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On fuzzy real-valued double A-sequence spaces defined by Orlicz function

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Abstract. The purpose of this paper is to introduce and study a new concept of strong fuzzy real-valued double A- convergence sequences with respect to an Orlicz function. Also, some properties of the resulting fuzzy real-valued sequence spaces are examined. In addition, we define the double A-statistical convergence and establish some connections between the spaces of strong double A-convergence sequence and double A-statistical convergence sequence.

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1. Introduction and background

After the pioneering work of Zadeh [25], a huge number of research papers have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set of modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It has a wide range of applications in various fields; population dynamics, chaos control, computer programming, nonlinear dynamical systems, etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down.

Statistical convergence of single sequences of fuzzy numbers was first deduced by Savas and Nuray [8]. Since the set of all real numbers can be embedded in the set of fuzzy numbers, statistical convergence in reals can be considered as a special case of those fuzzy numbers. However, since the set of fuzzy numbers is partially ordered and does not carry a group structure, most of the results known for the sequences of real numbers may not be valid in fuzzy setting. Therefore this theory should not be considered as a trivial extension of what has been known in a real case.

Savas [12] introduced and discussed fuzzy real-valued convergent double sequences and showed that the set of all fuzzy real-valued convergent double sequences of fuzzy numbers is complete. The concepts of the double lacunary strongly p-Cesàro summability and double lacunary statistical convergence of fuzzy real-valued sequences were studied in [13]. Also, bounded variation double sequence spaces of fuzzy real numbers were studied by Tripaty and Dutta in [20].

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In this paper, we introduce and study the concept of strong A-summability with respect to an Orlicz function. We also examine some properties of this sequence space.

Before we state our main results, first we shall present some definitions.

Since the theory of fuzzy numbers has been widely studied, it is impossible to find either a commonly accepted definition or a fixed notion. We therefore begin by introducing some notions and definitions which will be used throughout.

A fuzzy real number X is a fuzzy set on R , i.e., a mapping $X : R \to I(= [0, 1])$, associating each real number t with its grade of membership $X(t)$.

The $\alpha-$cut of fuzzy real number X is denoted by $[X]_\alpha, 0 < \alpha \leq 1$, where $[X]_\alpha = \{t \in R : X(t) \geq \alpha\}$. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called normal.

A fuzzy number X is said to be convex if $X(t) \geq X(s) \wedge X(r) = min(X(s), X(r))$, where $s < t < r$.

The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article by a fuzzy real number we mean that the number belongs to $R(I)$.

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$. Then we write

$X \leq Y$, if and only if $X^L \leq Y^L$ and $X^R \leq Y^R$, and

$$\rho(X, Y) = max \left\{|X^L - Y^L|, |X^R - Y^R|\right\}.$$

It is obvious that (D, ρ) is a complete metric space. Now we define the metric $d : R(I)\text{x}R(I) \to R$ by

$$d(X, Y) = \sup_{0 \leq \alpha \leq 1} \rho([X]_\alpha, [Y]_\alpha),$$

for $X, Y \in R(I)$.

Applying the notion of fuzzy real numbers, fuzzy real valued sequences were introduced and studied by Nanda [7], Nuray and Savas [8], Savas ([12, 13, 14, 16]), Savas and Patterson ([15]), Tripaty and Dutta ([18, 19]) and Tripaty and Sarma ([22, 23]). A fuzzy double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy real-valued double sequence by (X_{mn}), where X_{mn} are fuzzy real numbers for each $(m, n) \in \mathbb{N} \times \mathbb{N}$.

We now give the following definition:

Definition 1. *Let A denote a four-dimensional summability method that maps the complex double sequences x into a double sequence Ax, where the mn-th term of Ax is as follows:*

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In 1926 Robison presented a four- dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison [11] and Hamilton [4] presented a Silverman-Toeplitz type multidimensional characterization of regularity. The definition of the regularity for four-dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

Definition 2. *The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Theorem 1. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3: $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4: $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$, *is P-convergent*

and

RH_6: *there exist positive numbers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Recall in [5] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently, Orlicz function was used to define sequence spaces by Parashar and B.Choudhary [9] and others. An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x p(t)dt$, where p is known as a kernel of M, right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Let $s^{''}$ denote the set of all double sequences of fuzzy numbers.

We give the following definitions for fuzzy double sequences.

Definition 3 (see [10])**.** *A fuzzy real-valued double sequence $X = (X_{kl})$ is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$d\left(X_{kl}, X_0\right) < \epsilon \quad for \quad k, l > n_0,$$

and we denote by $P - limX = X_0$. The fuzzy number X_0 is called the Pringsheim limit of (X_{kl}).

Let $c^{''}(F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 4 (see [13])**.** *A fuzzy real-valued double sequence $X = (X_{kl})$ is bounded if there exists a positive number M such that $d\left(X_{kl}, \bar{0}\right) < M$ for all k and l. We will denote the set of all bounded double sequences by $\ell_{\infty}^{''}(F)$.*

2. Main results

Definition 5. *Let M be an Orlicz function and $A = (a_{m,n,k,l})$ a nonnegative RH-regular summability matrix method. We now present the following sets of double sequence spaces:*

$$\omega_0^{''}(A, M, p)(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(A, M, p)(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\omega_\infty^{''}(A, M, p)(F) = \left\{ X \in s^{''} : \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}.$$

Let us consider a few special cases of the above sets.

1. If $M(x) = x$ for all $x \in [0, \infty)$, then the above classes of sequences are denoted by $\omega_0^{''}(A, p)(F), \omega^{''}(A, p)(F)$, and $\omega_\infty^{''}(A, p)(F)$, respectively.

2. If $p_{k,l} = 1$ for all $(k, l) \in N \times N$, then we denote the above classes of sequences by $\omega_0^{''}(A, M)(F), \omega^{''}(A, M)(F)$, and $\omega_\infty^{''}(A, M)(F)$, respectively.

3. If $M(x) = x$ for all $x \in [0, \infty)$, and $p_{k,l} = 1$ for all $(k, l) \in N \times N$, then we denote the above spaces by $\omega_0^{''}(A)(F), \omega^{''}(A)(F)$, and $\omega_\infty^{''}(A)(F)$, respectively.

4. If we take $A = (C, 1, 1)$,i.e., a double Cesàro matrix, we denote the above classes of sequences by $\omega_0^{''}(M, p)(F), \omega^{''}(M, p)(F)$ and $\omega_\infty^{''}(M, p)(F)$, respectively.

5. If we take $A = (C, 1, 1)$ and $p_{k,l} = 1$ for all $(k, l) \in N \times N$, then we denote the above classes of sequences by $\omega_0^{''}(M)(F)$, and $\omega_\infty^{''}(M)(F)$, respectively.

6. If we take $A = (C, 1, 1), M(x) = x$, for all $x \in [0, \infty)$ and $p_{k,l} = 1$ for all $(k, l) \in N \times N$, then we denote the above classes of sequences by $\omega_0^{''}(F), \omega^{''}(F)$, and $\omega_\infty^{''}(F)$, respectively.

7. Let us consider the following notations and definitions. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$

$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and let $\bar{h}_{r,s} = h_r h_s$, $\theta_{r,s}$ be determined by

$$I_{r,s} = \{(i,j) : k_{r-1} < i \le k_r \ \& \ l_{s-1} < j \le l_s\}.$$

If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}}, & \text{if } (k,l) \in I_{r,s}; \\ 0, & \text{otherwise.} \end{cases}$$

We write (see [16])

$$\omega_0^{''}(\theta, M, p)(F)$$
$$= \left\{ X \in s^{''} : P - \lim_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(\theta, M, p)(F)$$
$$= \left\{ X \in s^{''} : P - \lim_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \right\},$$

and

$$\omega_\infty^{''}(\theta, M, p)(F)$$
$$= \left\{ X \in s^{''} : \sup_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}.$$

As a final illustration let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in L_j = [j - \lambda_j + 1, j] \\ 0, & \text{otherwise} \end{cases}$$

where we shall denote $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tends to ∞ and $\lambda_{i+1} \le \lambda_i + 1, \lambda_1 = 0$ and $\mu_{j+1} \le \mu_j + 1, \mu_1 = 0$. Then our definition reduces to the following

$$\omega_0^{''}(\bar{\lambda}, M, p)(F)$$
$$= \left\{ X \in s^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(\bar{\lambda}, M, p)(F)$$
$$= \left\{ X \in s^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\omega_{\infty}^{''}(\bar{\lambda}, M, p)(F)$$
$$= \left\{ X \in s^{''} : \sup_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\},$$

which were defined in [14]. Let $p = (p_{k,l})$ be a sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $C = \max\{1; 2^{H-1}\}$. Now we give the following theorem.

Theorem 2. *If M is an Orlicz function, then $\omega_0^{''}(A, M, p)(F) \subset \omega^{''}(A, M, p)(F)$.*

Proof. The proof is easy and therefore omitted. □

Theorem 3.

1. *If $0 < \inf p_{k,l} \leq p_{k,l} < 1$, then*
$$\omega^{''}(A, M, p)(F) \subset \omega^{''}(A, M)(F)$$
2. *If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then*
$$\omega^{''}(A, M)(F) \subset \omega^{''}(A, M, p)(F)$$

Proof. (1) Let $X \in \omega^{''}(A, M, p)(F)$; since $0 < \inf p_{k,l} \leq 1$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}}$$

and hence $X \in \omega^{''}(A, M, p)(F)$.

(2) Let $p_{k,l} \geq 1$ for each (k,l) and $\sup_{k,l} p_{k,l} < \infty$. Let $X \in \omega^{''}(A, M)(F)$. Then for each $0 < \epsilon < 1$ there exists a positive integer n_0 such that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \leq \epsilon < 1$$

for all $m, n \geq n_0$. This implies that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right).$$

Thus $X \in \omega^{''}(A, M, p)(F)$. □

The following corollary follows immediately from the above theorem.

Corollary 1. *Let $A = (C, 1, 1)$ be a double Cesàro matrix and let M be an Orlicz function.*

1. *If $0 < \inf p_{k,l} \leq p_{k,l} < 1$, then $\omega^{''}(M, p)(F) \subset \omega^{''}(M)(F)$.*
2. *If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then $\omega^{''}(M)(F) \subset \omega^{''}(M, p)(F)$.*

3. A-statistical convergence

Natural density was generalized by Freedman and Sember in [3] by replacing C_1 with a nonnegative regular summability matrix $A = (a_{n,k})$. Thus, if K is a subset of N, then the A-density of K is given by $\delta_A(K) = \lim_n \sum_{k \in K} a_{n,k}$ if the limit exists.

A sequence of real number $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set. In this case, we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$. Statistical convergence is a generalization of the usual notion of convergence for real valued sequences that parallels the usual theory of convergence. The idea of statistical convergence was first introduced by Fast [2]. Today, statistical convergence has become one of the most active area of research in the field of summability theory.

Before we present a new definition and the main theorems, we shall state a few known results. The following definition was presented by Nuray and Savaş [8] for a single sequence of fuzzy numbers. A sequence X is said to be statistically convergent to X_0 or st_1-convergent to X_0, if for every $\epsilon > 0$

$$\lim_n \frac{1}{n} \Big| \{k < n : d(X_k, X_0) \geq \epsilon\} \Big| = 0,$$

where the vertical bars indicate the numbers of elements in the enclosed set. In this case, we write $s - \lim X = X_0$ or $X_k \to X_0(st_1)$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m, n)$ denote the numbers of (k, l) in K such that $k \leq m$ and $l \leq n$. The two-dimensional analogues of natural density can be defined as follows: The lower asymptotic density of a set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_*^2(K) = \liminf_{m,n} \frac{K(m, n)}{mn}.$$

In case the double sequence $\frac{K(m,n)}{mn}$ has a limit in the Pringsheim sense, we say that K has a double natural density defined as

$$P - \lim_{m,n} \frac{K(m, n)}{mn} = \delta^2(K).$$

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. Then the A-density of K is given by

$$\delta_A^2(K) = P - \lim_{m,n} \sum_{(k,l) \in K} a_{m,n,k,l}$$

provided the limit exists.

Savas and Mursaleen [17] have recently introduced statistical convergence for a fuzzy real-valued double sequence as follows:

Definition 6. *A fuzzy real-valued double sequence $X = (X_{kl})$ is said to be statistically convergent to X_0 provided for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{nm} |\{(k,l); k \leq m \text{ and } l \leq n : d(X_{kl}, X_0) \geq \varepsilon\}| = 0.$$

In this case, we write $st_2 - \lim_{k,l} X_{k,l} = X_0$ and denote the set of all double statistically convergent fuzzy real-valued double sequences by $st^2(F)$.
We now have

Definition 7. *A fuzzy real-valued double sequence X is said to be A-statistically convergent to L if for every positive ϵ*

$$\delta_A^2(\{(k,l) : d(X_{k,l}, X_0) \geq \epsilon\}) = 0.$$

In this case, we write $X_{k,l} \to X_0(st^2(A)(F))$ or $st^2(A)(F) - \lim X = X_0$ and

$$st^2(A)(F) = \{X : \exists X_0 \in R(I), st^2(A)(F) - \lim X = X_0\}.$$

If $A = (C,1,1)$ then $(st^2(A)(F)$ reduces to $(st^2)(F)$, which is defined above.

If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}}, & \text{if } k \in I_r = (k_{r-1}, k_r] \text{ and } l \in L_s = (l_{s-1}, l_s] \\ 0, & \text{otherwise ,} \end{cases}$$

where the double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ and $\bar{h}_{r,s}$ are defined above. Then our definition reduces to the following: A fuzzy real-valued double sequence X is said to be lacunary θ-statistically convergent to X_0, if for every positive $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{\bar{h}_{r,s}} |\{(k,l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\}| = 0,$$

which was defined in [13].

Finally, if we write

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in L_j = [j - \lambda_j + 1, j]; \\ 0, & \text{otherwise .} \end{cases}$$

Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be defined as above. A fuzzy real-valued double sequence X is said to be $\bar{\lambda}$-statistically convergent to X_0, if for every positive $\epsilon > 0$

$$P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} |\{k \in I_i \text{ and } l \in L_j : d(X_{k,l}, X_0) \geq \varepsilon\}| = 0,$$

which was defined in [14].

Theorem 4. *If M is an Orlicz function and $\sup_{k,l} p_{k,l} = H$, then $\omega''(A, M, p)(F) \subset st^2(A)(F)$.*

Proof. If $X \in \omega^{''}(A, M, p)(F)$, then there exists $\rho > 0$ such that

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Then, we obtain for a given $\varepsilon > 0$ and $\varepsilon_1 = \frac{\epsilon}{\rho}$ that

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&= \sum_{k,l=0,0; d(X_{k,l},X_0) \geq \varepsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&\quad + \sum_{k,l=0,0; d(X_{k,l},X_0) < \varepsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&\geq \sum_{k,l=0,0; d(X_{k,l},X_0) \geq \varepsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&\geq \sum_{k,l=0,0; d(X_{k,l},X_0) \geq \varepsilon}^{\infty,\infty} a_{m,n,k,l} \min\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\} \\
&\geq \left(\min \left\{ [M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \right\} \right) \sum_{k,l=0,0; d(X_{k,l},X_0) \geq \varepsilon}^{\infty,\infty} a_{m,n,k,l} \\
&\geq \min\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\} \delta_A^2 \left(\{(k,l) : d(X_{k,l}, X_0) \geq \varepsilon\} \right).
\end{aligned}$$

Hence $X \in st^2(A)(F)$. □

Theorem 5. *Let M be an Orlicz function and $X = (X_{kl})$ a fuzzy real-valued bounded sequence and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, then $st^2(A)(F) \subset \omega^{''}(A, M, p)(F)$.*

Proof. Suppose that $X \in l_\infty^{''}(F)$ and $X_{k,l} \to X_0(st^2(A))(F)$. Since $X \in l_\infty^{''}(F)$, there is a constant $K > 0$ such that $d(X_{k,l}, \bar{0}) < K$ for all k, l . Given $\varepsilon > 0$ we have

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&= \sum_{k,l=0,0; d(X_{k,l},X_0) \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
&\quad + \sum_{k,l=0,0; d(X_{k,l},X_0) < \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}}
\end{aligned}$$

$$\leq \sum_{k,l=0,0;d(X_{k,l},X_0)\geq\varepsilon}^{\infty,\infty} a_{m,n,k,l} \max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h, \left[M\left(\frac{K}{\rho}\right)\right]^H\right\}$$

$$+ \sum_{k,l=0,0;d(X_{k,l},X_0)<\varepsilon}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_{k,l}}$$

$$\leq \delta_A^2\left(\{(k,l) : d(X_{k,l}, X_0) \geq \varepsilon\}\right) \max\left\{[M(T)]^h, [M(T)]^H\right\}$$

$$+ \max\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}, \frac{K}{\rho} = T.$$

Thus $X \in \omega''(A, M, p)(F)$. □

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