Neke konačne sume

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Sažetak. *Razmatraju se konačne sume, koje su ilustrirane na nizu zanimljivih zadataka prilagođenim učenicima srednjih škola.*

Ključne riječi: *konačne sume*

Some finite sums

Abstract. *Finite sums are considered. These applications are illustrated on a number of interesting tasks adapted for high school students.*

Key words: *finite sums*

Zadanoj n-torki brojeva a_1, a_2, …, a_n pridružimo sumu

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Primjerice:

$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k,$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2,$$

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = \sum_{k=1}^{n} (2k-1)^3,$$

$$c + c + \cdots + c = nc = \sum_{k=1}^{n} c,$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \sum_{k=1}^{5} \frac{1}{k(k+1)}.$$

Lako se dokazuje da vrijede sljedeće formule:

1. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$,

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2. $\sum_{k=1}^{n}(c\cdot a_k)=c\cdot\sum_{k=1}^{n}a_k$,
3. $\sum_{k=1}^{n}(a_k-a_{k-1})=a_n-a_0$.

Sa $S_k(n)$ označimo sumu k-tih potencija prvih n prirodnih brojeva, tj.

$$S_k(n)=1^k+2^k+\cdots+n^k,\qquad k\in\mathbb{N}$$

i odredimo formule pomoću kojih ćemo izračunavati te sume.

Primjer 1. *Odredimo sumu prvih n prirodnih brojeva.*

Polazimo od jednakosti $(a+1)^2=a^2+2a+1$ u koju umjesto a redom uvrštavamo brojeve $0,\,1,\,2,\,\ldots,\,n-1,\,n$:

$$\begin{aligned}(0+1)^2 &= 0^2+2\cdot 0+1,\\ (1+1)^2 &= 1^2+2\cdot 1+1,\\ (2+1)^2 &= 2^2+2\cdot 2+1,\\ &\vdots\\ (n-1+1)^2 &= (n-1)^2+2(n-1)+1,\\ (n+1)^2 &= n^2+2n+1.\end{aligned}$$

Zbrajanjem ovih jednakosti dobivamo

$$\begin{aligned}1^2+2^2+3^2+\cdots+n^2+(n+1)^2 &= 1^2+2^2+\cdots+(n-1)^2+n^2\\ &\quad+2\big(1+2+\cdots+(n-1)+n\big)\\ &\quad+n+1,\end{aligned}$$

odakle slijedi

$$(n+1)^2=2S_1(n)+n+1,$$

tj.

$$S_1(n)=\frac{n(n+1)}{2}.$$

Primjer 2. *Odredimo sumu kvadrata prvih n prirodnih brojeva.*

U jednakost $(a+1)^3=a^3+3a^2+3a+1$ umjesto a redom uvrštavamo brojeve $0,\,1,\,2,\,\ldots,\,n-1,\,n$. Dobivamo:

$$\begin{aligned}1^3 &= 0^3+3\cdot 0^2+3\cdot 0+1,\\ 2^3 &= 1^3+3\cdot 1^2+3\cdot 1+1,\\ 3^3 &= 2^3+3\cdot 2^2+3\cdot 2+1,\\ &\vdots\\ n^3 &= (n-1)^3+3(n-1)^2+3(n-1)+1,\\ (n+1)^3 &= n^3+3n^2+3n+1.\end{aligned}$$

Zbrajanjem dobivamo

$$(n+1)^3 = 3S_2(n) + 3S_1(n) + (n+1).$$

Odatle imamo, redom,

$$(n+1)^3 = 3S_2(n) + 3\cdot\frac{n(n+1)}{2} + (n+1),$$

$$3S_2(n) = (n+1)\big((n+1)^2 - \frac{3}{2}n - 1\big),$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Primjer 3. *Odredimo sumu kubova prvih n prirodnih brojeva.*

U jednakost $(a+1)^4 = a^4 + 4a^3 + 6a^2 + 4a + 1$ umjesto a redom uvrštavamo brojeve $0, 1, 2, \ldots, n-1, n$. Zbrajanjem dobivamo

$$(n+1)^4 = 4S_3(n) + 6S_2(n) + 4S_1(n) + (n+1),$$

odakle je

$$S_3(n) = \left(\frac{n(n+1)}{2}\right)^2.$$

Analogno bismo dobili formulu za sumu četvrtih, petih, … potencija prvih n prirodnih brojeva.

Primjer 4. *Odredimo rekurzivnu formulu za sumu k-tih potencija prvih n prirodnih brojeva, tj.*

$$S_k(n) = 1^k + 2^k + \cdots + n^k, \qquad k \in \mathbb{N}.$$

Prema binomnom poučku je

$$\begin{aligned}(a+1)^{k+1} &= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k + \binom{k+1}{2}a^{k-1} + \cdots\\ &\quad + \binom{k+1}{k-1}a^2 + \binom{k+1}{k}a + \binom{k+1}{k+1}\cdot 1\\ &= a^{k+1} + \binom{k+1}{1}a^k + \binom{k+1}{2}a^{k-1} + \cdots\\ &\quad + \binom{k+1}{2}a^2 + \binom{k+1}{1}a + 1.\end{aligned}$$

Uvrštavajući umjesto a redom $0, 1, 2, \ldots, n-2, n-1, n$ dobivamo

$$1^{k+1} = 0 + 0 + 0 + \cdots + 0 + 0 + 1,$$

$$2^{k+1} = 1^{k+1} + \binom{k+1}{1}\cdot 1^k + \binom{k+1}{2}\cdot 1^{k-1} + \cdots + \binom{k+1}{2}\cdot 1^2 + \binom{k+1}{1}\cdot 1 + 1,$$

$$3^{k+1} = 2^{k+1} + \binom{k+1}{1} \cdot 2^k + \binom{k+1}{2} \cdot 2^{k-1} + \cdots + \binom{k+1}{2} \cdot 2^2 + \binom{k+1}{1} \cdot 2 + 1,$$

$$\vdots$$

$$\begin{aligned}(n-1)^{k+1} &= (n-2)^{k+1} + \binom{k+1}{1} \cdot (n-2)^k + \binom{k+1}{2} \cdot (n-2)^{k-1} + \cdots \\ &+ \binom{k+1}{2} \cdot (n-2)^2 + \binom{k+1}{1} \cdot (n-2) + 1,\end{aligned}$$

$$\begin{aligned}n^{k+1} &= (n-1)^{k+1} + \binom{k+1}{1} \cdot (n-1)^k + \binom{k+1}{2} \cdot (n-1)^{k-1} + \cdots \\ &+ \binom{k+1}{2} \cdot (n-1)^2 + \binom{k+1}{1} \cdot (n-1) + 1,\end{aligned}$$

$$\begin{aligned}(n+1)^{k+1} &= n^{k+1} + \binom{k+1}{1} \cdot n^k + \binom{k+1}{2} \cdot n^{k-1} + \cdots \\ &+ \binom{k+1}{2} \cdot n^2 + \binom{k+1}{1} \cdot n + 1.\end{aligned}$$

Zbrajanjem dobivamo

$$\begin{aligned}(n+1)^{k+1} &= \binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \cdots \\ &+ \binom{k+1}{2} S_2(n) + \binom{k+1}{1} S_1(n) + S_0(n),\end{aligned}$$

gdje je $S_0(n) = n + 1$. Odatle lako nalazimo $S_k(n)$ kada znamo $S_1(n)$, $S_2(n)$, $\ldots$, $S_{k-1}(n)$.

Primjer 5. *Odredimo sumu četvrtih potencija prvih n prirodnih brojeva.*

Prema rekurzivnoj formuli je

$$(n+1)^5 = \binom{5}{1} S_4(n) + \binom{5}{2} S_3(n) + \binom{5}{2} S_2(n) + \binom{5}{1} S_1(n) + n + 1.$$

Dalje imamo redom

$$(n+1)^5 = 5S_4(n) + 10 \cdot \frac{n^2(n+1)^2}{4} + 10 \cdot \frac{n(n+1)(2n+1)}{6} + 5 \cdot \frac{n(n+1)}{2} + n + 1,$$

$$5S_4(n) = (n+1) \cdot \left((n+1)^4 - \frac{5n^2(n+1)}{2} - \frac{5n(2n+1)}{3} - \frac{5n}{2} - 1 \right),$$

$$5S_4(n) = (n+1) \cdot \left(n^4 + 4n^3 + 6n^2 + 4n + 1 - \frac{5n^3 + 5n^2}{2} - \frac{10n^2 + 5n}{3} - \frac{5n}{2} - 1 \right),$$

$$5S_4(n) = (n+1) \cdot \frac{6n^4 + 9n^3 + n^2 - n}{6},$$

$$S_4(n) = \frac{(n+1)n(6n^3+9n^2+n-1)}{30}.$$

Rastavimo sada polinom $6n^3+9n^2+n-1$ na proste faktore. Moguća racionalna nultočka tog polinoma je iz skupa kvocijenata djelitelja slobodnog člana -1 i vodećeg koeficijenta 6, tj. $n \in \{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}\}$. Probanjem se vidi da je $-\frac{1}{2}$ nultočka, pa dijeljenjem $6n^3+9n^2+n-1$ sa $(n+\frac{1}{2})$ dobivamo $6n^2+6n-2$, pa je $6n^3+9n^2+n-1 = (2n+1)(3n^2+3n-1)$. Stoga je

$$S_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Riješimo sada nekoliko zadataka.

U zadacima 1–14 izračunajte zadanu konačnu sumu.

Zadatak 1. $1+3+5+\cdots+(2n-1)$

Rješenje.

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= \sum_{k=1}^{n}(2k-1) \\ = \quad 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 &= 2\cdot\frac{n(n+1)}{2} - n = n^2 \end{aligned}$$

Zadatak 2. $2^2+4^2+6^2+\cdots+(2n)^2$.

Rješenje.

$$\begin{aligned} 2^2+4^2+6^2+\cdots+(2n)^2 &= \sum_{k=1}^{n}(2k)^2 = 4\sum_{k=1}^{n}k^2 \\ = \quad 4\cdot\frac{n(n+1)(2n+1)}{6} &= \frac{2n(n+1)(2n+1)}{3} \end{aligned}$$

Zadatak 3. $1^2+3^2+5^2+\cdots+(2n-1)^2$.

Rješenje.

$$\begin{aligned} 1^2+3^2+5^2+\cdots+(2n-1)^2 &= \sum_{k=1}^{n}(2k-1)^2 = \sum_{k=1}^{n}(4k^2-4k+1) \\ = \quad 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 &= 4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + n \\ = \quad \frac{4n^3-n}{3} &= \frac{n(2n-1)(2n+1)}{3} \end{aligned}$$

Zadatak 4. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$.

Rješenje.

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\ = \quad & \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3} \end{aligned}$$

Zadatak 5. $1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1)$.

Rješenje.

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1) = \sum_{k=1}^{n} k(3k-1) = \sum_{k=1}^{n} (3k^2 - k) \\ = \quad & 3\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k = 3 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = n^2(n+1) \end{aligned}$$

Zadatak 6. $\sum_{k=1}^{n} (2k)^3$.

Rješenje.

$$\sum_{k=1}^{n} (2k)^3 = 8\sum_{k=1}^{n} k^3 = 8 \cdot \left(\frac{n(n+1)}{2}\right)^2 = 2n^2(n+1)^2$$

Zadatak 7. $\sum_{k=1}^{n} (2k-1)^3$.

Rješenje.

$$\begin{aligned} & \sum_{k=1}^{n} (2k-1)^3 = \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1) \\ = \quad & 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6k - \sum_{k=1}^{n} 1 \end{aligned}$$

$$\begin{aligned} = \quad & 8 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 12 \cdot \frac{n(n+1)(2n+1)}{6} + 6 \cdot \frac{n(n+1)}{2} - n \\ = \quad & n^2(2n^2 - 1) \end{aligned}$$

Zadatak 8. $\sum_{k=1}^{n} k(k+1)(k+2)$.

Rješenje.

$$\begin{aligned}
& \sum_{k=1}^{n} k(k+1)(k+2) = \sum_{k=1}^{n}(k^3+3k^2+2k) \\
= & \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
= & \left(\frac{n(n+1)}{2}\right)^2 + 3\cdot\frac{n(n+1)(2n+1)}{6} + 2\cdot\frac{n(n+1)}{2} \\
= & \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}$$

Zadatak 9. $1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+n)$.

Rješenje. Kako je suma prvih n prirodnih brojeva jednaka $\frac{n(n+1)}{2}$, to je

$$\begin{aligned}
& 1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+n) \\
= & \sum_{k=1}^{n}\frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{2}\sum_{k=1}^{n}k \\
= & \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2} \\
= & \frac{n(n+1)(n+2)}{6}.
\end{aligned}$$

Zadatak 10. $1+(1+3)+(1+3+5)+(1+3+5+7)+\cdots+\big(1+3+5+\cdots+(2n-1)\big)$.

Rješenje. Kako je suma prvih n neparnih prirodnih brojeva jednaka n^2, to je

$$\begin{aligned}
& 1+(1+3)+(1+3+5)+(1+3+5+7)+\cdots \\
& +\big(1+3+5+\cdots+(2n-1)\big) \\
= & \sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Zadatak 11. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots+(1^2+2^2+3^2+\cdots+n^2)$.

Rješenje. Kako je suma kvadrata prvih n prirodnih brojeva jednaka $\frac{n(n+1)(2n+1)}{6}$, to je

$$\begin{aligned}
& 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots + (1^2 + 2^2 + 3^2 + \cdots + n^2) \\
= & \sum_{k=1}^{n} \frac{k(k+1)(2k+1)}{6} = \frac{1}{6}\sum_{k=1}^{n}(2k^3 + 3k^2 + k) \\
= & \frac{1}{6}\Big(2\sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\Big) \\
= & \frac{1}{6} \cdot \left(2 \cdot \frac{n^2(n+1)^2}{4} + 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) \\
= & \frac{n(n+1)^2(n+2)}{12}.
\end{aligned}$$

Zadatak 12. $\sum_{k=1}^{n} \frac{2k^2+2k+1}{k(k+1)}$.

Rješenje.

$$\begin{aligned}
& \sum_{k=1}^{n} \frac{2k^2 + 2k + 1}{k(k+1)} = \sum_{k=1}^{n}\Big(2 + \frac{1}{k(k+1)}\Big) \\
= & \sum_{k=1}^{n} 2 + \sum_{k=1}^{n} \frac{1}{k(k+1)} = 2n + \sum_{k=1}^{n}\Big(\frac{1}{k} - \frac{1}{k+1}\Big) \\
= & 2n + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\
& + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
= & 2n + \left(1 - \frac{1}{n+1}\right) = 2n + \frac{n}{n+1} = \frac{n(2n+3)}{n+1}
\end{aligned}$$

Zadatak 13. $\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)}$.

Rješenje.

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \frac{k^2}{4k^2-1} = \frac{1}{4}\sum_{k=1}^{n} \frac{4k^2}{4k^2-1} \\
= \; & \frac{1}{4}\sum_{k=1}^{n} \frac{4k^2-1+1}{4k^2-1} = \frac{1}{4}\Big(\sum_{k=1}^{n} 1 + \frac{1}{4k^2-1}\Big) \\
= \; & \frac{1}{4}\Big(n + \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}\Big) = \frac{1}{4}\Big(n + \sum_{k=1}^{n} \frac{1}{2}\Big(\frac{1}{2k-1} - \frac{1}{2k+1}\Big)\Big) \\
= \; & \frac{1}{4}n + \frac{1}{8}\sum_{k=1}^{n}\Big(\frac{1}{2k-1} - \frac{1}{2k+1}\Big) \\
= \; & \frac{1}{4}n + \frac{1}{8}\Big(\Big(\frac{1}{1} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{5}\Big) + \cdots \\
& + \Big(\frac{1}{2n-3} - \frac{1}{2n-1}\Big) + \Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big)\Big) \\
= \; & \frac{1}{4}n + \frac{1}{8}\Big(1 - \frac{1}{2n+1}\Big) = \frac{1}{4}n + \frac{1}{8}\cdot\frac{2n}{2n+1} \\
= \; & \frac{1}{4}n\cdot\Big(1 + \frac{1}{2n+1}\Big) = \frac{1}{4}n\cdot\frac{2n+2}{2n+1} = \frac{n(n+1)}{2(2n+1)}
\end{aligned}$$

Zadatak 14. $\sum_{k=1}^{n} \big(n^2 - (2k-1)n\big)$.

Rješenje.

$$\begin{aligned}
&\sum_{k=1}^{n}\big(n^2 - (2k-1)n\big) = \sum_{k=1}^{n} n^2 - \sum_{k=1}^{n}(2k-1)n \\
= \; & n^2\sum_{k=1}^{n} 1 - n\sum_{k=1}^{n}(2k-1) = n^2\cdot n - n\cdot n^2 = 0
\end{aligned}$$

Zadaci za vježbu

Izračunajte sljedeće sume:

1. $\sum_{k=1}^{n} k(2k-1)$

   Rješenje. $\frac{n(n+1)(4n-1)}{6}$

2. $\sum_{k=1}^{n} 2(3k-1)$

   Rješenje. $n(3n+1)$

3. $\sum_{k=1}^{n} n(3k-1)$

   Rješenje. $n^2(n+1)$

4. $\sum_{k=1}^{n} k(k+1)^2$

   Rješenje. $\frac{n(n+1)(n+2)(3n+5)}{12}$

5. $\sum_{k=1}^{n}(k+1)k^2$

   Rješenje. $\frac{n(n+1)(n+2)(3n+1)}{12}$

6. $\sum_{k=1}^{n} \frac{k(3k+1)}{2}$

   Rješenje. $\frac{n(n+1)^2}{2}$

7. $\sum_{k=1}^{n}(2k-1)(k+2)$

   Rješenje. $\frac{n(4n^2+15n-1)}{6}$

8. $\sum_{k=1}^{n}(k^2+2k-1)$

   Rješenje. $\frac{n(2n^2+9n+1)}{6}$

9. $\sum_{k=1}^{n}(k+1)(2k+1)$

   Rješenje. $\frac{n(4n^2+15n+17)}{6}$

10. $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\cdots+\frac{1^2+2^2+\cdots+n^2}{n}$

    Rješenje. $\frac{n(4n^2+15n+17)}{36}$

11. $\sum_{k=1}^{n}(2k-1)(2k+1)(2k+3)$

    Rješenje. $n(n+2)(2n^2+4n-1)$

12. $\sum_{k=1}^{n} \frac{k^2+k+1}{k(k+1)}$

    Rješenje. $\frac{n(n+2)}{n+1}$

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