \vec{w} \rangle = -\|\vec{v}\|\, \|\vec{w}\| \cosh\theta, \tag{2}$$

where the number θ is called an angle between the timelike vectors, [14].
Let $\vec{v}$ and $\vec{w}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike subspace. We have that

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\, \|\vec{w}\|$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent. Hence, there is a unique angle $0 \leq \theta \leq \pi$ such that

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\|\, \|\vec{w}\| \cos\theta, \tag{3}$$

where the number θ is called the Lorentzian spacelike angle between spacelike vectors $\vec{v}$ and $\vec{w}$, [16].
Let $\vec{v}$ and $\vec{w}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike subspace. We have that

$$|\langle \vec{v}, \vec{w} \rangle| > \|\vec{v}\|\, \|\vec{w}\|\,.$$

Hence, there is a unique real number $\theta > 0$ such that

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \, \|\vec{w}\| \cosh \theta. \tag{4}$$

The Lorentzian timelike angle between spacelike vectors $\vec{v}$ and $\vec{w}$ is defined to be θ, [16].
Let $\vec{v}$ be a spacelike vector and $\vec{w}$ a timelike vector in $\mathbb{R}_1^3$. Then there is a unique real number $\theta \geq 0$ such that

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \, \|\vec{w}\| \sinh \theta. \tag{5}$$

The Lorentzian timelike angle between $\vec{v}$ and $\vec{w}$ is defined to be θ, [16].
For any vectors $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}_1^3$, the Lorentzian product $\vec{v} \wedge \vec{w}$ of $\vec{v}$ and $\vec{w}$ is defined as [16]

$$\vec{v} \wedge \vec{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_2 w_1 - v_1 w_2) \, . \tag{6}$$

3. Ruled surface in $\mathbb{R}_1^3$

A ruled surface $M \in \mathbb{R}_1^3$ is a regular surface that has a parametrization $\varphi : (I \times \mathbb{R}) \to \mathbb{R}_1^3$ of the form

$$\varphi (u, v) = \vec{\alpha} (u) + v \vec{\gamma} (u) \, , \tag{7}$$

where $\vec{\alpha}$ and $\vec{\gamma}$ are curves in $\mathbb{R}_1^3$ with $\vec{\alpha}'$ that never vanishes. The curve α is called the base curve. The rulings of a ruled surface are the straight lines $v \to \vec{\alpha} (u) + v \vec{\gamma} (u)$. If consecutive rulings of a ruled surface in $\mathbb{R}_1^3$ intersect, then the surface is said to be developable. All other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two consecutive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a striction point. The set of striction points on a ruled surface defines the striction curve, [19].
The striction curve, $\beta (u)$ can be written in terms of the base curve $\alpha (u)$ as

$$\vec{\beta} (u) = \vec{\alpha} (u) - \frac{\langle \vec{\alpha}' (u) , \vec{\gamma}' (u) \rangle}{\langle \vec{\gamma}', \vec{\gamma}' \rangle} \, \vec{\gamma} (u) \, . \tag{8}$$

A ruled surface given by (7) is called non-cylindrical if $\vec{\gamma} \wedge \vec{\gamma}'$ is nowhere zero. Thus, the rulings are always changing directions on a non-cylindrical ruled surface. A non-cylindrical ruled surface always has a parameterization of the form

$$\tilde{\varphi} (u, v) = \vec{\beta} (u) + v \, \vec{e} (u) \, , \tag{9}$$

where $\vec{e} (u) = \dfrac{\vec{\gamma} (u)}{\|\vec{\gamma} (u)\|}$, $\|\vec{e}\| = 1$, $\left\langle \overrightarrow{\beta}' (u) \, , \overrightarrow{e}' (u) \right\rangle = 0$ and $\vec{\beta} (u)$ is a striction curve of $\tilde{\varphi}$, [4].
The distribution parameter (Drall) of a non-cylindrical ruled surface given by equation (9), is a function D defined by

$$D = \frac{\det \left(\overrightarrow{\beta}', \overrightarrow{e}, \overrightarrow{e}' \right)}{\langle \overrightarrow{e}', \overrightarrow{e}' \rangle}, \tag{10}$$

where $\vec{\beta}$ is the striction curve and $\vec{e}$ is the ruling of ruled surface.
Moreover, the Gaussian curvature of the non-cylindrical ruled surface $\tilde{\varphi}(u,v)$ is

$$K = \langle \vec{\eta}, \vec{\eta} \rangle \frac{LN - M^2}{EG - F^2}, \tag{11}$$

where E, F and G are the coefficients of the first fundamental form, whereas L, M and N are the coefficients of the second fundamental form of a non-cylindrical ruled surface. Here

$$\vec{\eta}(u,v) = \frac{\tilde{\varphi}_u \wedge \tilde{\varphi}_v}{\|\tilde{\varphi}_u \wedge \tilde{\varphi}_v\|} \tag{12}$$

is a unit normal vector on the non-cylindrical ruled surface, [3].
A surface in the 3−dimensional Minkowski space-time $\mathbb{R}^3_1$ is called a timelike surface if metric induced on the surface is a Lorentzian metric, i.e. the normal on the surface is a spacelike vector, [19].
In $\mathbb{R}^3_1$, there are five different kinds of non-null ruled surfaces according to the character of the base curve and the tangent of ruling, [8]–[10]. In this paper, the classes of ruled surfaces having null tangent of ruling are excluded and non-null ruled surfaces are classified into three different groups as follows: As a spacelike ruling moves along a spacelike curve, it generates a spacelike ruled surface that will be denoted by M_1. Furthermore, the movement of a timelike ruling along a spacelike curve and the movement of a spacelike ruling along a timelike curve generate timelike ruled surfaces. Let us denote these timelike ruled surfaces by M_2 and M_3, respectively. Now, we will establish Lamarle formula for these ruled surfaces M_1, M_2, M_3 separately.

4. Lamarle formula for the spacelike ruled surface

Let M_1 be a spacelike ruled surface parametrized by

$$\begin{aligned} &\varphi_1 : \mathrm{I} \times \mathbb{R} \to \mathbb{R}^3_1 \\ &(u,v) \to \varphi_1(u,v) = \vec{\alpha}_1(u) + v\,\vec{e}_1(u). \end{aligned}$$

If we choose $\|\vec{e}_1\| = 1$, $\vec{n}_1 = \dfrac{\overrightarrow{e}'_1}{\|\overrightarrow{e}'_1\|}$ and $\vec{\xi}_1 = \dfrac{\overrightarrow{e}_1 \wedge \overrightarrow{e}'_1}{\|\overrightarrow{e}_1 \wedge \overrightarrow{e}'_1\|}$, we obtain the orthonormal frame field $\left\{\vec{e}_1, \vec{n}_1, \vec{\xi}_1\right\}$. Suppose that the orthonormal frame field forms a right-handed system and in $\{\text{space}, \text{time}, \text{space}\}$ type. In this case we may write

$$\begin{aligned} &\langle \vec{e}_1, \vec{e}_1 \rangle = 1\ ,\ \langle \vec{n}_1, \vec{n}_1 \rangle = -1\ ,\ \left\langle \vec{\xi}_1, \vec{\xi}_1 \right\rangle = 1, \\ &\langle \vec{e}_1, \vec{n}_1 \rangle = \left\langle \vec{n}_1, \vec{\xi}_1 \right\rangle = \left\langle \vec{\xi}_1, \vec{e}_1 \right\rangle = 0 \end{aligned} \tag{13}$$

and

$$\vec{e}_1 \wedge \vec{n}_1 = \vec{\xi}_1 \quad , \quad \vec{n}_1 \wedge \vec{\xi}_1 = \vec{e}_1 \quad , \quad \vec{\xi}_1 \wedge \vec{e}_1 = -\vec{n}_1. \tag{14}$$

The Frenet formula of this orthonormal frame along $\vec{e}_1$ becomes

$$\overrightarrow{e}'_1 = \kappa_1\,\vec{n}_1 \quad , \quad \overrightarrow{n}'_1 = \kappa_1\,\vec{e}_1 + \tau_1\,\vec{\xi}_1 \quad , \quad \overrightarrow{\xi}'_1 = \tau_1\,\vec{n}_1. \tag{15}$$

Let $\vec{\beta}_1(u)$ be a striction curve of the spacelike ruled surface M_1 given by equation (8) in $\mathbb{R}^3_1$. In this case $\left\langle \overrightarrow{\beta}'_1, \overrightarrow{e}'_1 \right\rangle = 0$, that is, $\left\langle \overrightarrow{\beta}'_1, \kappa_1 \vec{n}_1 \right\rangle = 0$. This means that the tangent vector $\overrightarrow{\beta}'_1$ stays in the plane spanned by $\vec{e}_1$ and $\vec{\xi}_1$. Since $\langle \vec{e}_1, \vec{e}_1 \rangle \left\langle \vec{\xi}_1, \vec{\xi}_1 \right\rangle - \left\langle \vec{e}_1, \vec{\xi}_1 \right\rangle^2 > 0$, then the plane spanned by $\vec{e}_1$ and $\vec{\xi}_1$ is spacelike and the angle between the spacelike vectors $\vec{\beta}'_1$ and $\vec{e}_1$ is a Lorentzian spacelike angle. Taking the angle σ_1 to be the angle between $\vec{\beta}'_1$ and $\vec{e}_1$ the tangent vector of striction curve of M_1 is

$$\overrightarrow{\beta}'_1 = \cos\sigma_1 \vec{e}_1 + \sin\sigma_1 \vec{\xi}_1,$$

then we find the striction curve of M_1 to be

$$\vec{\beta}_1 = \int \left(\cos\sigma_1 \, \vec{e}_1 + \sin\sigma_1 \, \vec{\xi}_1 \right) du.$$

The spacelike non-cylindrical ruled surface M_1 is parametrized by

$$\tilde{\varphi}_1(u, v) = \int \left(\cos\sigma_1 \, \vec{e}_1 + \sin\sigma_1 \, \vec{\xi}_1 \right) du + v \, \vec{e}_1.$$

From equation (10) the distribution parameter (Drall) of M_1 is found to be

$$D = \frac{\det\left(\cos\sigma_1 \vec{e}_1 + \sin\sigma_1 \vec{\xi}_1 \, , \vec{e}_1, \kappa_1 \vec{n}_1 \right)}{\langle \kappa_1 \vec{n}_1 \, , \kappa_1 \vec{n}_1 \rangle} = -\frac{\sin\sigma_1}{\kappa_1}.$$

By taking $\kappa_1 = \frac{1}{\rho_1}$ we get the Drall as follows

$$D = -\rho_1 \, \sin\sigma_1. \tag{16}$$

Considering equations (14) and (12) we write the unit normal vector of the spacelike non-cylindrical ruled surface M_1 as

$$\vec{\eta}_1 = -\frac{\sin\sigma_1 \, \vec{n}_1 + v\kappa_1 \, \vec{\xi}_1}{\sqrt{\left| -\sin^2\sigma_1 + v^2\kappa_1^2 \right|}}. \tag{17}$$

Taking into consideration that $\kappa_1 = \frac{1}{\rho_1}$ and the equation (16) we obtain

$$\vec{\eta}_1 = \frac{D\vec{n}_1 - v\vec{\xi}_1}{\sqrt{\left| -D^2 + v^2 \right|}}. \tag{18}$$

Furthermore, since the unit normal vector $\vec{\eta}_1$ of a spacelike surface M_1 is timelike, we find that $-D^2 + v^2 < 0$, that is, $|v| < |D|$.
The partial differentiations of $\tilde{\varphi}_1$ with respect to u and v from equation (15) are as follows

$$\begin{aligned} \tilde{\varphi}_{1u} &= \cos\sigma_1 \; \vec{e}_1 + \sin\sigma_1 \; \vec{\xi}_1 + \; v\,\kappa_1 \vec{n}_1, \\ \tilde{\varphi}_{1v} &= \vec{e}_1. \end{aligned} \tag{19}$$

Therefore, we find the first fundamental form coefficients of M_1 to be

$$\begin{aligned} E &= \langle \tilde{\varphi}_{1u}, \tilde{\varphi}_{1u} \rangle = \cos^2 \sigma_1 + \sin^2 \sigma_1 - v^2 \kappa_1^2 = 1 - v^2 \kappa_1^2, \\ F &= \langle \tilde{\varphi}_{1u}, \tilde{\varphi}_{1v} \rangle = \cos \sigma_1, \\ G &= \langle \tilde{\varphi}_{1v}, \tilde{\varphi}_{1v} \rangle = 1. \end{aligned} \tag{20}$$

In addition to these, the second order partial differentials of $\tilde{\varphi}_2$ are found to be

$$\begin{aligned} \tilde{\varphi}_{1uu} &= \left(-\sigma_1' \sin \sigma_1 + v\kappa_1^2 \right) \vec{e}_1 + \left(\kappa_1 \cos \sigma_1 + \tau_1 \sin \sigma_1 + v\kappa_1' \right) \vec{n}_1 \\ &\quad + \left(\sigma_1' \cos \sigma_1 + v\kappa_1 \tau_1 \right) \vec{\xi}_1, \\ \tilde{\varphi}_{1uv} &= \kappa_1 \vec{n}_1, \\ \tilde{\varphi}_{1vv} &= 0. \end{aligned}$$

From equation (17) and the last equations we get the coefficients of second fundamental of M_1 as

$$\begin{aligned} L &= \langle \tilde{\varphi}_{1uu}, \vec{\eta}_1 \rangle = \frac{\kappa_1 \cos \sigma_1 \sin \sigma_1 + \tau_1 \sin^2 \sigma_1 + v\kappa_1' \sin \sigma_1 - \sigma_1' \cos \sigma_1 v\kappa_1 - v^2 \kappa_1^2 \tau_1}{\sqrt{\left| -\sin^2 \sigma_1 + v^2 \kappa_1^2 \right|}}, \\ M &= \langle \tilde{\varphi}_{1uv}, \vec{\eta}_1 \rangle = \frac{\kappa_1 \sin \sigma_1}{\sqrt{\left| -\sin^2 \sigma_1 + v^2 \kappa_1^2 \right|}}, \\ N &= \langle \tilde{\varphi}_{1vv}, \vec{\eta}_1 \rangle = 0. \end{aligned} \tag{21}$$

Considering equations (20) and (21) together, we give the following theorem for the Gaussian curvature of the spacelike ruled surface M_1.

Theorem 1. *Let M_1 be a spacelike non-cylindrical ruled surface in $\mathbb{R}_1^3$. The Gaussian curvature of the spacelike non-cylindrical ruled surface M_1 is given in terms of its Drall D by*

$$K = \frac{D^2}{\left(D^2 - v^2 \right)^2}, \tag{22}$$

where $|v| < |D|$.

Proof. Substituting equations (20) and (21) into equation (11) and making appropriate simplifications, we find the Gaussian curvature of M_1 to be

$$K = \langle \vec{\eta}_1, \vec{\eta}_1 \rangle \frac{\kappa_1^2 \sin^2 \sigma_1}{\left(\sin^2 \sigma_1 - v^2 \kappa_1^2 \right)^2},$$

where $\vec{\eta}_1$ is the timelike unit normal vector of the spacelike surface M_1. Considering $\kappa_1 = \frac{1}{\rho_1}$ and the equation (16) completes the proof. □

The relation between the Gaussian curvature and the distribution parameter (Drall) of M_1 given by equation (22) is called the *Lorentzian Lamarle formula* for the spacelike non-cylindrical ruled surface M_1. In addition, if we choose the orthonormal frame field $\left\{\vec{e}_1, \vec{n}_1, \vec{\xi}_1\right\}$ as in $\{\text{space}, \text{space}, \text{time}\}$ type for the spacelike non-cylindrical ruled surface M_1, then the plane spanned by $\vec{e}_1$ and $\vec{\xi}_1$ becomes timelike and the angle σ_1 between the tangent vector of the striction curve $\vec{\beta}_1'$ and $\vec{e}_1$ is a hyperbolic angle. Thus, the spacelike non-cylindrical ruled surface M_1 is parametrized by

$$\tilde{\varphi}_1(u,v) = \int \left(\cosh\sigma_1\,\vec{e}_1 + \sinh\sigma_1\,\vec{\xi}_1\right) du + v\,\vec{e}_1.$$

After similar calculations, the Gaussian curvature of M_1 is obtained as in equation (22) and the Lorentzian Lamarle formula for the spacelike non-cylindrical ruled surface remains unchanged.
Therefore we give the following corollary.

Corollary 1. *Let M_1 be a spacelike non-cylindrical ruled surface with Drall D and the Gaussian curvature K in $\mathbb{R}^3_1$.*

1. *Along a ruling as $v \to \mp D$, the Gaussian curvature $K(u,v) \to \infty$.*
2. *$K(u,v) = 0$ if and only if $D = 0$.*
3. *If the Drall D never vanishes, then $K(u,v)$ is continuous and when $v = 0$, i.e. at the central point on each ruling, $K(u,v)$ takes its minimum value.*

Example 1. *In a $3-$dimensional Lorentz space $\mathbb{R}^3_1$ let us define a non-cylindrical ruled surface as*

$$\varphi(u,v) = (\,-v\cosh u\,,\,u\,,\,-v\sinh u\,)\,,$$

that is, a 2^{nd} type helicoid and a spacelike surface where $-1 < v < 1$, see Figure 1. The Gaussian curvature of this 2^{nd} type helicoid is $K = \frac{1}{(1-v^2)^2}, |v| < 1$, see Figure 2.

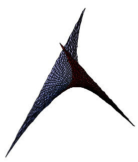

Figure 1: 2^{nd} *type helicoid*

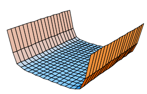

Figure 2: *The Gaussian curvature of the 2^{nd} type helicoid*

5. Lamarle formula for the timelike ruled surface with a spacelike base curve and timelike ruling

Let M_2 be a timelike ruled surface with a spacelike base curve and timelike ruling in the 3−dimensional Lorentz space, $\mathbb{R}^3_1$. Thus, this ruled surface is parametrized as follows;

$$\begin{aligned} &\varphi_2 : \mathrm{I} \times \mathbb{R} \to \mathbb{R}^3_1 \\ &(u, v) \to \varphi_2(u, v) = \vec{\alpha}_2(u) + v\, \vec{e}_2(u) . \end{aligned}$$

Here, taking $\|\vec{e}_2\| = 1$, $\vec{n}_2 = \dfrac{\overrightarrow{e}'_2}{\|\overrightarrow{e}'_2\|}$ and $\vec{\xi}_2 = \dfrac{\overrightarrow{e}_2 \wedge \overrightarrow{e}'_2}{\|\overrightarrow{e}_2 \wedge \overrightarrow{e}'_2\|}$, we reach the orthonormal frame field $\left\{\vec{e}_2, \vec{n}_2, \vec{\xi}_2\right\}$. This forms a right-handed system and in $\{\text{time, space, space}\}$ type. Therefore,

$$\begin{aligned} -\langle \vec{e}_2, \vec{e}_2 \rangle = \langle \vec{n}_2, \vec{n}_2 \rangle = \left\langle \vec{\xi}_2, \vec{\xi}_2 \right\rangle = 1 \\ \langle \vec{e}_2, \vec{n}_2 \rangle = \left\langle \vec{n}_2, \vec{\xi}_2 \right\rangle = \left\langle \vec{\xi}_2, \vec{e}_2 \right\rangle = 0 \end{aligned} \tag{23}$$

and

$$\vec{e}_2 \wedge \vec{n}_2 = \vec{\xi}_2 \quad , \quad \vec{n}_2 \wedge \vec{\xi}_2 = -\vec{e}_2 \quad , \quad \vec{\xi}_2 \wedge \vec{e}_2 = \vec{n}_2. \tag{24}$$

The differential formula of this orthonormal system is

$$\overrightarrow{e}'_2 = \kappa_2\, \vec{n}_2 \quad , \quad \overrightarrow{n}'_2 = \kappa_2\, \vec{e}_2 - \tau_2\, \vec{\xi}_2 \quad , \quad \overrightarrow{\xi}'_2 = \tau_2\, \vec{n}_2. \tag{25}$$

Now, let the striction curve given by equation (8) of the timelike ruled surface M_2 be $\vec{\beta}_2(u)$. $\vec{\beta}_2(u)$ is a spacelike curve and the tangent vector of this curve $\vec{\beta}'_2$ stays in the timelike plane $\left(\vec{e}_2, \vec{\xi}_2\right)$. Taking the hyperbolic angle σ_2 between $\vec{\beta}'_2$ and $\vec{e}_2$ we write

$$\overrightarrow{\beta}'_2 = \sinh \sigma_2\, \vec{e}_2 + \cosh \sigma_2\, \vec{\xi}_2 \, .$$

From the last equation we write for the striction curve of M_2

$$\vec{\beta}_2 = \int (\sinh \sigma_2\, \vec{e}_2 + \cosh \sigma_2\, \xi_2)\, du.$$

Let M_2 be a timelike non-cylindrical ruled surface with a spacelike base curve and timelike ruling in $\mathbb{R}^3_1$. In this case we reparametrize M_2 such as

$$\tilde{\varphi}_2(u, v) = \int \left(\sinh \sigma_2\, \vec{e}_2 + \cosh \sigma_2\, \vec{\xi}_2\right) du + v\, \vec{e}_2.$$

Considering the equation (10) we find the distribution parameter (Drall) of the timelike non-cylindrical ruled surface M_2 to be

$$D = \frac{\det\left(\sinh \sigma_2\, \vec{e}_2 + \cosh \sigma_2\, \vec{\xi}_2\, , \vec{e}_2, \kappa_2\, \vec{n}_2\right)}{\langle \kappa_2\, \vec{n}_2, \kappa_2\, \vec{n}_2 \rangle} = \frac{\cosh \sigma_2}{\kappa_2}.$$

Taking $\kappa_2 = \frac{1}{\rho_2}$ we rewrite the Drall of M_2 as

$$D = \rho_2 \cosh \sigma_2. \tag{26}$$

If we consider equation (24), from equation (12) we see that the timelike non-cylindrical ruled surfaces unit normal vector becomes

$$\vec{\eta}_2 = \frac{\cosh \sigma_2 \, \vec{n}_2 - v \kappa_2 \, \vec{\xi}_2}{\sqrt{\left|\cosh^2 \sigma_2 + v^2 \kappa_2^2\right|}}. \tag{27}$$

Since $\kappa_2 = \frac{1}{\rho_2}$, from equation (26) we find

$$\vec{\eta}_2 = \frac{D\vec{n}_2 - v\vec{\xi}_2}{\sqrt{D^2 + v^2}}. \tag{28}$$

From equation (25), partial differentials of $\tilde{\varphi}_2$ with respect to u and v are

$$\begin{aligned} \tilde{\varphi}_{2u} &= \sinh \sigma_2 \, \vec{e}_2 + \cosh \sigma_2 \, \vec{\xi}_2 + v \kappa_2 \vec{n}_2, \\ \tilde{\varphi}_{2v} &= \vec{e}_2. \end{aligned} \tag{29}$$

Considering the last equations with equation (23) we find the coefficients of the first fundamental form of M_2 to be

$$\begin{aligned} E &= \langle \tilde{\varphi}_{2u}, \tilde{\varphi}_{2u} \rangle = -\sinh^2 \sigma_2 + \cosh^2 \sigma_2 + v^2 \kappa_2^2 = 1 + v^2 \kappa_2^2, \\ F &= \langle \tilde{\varphi}_{2u}, \tilde{\varphi}_{2v} \rangle = -\sinh \sigma_2, \\ G &= \langle \tilde{\varphi}_{2v}, \tilde{\varphi}_{2v} \rangle = -1. \end{aligned} \tag{30}$$

Furthermore, if we consider equation (29), we reach that the second order partial differentials of $\tilde{\varphi}_2$

$$\begin{aligned} \tilde{\varphi}_{2uu} &= \left(\sigma_2' \cosh \sigma_2 + v \kappa_2^2\right) \vec{e}_2 + \left(\kappa_2 \sinh \sigma_2 + \tau_2 \cosh \sigma_2 + v \kappa_2'\right) \vec{n}_2 \\ &\quad + \left(\sigma_2' \sinh \sigma_2 - v \kappa_2 \tau_2\right) \vec{\xi}_2, \\ \tilde{\varphi}_{2uv} &= \kappa_2 \vec{n}_2, \\ \varphi_{2vv} &= 0. \end{aligned}$$

From equation (27) and the last equations we find the second fundamentals form coefficients as follows

$$\begin{aligned} L &= \langle \tilde{\varphi}_{2uu}, \vec{\eta}_2 \rangle \\ &= \frac{\kappa_2 \sinh \sigma_2 \cosh \sigma_2 + \tau_2 \cosh^2 \sigma_2 + v \kappa_2' \cosh \sigma_2 - v \kappa_2 \sigma_2' \sinh \sigma_2 + v^2 \kappa_2^2 \tau_2}{\sqrt{\cosh^2 \sigma_2 + v^2 \kappa_2^2}}, \\ M &= \langle \tilde{\varphi}_{2uv}, \vec{\eta}_2 \rangle = \frac{\kappa_2 \cosh \sigma_2}{\sqrt{\cosh^2 \sigma_2 + v \kappa_2^2}}, \\ N &= \langle \tilde{\varphi}_{2vv}, \vec{\eta}_2 \rangle = 0. \end{aligned} \tag{31}$$

Therefore, for the Gaussian curvature of the timelike ruled surface M_2, we give the following theorem.

Theorem 2. *Let M_2 be a timelike non-cylindrical ruled surface with a spacelike base curve and timelike ruling in $\mathbb{R}^3_1$. The Gaussian curvature of M_2 is*

$$K = \frac{D^2}{\left(D^2 + v^2\right)^2}, \tag{32}$$

where D is the Drall of M_2.

Proof. Substituting equations (30) and (31) into equation (11) we find the Gaussian curvature of M_2 to be

$$K = \langle \vec{\eta}_2, \vec{\eta}_2 \rangle \frac{\kappa_2^2 \cosh^2 \sigma_2}{\left(\cosh^2 \sigma_2 + \kappa_2^2 v^2\right)^2},$$

where $\vec{\eta}_2$ is the spacelike unit normal vector of the timelike surface M_2. Here, considering

$$\kappa_2 = \frac{1}{\rho_2}$$

and equation (26) completes the proof. □

The relation between the Gaussian curvature and the distribution parameter (Drall) of M_2 given by equation (32) is called the *Lorentzian Lamarle formula* for the timelike non-cylindrical ruled surface with a spacelike base curve and timelike ruling.
The Lamarle formula for the timelike ruled surface in $\mathbb{R}^3_1$ is non–negative. So, we give the following corollary.

Corollary 2. *Let D be a Drall and K be a Gaussian curvature of a timelike non-cylindrical ruled surface M_2 with spacelike base curve and timelike ruling in $\mathbb{R}^3_1$. In this case*

1. *Along ruling as $v \to \mp\infty$, $K(u,v) \to 0$.*
2. *$K(u,v) = 0$ if and only if $D = 0$.*
3. *If the Drall of M_2 never vanishes, then $K(u,v)$ is continuous and as $v = 0$, i.e. at the central point on each ruling $K(u,v)$ takes its maximum value.*

Example 2. *In a 3−dimensional Lorentz space $\mathbb{R}^3_1$*

$$\varphi(u,v) = (\, -v \sinh u\, ,\, u\, ,\, -v \cosh u)$$

is a 3^{rd} type helicoid and a timelike non-cylindrical ruled surface with a spacelike base curve and timelike ruling, see Figure 3. The Gaussian curvature of this 3^{rd} type helicoid is

$$K = \frac{1}{\left(1 + v^2\right)^2},$$

see Figure 4.

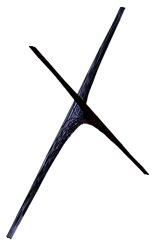

Figure 3: 3^{rd} *type helicoid*

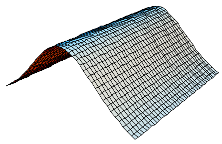

Figure 4: *The Gaussian curvature of the* 3^{rd} *type helicoid*

6. Lamarle formula for the timelike ruled surface with the timelike base curve and spacelike ruling

Suppose that the timelike ruled surface M_3 with the timelike base curve and spacelike ruling in a three-dimensional Lorentz space $\mathbb{R}_1^3$ is parametrized as follows

$$\begin{array}{l}\varphi_3 : \mathrm{I} \times \mathbb{R} \to \mathbb{R}_1^3 \\ (u,v) \to \varphi_3\left(u,v\right) = \vec{\alpha}_3\left(u\right) + v\,\vec{e}_3\left(u\right).\end{array}$$

Considering that $\|\vec{e}_3\| = 1$, $\vec{n}_3 = \dfrac{\overrightarrow{e}'_3}{\|\overrightarrow{e}'_3\|}$ and $\vec{\xi}_3 = \dfrac{\overrightarrow{e}_3 \wedge \overrightarrow{e}'_3}{\|\overrightarrow{e}_3 \wedge \overrightarrow{e}'_3\|}$, we reach the orthonormal frame field $\left\{\vec{e}_3, \vec{n}_3, \vec{\xi}_3\right\}$. Suppose that this orthonormal frame field is in type $\{\text{space}, \text{space}, \text{time}\}$. Thus we write

$$\begin{array}{c}\langle\vec{e}_3, \vec{e}_3\rangle = \langle\vec{n}_3, \vec{n}_3\rangle = -\left\langle\vec{\xi}_3, \vec{\xi}_3\right\rangle = 1 \\ \langle\vec{e}_3, \vec{n}_3\rangle = \left\langle\vec{n}_3, \vec{\xi}_3\right\rangle = \left\langle\vec{\xi}_3, \vec{e}_3\right\rangle = 0\end{array} \tag{33}$$

and cross product is defined to be

$$\vec{e}_3 \wedge \vec{n}_3 = \vec{\xi}_3 \quad , \quad \vec{n}_3 \wedge \vec{\xi}_3 = -\vec{e}_3 \quad , \quad \vec{\xi}_3 \wedge \vec{e}_3 = -\vec{n}_3. \tag{34}$$

A differential formula for this orthonormal system is expressed by

$$\overrightarrow{e}'_3 = \kappa_3\,\vec{n}_3 \quad , \quad \overrightarrow{n}'_3 = -\kappa_3\,\vec{e}_3 + \tau_3\,\vec{\xi}_3 \quad , \quad \overrightarrow{\xi}'_3 = \tau_3\,\vec{n}_3. \tag{35}$$

Let the striction curve of the timelike ruled surface given by equation (8) be $\vec{\beta}_3\left(u\right)$. This curve is a timelike curve and the tangent vector of this curve stays within the timelike plane $\left(\vec{e}_3, \vec{\xi}_3\right)$. Taking the hyperbolic angle σ_3 to be the angle between $\vec{\beta}_3'$ and $\vec{e}_3$ we may write

$$\overrightarrow{\beta}'_3 = \sinh\sigma_3\,\vec{e}_3 + \cosh\sigma_3\,\vec{\xi}_3$$

yielding the striction curve of M_3 to be

$$\vec{\beta}_3 = \int \left(\sinh\sigma_3\,\vec{e}_3 + \cosh\sigma_3\,\vec{\xi}_3\right) du.$$

The timelike non-cylindrical ruled surface M_3 with the timelike base curve and spacelike ruling is reparametrized by

$$\tilde{\varphi}_3\,(u,v) = \int \left(\sinh\sigma_3\,\vec{e}_3 + \cosh\sigma_3\,\vec{\xi}_3\right) du + v\,\vec{e}_3.$$

The Drall of this ruled surface is found to be

$$D = \frac{\det\left(\sinh\sigma_3\,\vec{e}_3 + \cosh\sigma_3\,\vec{\xi}_3, \vec{e}_3, \kappa_3\,\vec{n}_3\right)}{\langle \kappa_3\,\vec{n}_3, \kappa_3\,\vec{n}_3\rangle} = -\frac{\cosh\sigma_3}{\kappa_3}.$$

Then the Drall of M_3 becomes

$$D = -\rho_3\,\cosh\sigma_3, \tag{36}$$

where $\kappa_3 = \frac{1}{\rho_3}$. Taking equation (34) into consideration, we find from equation (12) that the unit normal vector of the timelike non-cylindrical ruled surface M_3 is

$$\vec{\eta}_3 = -\frac{\cosh\sigma_3\,\vec{n}_3 + v\,\kappa_3\,\vec{\xi}_3}{\sqrt{\left|\cosh^2\sigma_3 - v^2\kappa_3^2\right|}}. \tag{37}$$

Since $\kappa_3 = \frac{1}{\rho_3}$, from equation (36) we find

$$\vec{\eta}_3 = \frac{D\vec{n}_3 - v\vec{\xi}_3}{\sqrt{|D^2 - v^2|}}. \tag{38}$$

In addition to these, since the unit normal vector $\vec{\eta}_3$ of the timelike ruled surface M_3 is spacelike, then $D^2 - v^2 > 0$, i.e. $|D| > |v|$.
The partial differentials of $\tilde{\varphi}_3$ with respect to u and v (from equation (35)) become

$$\begin{aligned} \tilde{\varphi}_{3u} &= \sinh\sigma_3\,\vec{e}_3 + \cosh\sigma_3\,\vec{\xi}_3 + v\,\kappa_3\,\vec{n}_3\,, \\ \tilde{\varphi}_{3v} &= \vec{e}_3. \end{aligned} \tag{39}$$

Considering the last equations together with equation (33) the coefficients of the first fundamental form of M_3 are

$$\begin{aligned} E &= \langle \tilde{\varphi}_{3u}, \tilde{\varphi}_{3u}\rangle = \sinh^2\sigma_3 - \cosh^2\sigma_3 + v^2\,\kappa_3^2 = -1 + v^2\,\kappa_3^2, \\ F &= \langle \tilde{\varphi}_{3u}, \tilde{\varphi}_{3v}\rangle = \sinh\sigma_3, \\ G &= \langle \tilde{\varphi}_{3v}, \tilde{\varphi}_{3v}\rangle = 1. \end{aligned} \tag{40}$$

Furthermore, considering equation (39) we find for the second order partial differentials of $\tilde{\varphi}_3$ as

$$\begin{aligned}
\tilde{\varphi}_{3uu} &= \left(\sigma_3' \cosh\sigma_3 - v\,\kappa_3^2\right)\vec{e}_3 + \left(\kappa_3 \sinh\sigma_3 + \tau_3 \cosh\sigma_3 + v\,\kappa_3'\right)\vec{n}_3 \\
&\quad + \left(\sigma_3' \sinh\sigma_3 + v\,\kappa_3\,\tau_3\right)\vec{\xi}_3, \\
\tilde{\varphi}_{3uv} &= \kappa_3\vec{n}_3\,, \\
\varphi_{3vv} &= 0.
\end{aligned}$$

From equation (37) and the last equations, the coefficients of the second order principal form read

$$\begin{aligned}
L &= \langle\tilde{\varphi}_{3uu}, \vec{\eta}_3\rangle \\
&= \frac{-\kappa_3 \sinh\sigma_3 \cosh\sigma_3 - \tau_3 \cosh^2\sigma_3 - v\,\kappa_3' \cosh\sigma_3 + v\,\kappa_3\,\sigma_3' \sinh\sigma_3 + v^2\,\kappa_3^2\,\tau_3}{\sqrt{\left|\cosh^2\sigma_3 - v^2\kappa_3^2\right|}}, \\
M &= -\langle\tilde{\varphi}_{3uv}, \vec{\eta}_3\rangle = \frac{\kappa_3 \cosh\sigma_3}{\sqrt{\left|\cosh^2\sigma_3 - v\,\kappa_3^2\right|}}, \\
N &= \langle\tilde{\varphi}_{3vv}, \vec{\eta}_3\rangle = 0.
\end{aligned} \tag{41}$$

Taking equations (40) and (41) together into account, we can give the following theorem for the Gaussian curvature of timelike ruled surface M_3.

Theorem 3. *Let M_3 be a timelike non-cylindrical ruled surface with a timelike base curve and spacelike ruling in $\mathbb{R}_1^3$. Taking the Drall of M_3 is D, the Gaussian curvature of M_3 becomes*

$$K = \frac{D^2}{\left(D^2 - v^2\right)^2}, \tag{42}$$

where $D^2 - v^2 > 0$.

Proof. Substituting equations (40) and (41) into equation (11) we find the Gaussian curvature of M_3 to be

$$K = \langle\vec{\eta}_3, \vec{\eta}_3\rangle \frac{\kappa_3^2 \cosh^2\sigma_3}{\left(\cosh^2\sigma_3 - \kappa_3^2 v^2\right)^2},$$

where $\vec{\eta}_3$ is the spacelike unit normal vector of the timelike surface M_3. Considering equation (36) together with $\kappa_3 = \frac{1}{\rho_3}$ completes the proof. □

The relation between the Gaussian curvature and the distribution parameter (Drall) of the timelike ruled surface given by equation (42) is called the *Lorentzian Lamarle formula* for the timelike non-cylindrical ruled surface with the timelike base curve and spacelike ruling.
The Lamarle formula for the timelike ruled surface in $\mathbb{R}_1^3$ is non–negative. So, we can give the following corollary.

Corollary 3. *Let M_3 be a timelike non-cylindrical ruled surface with a timelike base curve and spacelike ruling in $\mathbb{R}_1^3$. Considering that D is the Drall and K is the Gaussian curvature we see that*

1. *Along ruling as $v \to \mp D$ the Gaussian curvature $K(u,v) \to +\infty$.*

2. *$K(u,v) = 0$ if and only if $D = 0$.*

3. *If the Drall of M_3 never vanishes, then $K(u,v)$ is continuous and as $v = 0$, i.e. at the central point on each ruling $K(u,v)$ takes its minimum value.*

Example 3. *Let us parametrize a 1^{st} type helicoid as*

$$\varphi(u,v) = (\ -v\cos u\ ,\ -v\sin u,\ u)\,,$$

which is a timelike non-cylindrical ruled surface with a timelike base curve and spacelike ruling in a $3-$dimensional Lorentz space $\mathbb{R}_1^3$ and here $-1 < v < 1$, see Figure 5. The Gaussian curvature of this 1^{st} type helicoid is $K = \frac{1}{(1-v^2)^2}$, $|v| < 1$, see Figure 6.

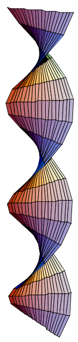

Figure 5: 1^{st} *type helicoid*

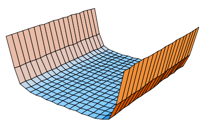

Figure 6: *The Gaussian curvature of the* 1^{st} *type helicoid*

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