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# RESPONSE AND DYNAMICAL STABILITY OF OSCILLATORS WITH DISCONTINUOUS OR STEEP FIRST DERIVATIVE OF RESTORING CHARACTERISTIC

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## ABSTRACT

Response and dynamical stability of oscillators with discontinuous or steep first derivative of restoring characteristic is considered in this paper. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic force excitation is analysed by the method of piecing the exact solutions (MPES) in the time domain and by the incremental harmonic balance method (IHBM) in the frequency domain. The stability of the periodic solutions obtained in the frequency domain by IHBM is estimated by the Floquet-Lyapunov theorem. Obtained frequency response characteristic is very complex and includes multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued and non-periodic solutions. Determining of frequency response characteristic in the time domain by MPES is exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions. In the frequency domain, IHBM is very efficient and very well suited for obtaining wide range frequency response characteristics, parametric studies and bifurcation analysis. On the other hand, neglecting of very small harmonic terms (which in-significantly influence the r.m.s. values of the response and are very small in comparison to other terms of the spectrum) can cause very large error in evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution. Moreover, frequency ranges are detected inside which the procedure of evaluation of eigenvalues of the monodromy matrix does not converge with increasing the number of harmonics included in the supposed approximate solution.

## KEY WORDS

dynamical stability, response characteristic, non-linear vibrations, piecewise-linear system

## CLASSIFICATION

ACM: J.2 Engineering

JEL: Z0

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## INTRODUCTION

Among the great number of various types of non-linear dynamic systems a very specific group constitutes non-linear systems described by differential equations which contain non-linear restoring characteristic with discontinuous or steep first derivative (for example systems with clearances, rolling bearings, gears, clutches, impacting oscillators, etc.). Frequency response characteristics of these systems are usually very complex and include multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued solutions, and possibility of non-periodic solutions. Both periodic and non-periodic responses can be determined in the time domain by using digital simulation. But procedures of that kind can be exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions (where many combinations of initial conditions have to be examined for obtaining all possible steady-state solutions), for lightly damped systems (since a great number of excitation periods must be simulated to obtain a steady-state response), and when the state of the system is near to bifurcation. These methods are not suitable for obtaining wide range frequency response characteristics, unstable solutions and for bifurcation analysis also. A very efficient method for solving strong non-linear differential equations in the frequency domain is the harmonic balance method (HBM) [1-6]. When the assumption of dominance of primary resonance in the response is satisfied, the HBM (single harmonic) is very accurate and numerically very efficient method for obtaining periodic response of non-linear systems with harmonic excitation. But it becomes very inaccurate if the influence of higher harmonics in the response is significant. Multi frequency harmonic balance methods (e.g., Incremental harmonic balance method or Newton-Raphson harmonic balance method [7-14]) provide the study of effects of superharmonics and subharmonics to response. These methods become exceptionally efficient in combination with path following techniques [15-17], and can be successfully applied to a wide range of non-linear problems. They are very well suited for parametric studies because a new solution can be sought by these methods, with the previous solution used as a very good approximation. Since these methods enable obtaining both dynamically stable and unstable solutions, determining of dynamical stability of these solutions should be reliable and numerically efficient. Since the estimation of dynamical stability of the steady state response by Floquet-Lyapunov theorem [18] is a sensitive procedure [19-21], the factors which can lead to incorrect prediction of the dynamical stability must be taken into consideration.

Responses determined in the time domain (MPES) and in the frequency domain (HBM and IHBM) are considered in this paper as well as problems which can occur in estimation of dynamical stability of the periodic solutions obtained in the frequency domain. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic excitation is analysed.

## MODEL OF A MECHANICAL SYSTEM WITH A CLEARANCE

Model of a simple mechanical system with clearance is shown in Figure 1. It consists of an inertia element $m$, a linear viscous damping parameter $c$, and a non-linear elastic element defined by a piecewise-linear function $g(x)$ and a coefficient $k$. When the system is excited by a periodic harmonic force $F(t)$, the motion of the system can be described by the non-linear differential equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+c\frac{\mathrm{d}x}{\mathrm{d}t}+kg\left(x\right)=F(t)=F_m+F_p\cos\left(\Omega t+\varphi_F\right)=f_0+f_C\cos\left(\Omega t\right)+f_S\sin\left(\Omega t\right), \quad (1)$$

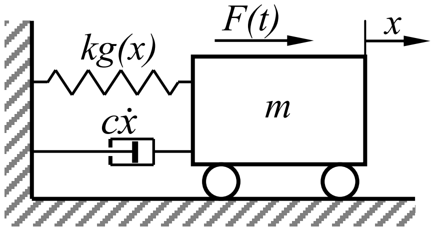



**Figure 1.** Model of vibration system.

where $f_0 = F_m$ represents mean transmitted force, $F_p = \sqrt{f_C^2 + f_S^2}$ is the amplitude of the vibratory component at frequency $\Omega$, while $f_C$ and $f_S$ are force component amplitudes of the corresponding harmonic terms and $\varphi_F$ is the excitation phase angle.

The piecewise linear function $g(x)$ and its derivative are shown in Figure 2(a) and Figure 2(b), respectivelly. Parameter b denotes one-half of the clearance space. Since the procedure of prediction of the dynamical stability is based on derivative of a non-linear function, expressions for non-linear function and its derivative are given:

$$g(x) = h^*(x - b^*), \tag{2}$$

$$\frac{\partial g(x)}{\partial x} = h^*, \tag{3}$$

where

$$h^* = \begin{Bmatrix} 1\,, & b < x \\ 0\,, & -b \le x \le b \\ 1\,, & x < -b \end{Bmatrix},\quad b^* = \begin{Bmatrix} b, & b < x \\ 0, & -b \le x \le b \\ -b, & x < -b \end{Bmatrix}. \tag{4}$$

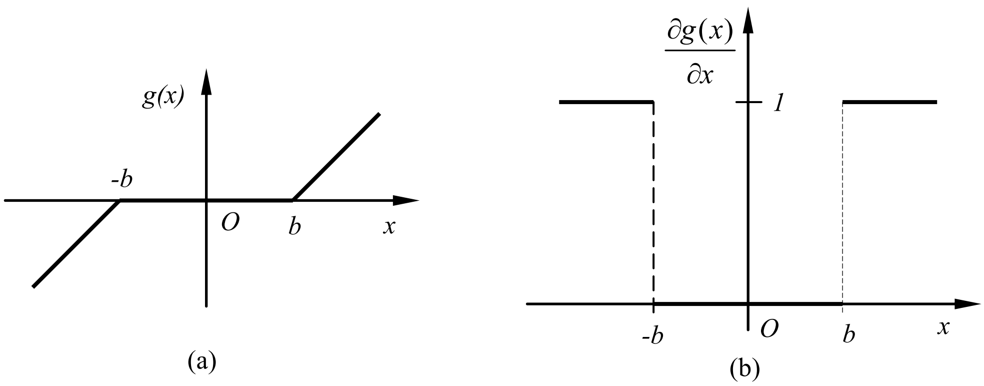



**Figure 2.** Graphs showing a) non-linear function $g(x)$ and b) its derivative $\partial g(x)/\partial x$.

## BRIEF DESCRIPTION OF THE APPLIED METHODS

### THE INCREMENTAL HARMONIC BALANCE METHOD (IHBM)

By introducing a non-dimensional time $\theta$ as a new independent variable, the differential equation (1) can be rewritten in the non-dimensional form:

$$\frac{\eta^2}{\nu^2}\frac{d^2\bar{x}}{d\theta^2} + \frac{2\zeta\eta}{\nu}\frac{d\bar{x}}{d\theta} + g(\bar{x}) = \bar{F}(\theta) = \sum_{n=0}^{M}(\bar{f}_n \cos n(\nu\theta) + \bar{g}_n \sin n(\nu\theta)), \tag{5}$$

with

$$\bar{x} = \frac{x}{l},\ \bar{b} = \frac{b}{l},\ \omega_0 = \sqrt{\frac{k}{m}},\ \zeta = \frac{c}{2m\omega_0},\ \bar{f}_C = \frac{f_C}{ml\omega_0^2},\ \bar{f}_S = \frac{f_S}{ml\omega_0^2},$$

$$\eta = \frac{\Omega}{\omega_0},\ \tau = \omega_0 \cdot t,\ \Omega t = \eta\tau = \nu\theta\ .$$

In this way, the period of the response (with $\nu$ subharmonics taken in consideration) is always 2π, making it possible (by using the IHBM) to consider any number of superharmonics and subharmonics included in the supposed approximate solution. Any characteristic dimension of the system is denoted by l here.

Supposed approximate solution is given by:

$$\bar{x} = \sum_{\mathrm{i}=0}^{\mathrm{N}} a_\mathrm{i} \cos \mathrm{i}\theta + b_\mathrm{i} \sin \mathrm{i}\theta = \boldsymbol{T}\,\boldsymbol{a},, \tag{6}$$

where

$$\boldsymbol{T} = \left[1,\ \cos\theta,\ \cos 2\theta, ..., \cos \mathrm{N}\theta,\ \sin\theta,\ \sin 2\theta, ..., \sin \mathrm{N}\theta\right],$$

$$\boldsymbol{a} = \left[a_0, a_1, ..., a_N, b_1, b_2, ..., b_N\right]^T.$$

The equation $N = \nu M$ represents the number of all harmonics included in the supposed solution, $\nu$ is the number of subharmonics and M is the number of superharmonics. By applying this method, which consists of two basic steps: incrementation and Galerkin's procedure, the non-linear differential equation (5) is transformed into the system of $2N + 1$ linearized incremental algebraic equations:

$$\mathrm{K}^{\mathrm{j}}\,\Delta\boldsymbol{a}^{\mathrm{j}+1} = \boldsymbol{r}^{\mathrm{j}}, \tag{7}$$

$$\boldsymbol{a}^{\mathrm{j}+1} = \boldsymbol{a}^{\mathrm{j}} + \Delta\boldsymbol{a}^{\mathrm{j}+1}, \tag{8}$$

with Fourier coefficients ($a_0$, $a_\mathrm{i}$, $b_\mathrm{i}$, $i = 1, \dots,$ N) as unknowns. In equations (7) and (8), the superscript j denotes the number of iterations. In each incremental step, only linear (i.e., linearized) algebraic equations have to be formed and solved. A solution is obtained from the iteration process when the corrective vector norm $\|r\|$ is smaller than a certain (arbitrary) convergence criterion. The comprehensive description of the method, its application to piecewise-linear systems and the way of determining elements of Jacobian matrix K and the corrector *r* in explicit form is given by Wong et al. in [10]. Generally, accuracy of the approximate solution obtained by using IHBM depends on the number of harmonics included in the solution, accuracy of procedures used for determining elements of K and *r*, and a value of convergence criterion. Since the IHBM described by Wong et al. ([10]) is used in this work, accuracy of the procedure of determining elements of K and *r* depends only on the precision of numerical determination of times $\theta_\mathrm{i}$ in which the system changes stage stiffness region (Fig. 3). Regarding the parameter *b*, the three stages in the problem are defined in accordance with (4).

## THE METHOD OF PIECING THE EXACT SOLUTIONS (MPES)

By introducing the non-dimensional time $\tau$ as an independent variable (what is convenient when one determines response in the time domain), the differential equation (1) can be rewritten in the non-dimensional form:

$$\frac{\mathrm{d}^2\,\bar{x}}{\mathrm{d}\,\tau^2} + 2\zeta\frac{\mathrm{d}\,\bar{x}}{\mathrm{d}\,\tau} + g(\bar{x}) = \bar{F}_m + \bar{F}_p \cos\left(\eta\,\tau + \varphi_F\right), \tag{9}$$

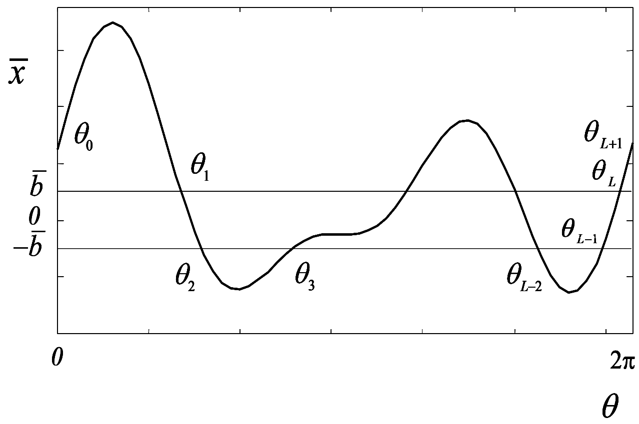



**Figure 3.** Solutions to the equations $\bar{x} = \bar{b}$ and $\bar{x} = -\bar{b}$ .

where

$$\bar{F}_m = \frac{F_m}{ml\omega_0^2}, \quad \bar{F}_p = \frac{F_p}{ml\omega_0^2} .$$

Force-displacement relationship $g(\bar{x})$, shown in Fig. 2(a), is piecewise-linear. Local solutions of differential equation (9) are known explicitly inside each of the stage stiffness, and can be repeatedly matched at $\bar{x} = \bar{b}$ and $\bar{x} = -\bar{b}$ , to obtain a global solution of (9). Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in a closed form. But, the matching of local solutions can be numerically done very easily. Only approximation done by the applying of this procedure is in the numerical determination of times in which the system changes stage stiffness region ($\bar{x} = \bar{b}$ , $\bar{x} = -\bar{b}$ ). Effective amplitudes $\bar{x}_p$ of the steady state time domain responses are calculated by using:

$$\bar{x}_m = \frac{1}{T}\int_{-T/2}^{T/2} \bar{x}(\tau)\, \mathrm{d}\tau \ , \quad \bar{x}_p = \sqrt{2}\,\sqrt{\frac{1}{T}\int_{-T/2}^{T/2} \left(\bar{x}(\tau) - \bar{x}_m\right)^2 \, \mathrm{d}\tau} \ , \tag{10}$$

where $T$ denotes the period of the response. The solutions of equation (9) inside each of the stage stiffness region are:

1. for $-\bar{b} \le \bar{x} \le \bar{b}$

$$\bar{x} = -\frac{C_1}{2\zeta}\mathrm{e}^{-2\zeta(\tau-\tau_0)} + \frac{\bar{F}_m}{2\zeta}\left(\tau - \tau_0\right) + \frac{Q_{1a}}{\eta}\sin\left(\eta\tau + \varphi_F - \varphi_R\right) + C_2 , \tag{11a}$$

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau} = C_1\mathrm{e}^{-2\zeta(\tau-\tau_0)} + \frac{\bar{F}_m}{2\zeta} + Q_{1a}\cos\left(\eta\tau + \varphi_F - \varphi_R\right) , \tag{11b}$$

where

$$Q_{1a} = \frac{\bar{F}_p}{\sqrt{\eta^2 + 4\zeta^2}} , \quad \varphi_R = \tan^{-1}\left(\frac{\eta}{2\zeta}\right) ,$$

$$C_1 = \left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 - \frac{\bar{F}_m}{2\zeta} - Q_{1a}\cos\left(\eta\tau_0 + \varphi_F - \varphi_R\right) ,$$

$$C_2 = \bar{x}_0 + \frac{C_1}{2\zeta} - \frac{Q_{1a}}{\eta}\sin\left(\eta\tau_0 + \varphi_F - \varphi_R\right) .$$

in which $\bar{x}_0$ and $(\mathrm{d}\bar{x}/\mathrm{d}\tau)_0$ are non-dimensional displacement and velocity at the initial time $\tau = \tau_0$, i.e. at time in which the motion is determined by the given equation.

2. for $\bar{x} \geq \bar{b}$

$$\bar{x} = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[A_2 \cos\sqrt{1-\zeta^2}(\tau-\tau_0) + B_2 \sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] + \bar{F}_m + \\ +(1-\alpha)\bar{b} + Q_2 \cos(\eta\tau + \varphi_F - \varphi_R), \tag{12a}$$

$$\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right) = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[\left(B_2\sqrt{1-\zeta^2} - \zeta A_2\right)\cos\sqrt{1-\zeta^2}(\tau-\tau_0) - \left(A_2\sqrt{1-\zeta^2} + \zeta B_2\right)\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] - \\ -\eta Q_2 \sin(\eta\tau + \varphi_F - \varphi_R), \tag{12b}$$

where

$$Q_2 = \frac{\bar{F}_p}{\sqrt{\left(1-\eta^2\right)^2 + (2\zeta\eta)^2}}, \quad \varphi_R = \tan^{-1}\left(\frac{2\zeta\eta}{1-\eta^2}\right),$$

$$A_2 = \bar{x}_0 - \bar{F}_m - (1-\alpha)\bar{b} - Q_2 \cos(\eta\tau_0 + \varphi_F - \varphi_R),$$

$$B_2 = \frac{1}{\sqrt{1-\zeta^2}}\left[\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 + \zeta A_2 + \eta Q_2 \sin(\eta\tau_0 + \varphi_F - \varphi_R)\right].$$

3. for $\bar{x} \leq -\bar{b}$

$$\bar{x} = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[A_3 \cos\sqrt{1-\zeta^2}(\tau-\tau_0) + B_3 \sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] + \bar{F}_m - \\ -(1-\alpha)\bar{b} + Q_3 \cos(\eta\tau + \varphi_F - \varphi_R), \tag{13a}$$

$$\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right) = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[\left(B_3\sqrt{1-\zeta^2} - \zeta A_3\right)\cos\sqrt{1-\zeta^2}(\tau-\tau_0) - \left(A_3\sqrt{1-\zeta^2} + \zeta B_3\right)\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] - \\ -\eta Q_3 \sin(\eta\tau + \varphi_F - \varphi_R), \tag{13b}$$

where

$$Q_{3j} = Q_{2j} = \frac{\bar{F}_p}{\sqrt{\left(1-\eta^2\right)^2 + (2\zeta\eta)^2}}, \quad \varphi_R = \tan^{-1}\left(\frac{2\zeta\eta}{1-\eta^2}\right),$$

$$A_3 = \bar{x}_0 - \bar{F}_m + (1-\alpha)\bar{b} - Q_3 \cos(\eta\tau_0 + \varphi_F - \varphi_R),$$

$$B_3 = \frac{1}{\sqrt{1-\zeta^2}}\left[\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 + \zeta A_3 + \eta Q_3 \sin(\eta\tau_0 + \varphi_F - \varphi_R)\right].$$

## THE STABILITY OF THE STEADY STATE SOLUTION

When the periodic solution is obtained, the stability of the given solution can be determined by examining the perturbed solution $\bar{x}^*$:

$$\bar{x}^* = \bar{x} + \Delta\bar{x}^*, \tag{14}$$

where $\Delta\bar{x}^*$ is a small perturbation of a periodic solution $\bar{x}$. By substitution of equation (14) into equation (5), and after expanding the non-linear function $g(\bar{x})$ in Taylor's series about the periodic solution with neglecting non-linear incremental terms, one obtains a linear homogeneous differential equation with time changing periodic coefficients $\partial g(\bar{x})/\partial\bar{x}$:

$$\frac{\eta^2}{\nu^2}\frac{\mathrm{d}^2\Delta\bar{x}^*}{\mathrm{d}\theta^2}+\frac{2\zeta\eta}{\nu}\frac{\mathrm{d}\Delta\bar{x}^*}{\mathrm{d}\theta}+\frac{\partial g(\bar{x})}{\partial\bar{x}}\Delta\bar{x}^*=0. \tag{15}$$

When the steady state solution $\bar{x}(\theta)$ is determined, the values of $\partial g(\bar{x})/\partial\bar{x}$ are known inside the period of the response. A very efficient and very often used method for determining the stability of the periodic solution is based on the Floquet-Lyapunov theorem [18, 22]. For that purpose equation (15) can be rewritten in the state variable form as:

$$\frac{\mathrm{d}\bar{\boldsymbol{X}}^*}{\mathrm{d}\theta}=\mathrm{A}(\theta)\,\bar{\boldsymbol{X}}^*, \tag{16}$$

where

$$\bar{\boldsymbol{X}}^*=\begin{Bmatrix}\Delta\bar{\boldsymbol{x}}^*\\ \dfrac{\mathrm{d}\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta}\end{Bmatrix},\quad \frac{\mathrm{d}\bar{\boldsymbol{X}}^*}{\mathrm{d}\theta}=\begin{Bmatrix}\dfrac{\mathrm{d}\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta}\\ \dfrac{\mathrm{d}^2\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta^2}\end{Bmatrix},\quad \mathrm{A}(\theta)=\begin{bmatrix}0 & 1\\ -\dfrac{\nu^2}{\eta^2}\left(\dfrac{\partial\boldsymbol{g}(\bar{\boldsymbol{x}})}{\partial\bar{\boldsymbol{x}}}\right) & -\dfrac{2\nu\zeta}{\eta}\end{bmatrix}. \tag{17}$$

Since the matrix $\mathrm{A}(\theta)$ is a periodic function of $\theta$ with period $2\pi$, the stability criteria are related to the eigenvalues of the monodromy matrix, which is defined as the state transition matrix at the end of one period. According to Floquet-Lyapunov theorem, the solution is stable if all the moduli of the eigenvalues of the monodromy matrix are less than unity. Otherwise the solution is unstable. Bifurcation occurs when one of the moduli of the eigenvalues of the monodromy matrix reaches unity. Generally, it is not possible to derive an analytic expression for the transition matrix. But, if the non-linear force-displacement relationship is piecewise-linear, its derivative $\partial g(x)/\partial x = h^*$ is, according to (4), constant inside each of the intervals $[\theta_i, \theta_{i+1}]$. Figure 3 shows a period of the response where $\theta_0 = 0$ and $\theta_{L+1} = 2\pi$. There are $L$ times denoted as $\theta_1, \theta_2, \ldots, \theta_L$, in which the system undergoes a stiffness change. Consequently, $\mathrm{A}(\theta_i, \theta_{i+1})$ is also a constant matrix inside that interval. According to [23], for the constant $\mathrm{A}(\theta_i, \theta_{i+1})$ (inside the interval $[\theta_i, \theta_{i+1}]$), transition matrix $\Phi(\theta_{i+1}, \theta_i)$ can be expressed as:

$$\Phi(\theta_{i+1},\theta_i)=\mathrm{e}^{\mathrm{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}; \tag{18}$$

and for the whole interval $[0, 2\pi]$ according to [10] one obtains:

$$\left[\Phi(2\pi,0)\right]=\prod_{i=0}^{L}\mathrm{e}^{\mathrm{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}. \tag{19}$$

Beside the precision of numerical determination of times θi in which the system changes stage stiffness region ($\bar{x}=\bar{b}$, $\bar{x}=-\bar{b}$), the only approximation occurring in this procedure is the accuracy of computation of the matrix exponential $\exp[\mathrm{A}(\theta_i, \theta_{i+1})\cdot(\theta_{i+1}-\theta_i)]$ and the product of matrix exponentials $\prod_{i=0}^{L}\mathrm{e}^{\mathrm{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}$.

## NUMERICAL EXAMPLES

Figures 4 and 5 show effective amplitude-frequency plots $\bar{x}_p=\bar{x}_p(\eta)$ obtained by MPES (both periodic and non-periodic solutions) for the parameter values: $\bar{b}=1$, $\zeta = 0{,}03$,

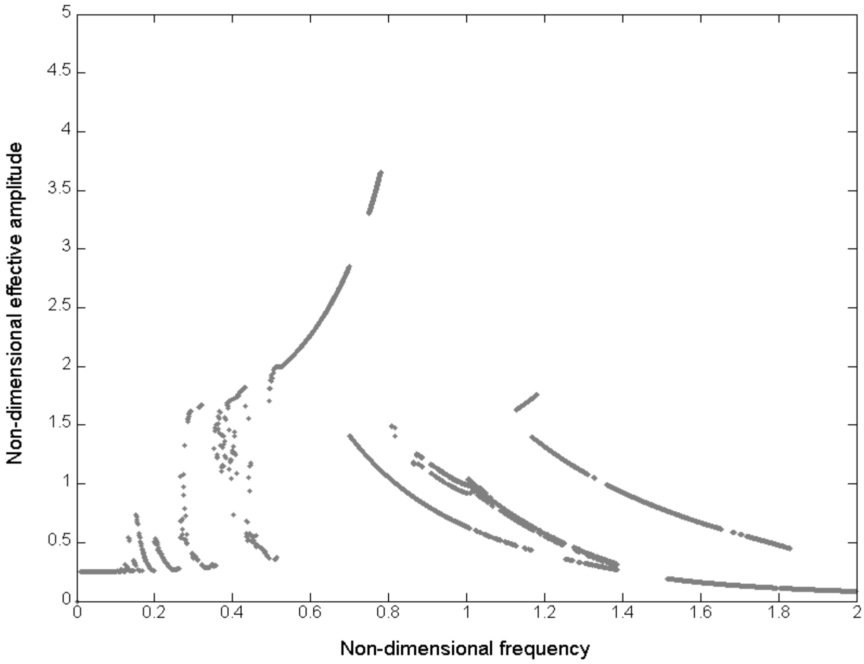



**Figure 4.** Effective amplitude-frequency plot $\bar{x}_p = \bar{x}_p(\eta)$ obtained by MPES: $\bar{x}_0 = 0$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 0$.

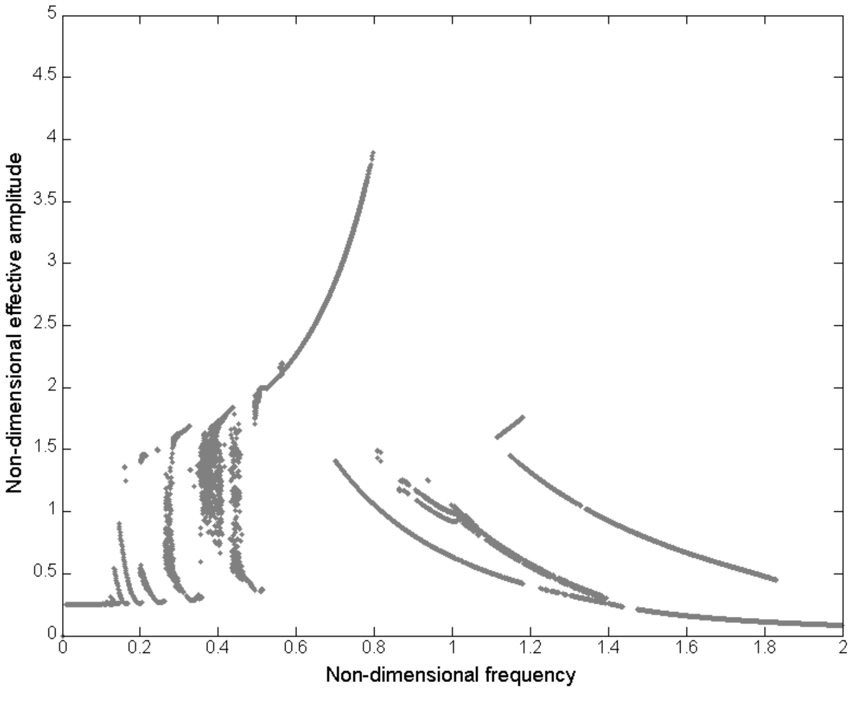



**Figure 5.** Effective amplitude-frequency plot $\bar{x}_p = \bar{x}_p(\eta)$ obtained by MPES for $\bar{x}_0 = 0$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 0$; $\bar{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 1$; $\bar{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$ and $\bar{x}_0 = -1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$.

$\bar{f}_0 = \bar{F}_m = 0.25$, $\bar{f}_C = \bar{F}_p = 0.25$, $\bar{f}_S = 0$ $(\varphi_F = 0)$. Figure 4 shows 1990 effective amplitudes $\bar{x}_p$ of the time domain responses obtained at 1990 non-dimensional frequencies $\eta$ for initial conditions: $\bar{x}_0 = 0$ and $(\mathrm{d}\bar{x}/\mathrm{d}\tau)_0 = 0$. Figure 5 shows 7960 effective amplitudes $\bar{x}_p$ obtained at the same 1990 non-dimensional frequencies $\eta$, for four different initial conditions: $\bar{x}_0 = 0$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 0$; $\bar{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 1$; $\bar{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$ and $\bar{x}_0 = -1$, $\left(\dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$. Figure 6 shows comparison of results obtained by MPES and those obtained by IHBM in the case when supposed approximate solution includes only a constant term and the first harmonic (single harmonic balance method). As one can see, a good agreement of the results obtained by these two methods is achieved, but only when the assumption of dominance of primary resonance in the response is satisfied.

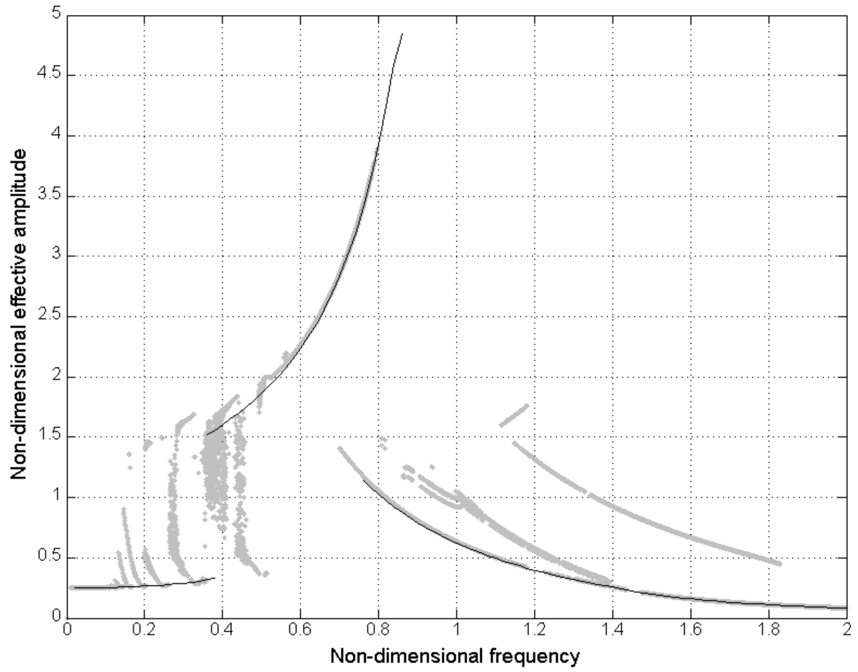



**Figure 6.** Comparison of the results obtained by MPES (dots) and by single harmonic balance method (line).

In Figure 7 the numerical results obtained by IHBM are compared with those obtained by MPES. Figure 7 shows excellent agreement between the results obtained by these methods. Non-periodic responses obtained by MPES are not found by the incremental harmonic balance method, because this method is limited only to consideration of periodic vibrations. Also, frequency response characteristics obtained by MPES are incomplete, because the results of MPES depend on given initial conditions, making it difficult to find all possible solutions.

As it is shown in [21], the accuracy of determining the eigenvalues of the monodromy matrix depend significantly on the number of harmonics included in the supposed approximate solution. Consequently, neglecting of very small harmonic terms of the actual time domain response can cause a very large error in evaluation of the eigenvalues of the monodromy matrix and can lead to incorrect prediction of the dynamical stability of the solution (Fig. 8). Figure 8(a) shows the relative differences of the effective amplitudes $\bar{x}_p$ (the a branch of the amplitude-frequency plot from Figure 7) obtained with N = 7, 12, 15, 18 and 30 harmonics with respect to the effective amplitudes $\bar{x}_p$ obtained with N = 100 harmonics

$$\left(\bar{x}_p\right)_{\mathrm{N\,diff}} = \frac{\left(\bar{x}_p\right)_{\mathrm{N}} - \left(\bar{x}_p\right)_{100}}{\left(\bar{x}_p\right)_{100}}, \qquad \mathrm{N} = 7,\ 12,\ 15,\ 18,\ 30\,. \tag{20}$$

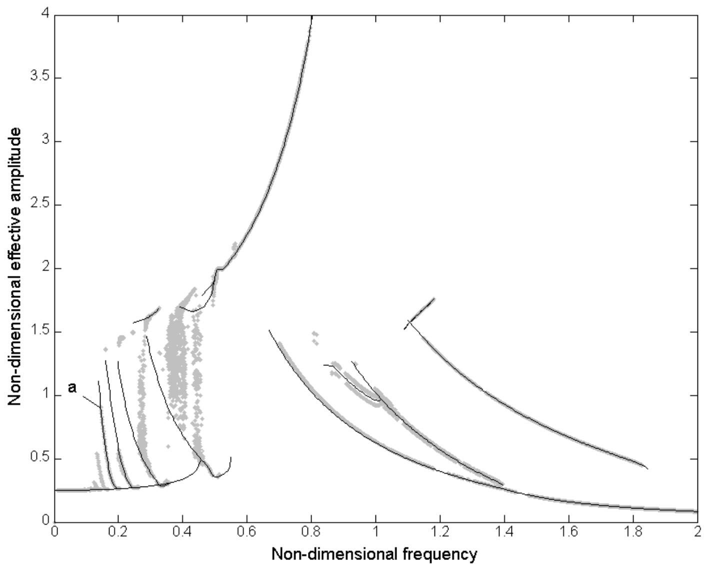



**Figure 7.** Comparison of the results obtained by MPES (dots) and by IHBM (line).

Figure 8(b) shows a corresponding plot of maximum modulus of the eigenvalues of the monodromy matrix $|\lambda_{\max}| = |\lambda_{\max}|(\eta, \mathrm{N})$ . A very specific situation occurs at $\eta$ = 0,176. In this case the value of $|\lambda_{\max}|$ is more precisely determined for N = 7 than for N = 12, N = 15 and N = 18, what can lead to incorrect estimation of dynamical stability of the response and can make bifurcation analysis difficult. The spectrum of the corresponding time domain response is shown in Fig. 9. In this example amplitudes of the harmonics for $\mathrm{N} > 8$ are exceptionally small in comparison to other terms of the spectrum and in-significantly influence effective amplitude $\bar{x}_p$ (amplitudes of the higher harmonics (N > 8) are less than 0.7 % of the amplitude of the dominant harmonic). Absence of convergence can be explained by essential difference between IHBM (the method used for obtaining approximate steady state solutions) and the procedure of evaluating the monodromy matrix (used for estimation of dynamical stability of the steady state solution by Floquet-Lyapunov theorem). Since the IHBM is based on the Galerkin's procedure, Fourier's coefficients of the supposed approximate solution ($a_0$, $a_\mathrm{i}$, $b_\mathrm{i}$, $\mathrm{i} = 1,\ldots,\mathrm{N}$) are determined in the way that differential equation (5) is satisfied in average, but not in every point of the response $\bar{x} = \bar{x}(\theta)$. On the other hand, stability estimation based on evaluation of the eigenvalues of the monodromy matrix depends only on the position of points $\theta_1, \theta_2, \ldots, \theta_\mathrm{L}$ in which the system undergoes a stiffness change (Fig. 3) and estimation of stability is influenced only by the differences between the approximate and exact positions of the points $\theta_1, \theta_2, \ldots, \theta_\mathrm{L}$. Increasing of number of harmonics N decreases average difference between the approximate and the exact solution and in this way increases the probability of more accurate determination of points $\theta_1, \theta_2, \ldots, \theta_L$.

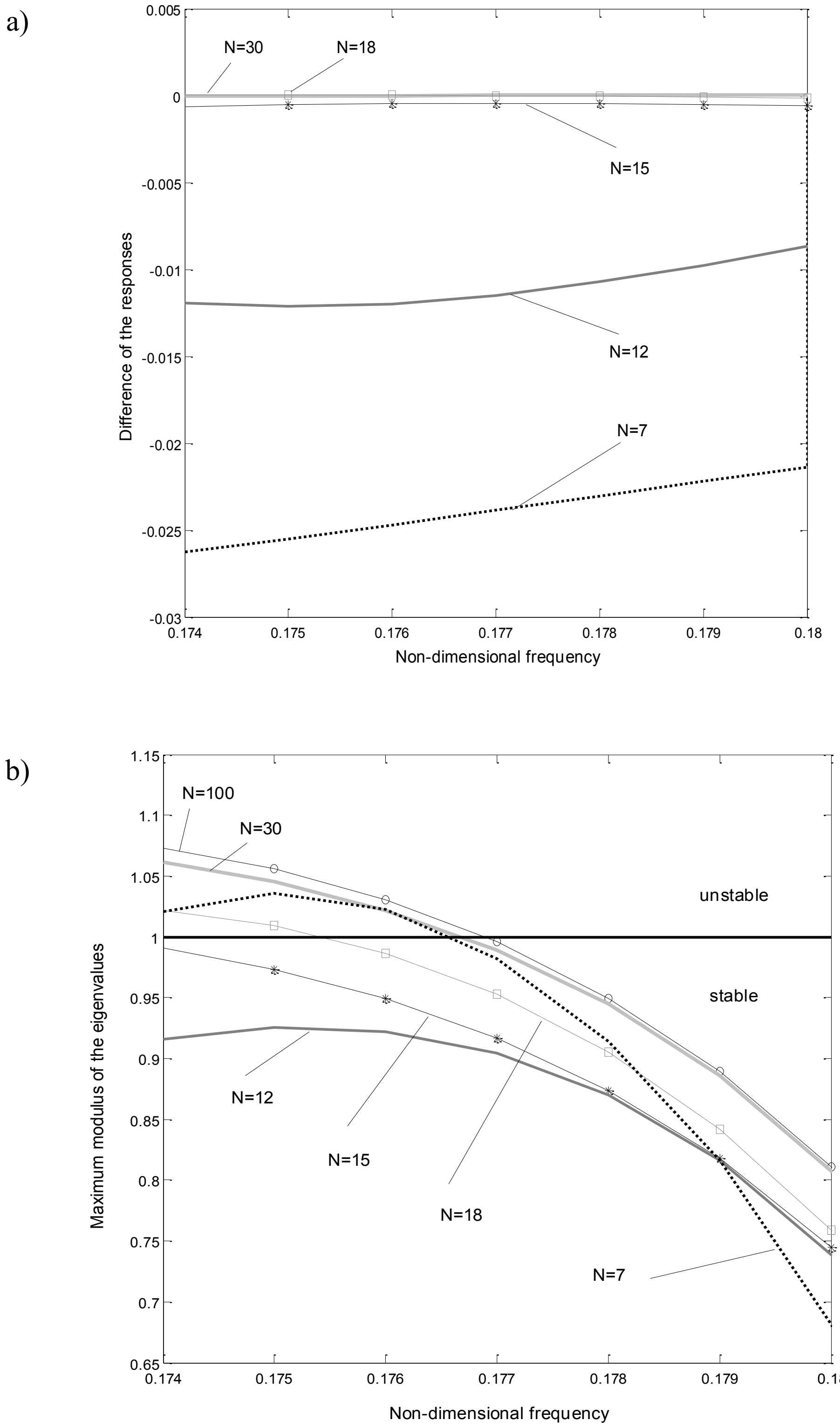
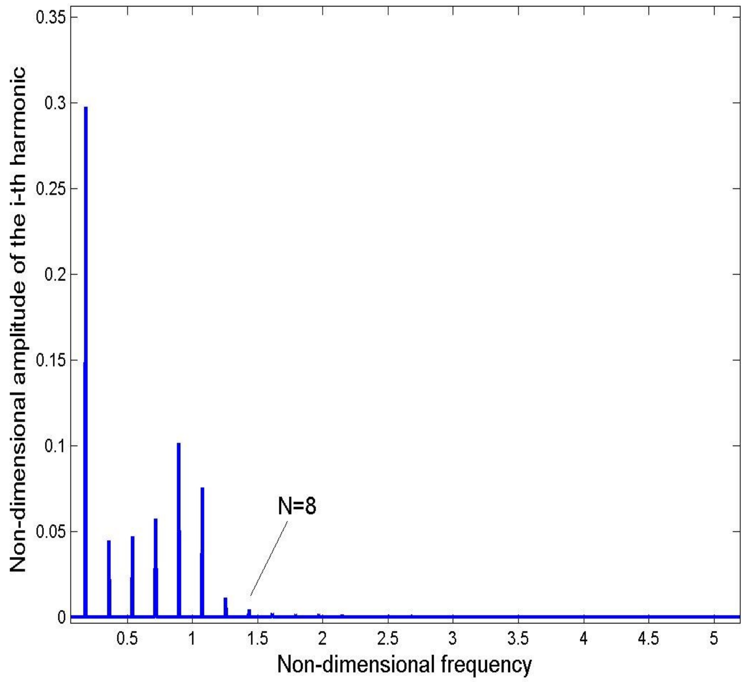



**Figure 8.** a) Differences of effective amplitudes $\left(\bar{x}_p\right)_{\mathrm{N\,diff}}$ (the a branch of the amplitude-frequency plot from Figure 7) obtained with N = 7, 12, 15, 18, and 30, and b) corresponding plot of maximum modulus of the eigenvalues of the monodromy matrix $|\lambda_{\max}| = |\lambda_{\max}|(\eta, \mathrm{N})$ .



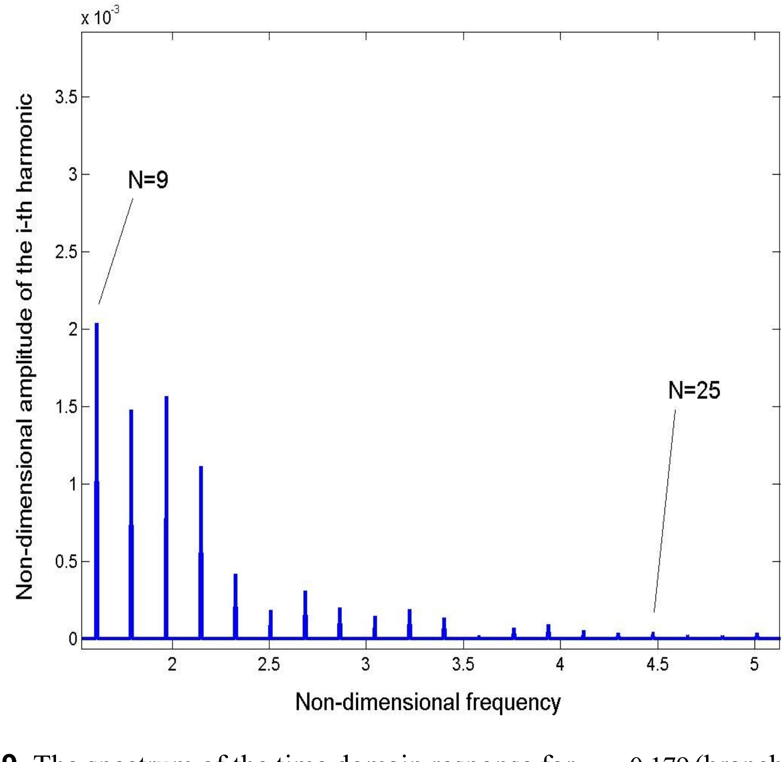




**Figure 9.** The spectrum of the time domain response for $\eta = 0{,}179$ (branch a in Fig 7).

## CONCLUSIONS

Response and dynamical stability of oscillators with discontinuous or steep first derivative of restoring characteristic is considered in this paper. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic force excitation is analysed by the method of piecing the exact solutions (MPES) in the time domain and by the incremental harmonic balance method (IHBM) in the frequency domain. The stability of the periodic solutions obtained in the frequency domain by IHBM is estimated by the Floquet-Lyapunov theorem.

The considerable advantage of using this piecewise-linear model is in the possibility of expressing monodromy matrix exactly as a product of matrix exponentials, what is not possible for a general non-linear function. In this way, the inaccuracy of evaluating monodromy matrix can be caused only by insufficient precision of numerical determination of the times in which the system changes stage stiffness region, and by numerical procedures of evaluation matrix exponential and product of matrix exponentials. On the other hand, local solutions of differential equation (1) are known explicitly inside each of the stage stiffness, and can be repeatedly matched at $x = b$ and $x = -b$, to obtain a global solution of (1) in the time domain. Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in a closed form. But, the matching of local solutions can be numerically done very easily. Only approximation done by applying this procedure is in the precision of numerical determination of times in which the system changes stage stiffness region ($x = b$, $x = -b$).

Obtained frequency response characteristic is very complex and includes multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued and non-periodic solutions. Determining of frequency response characteristic in the time domain by MPES is exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions, where many combinations of initial conditions have to be examined for obtaining complete frequency response characteristic. In the frequency domain, IHBM is very efficient and very well suited for obtaining wide range frequency response characteristics, parametric studies and bifurcation analysis. On the other hand, neglecting of very small harmonic terms (which in-significantly influence the r.m.s. values of the response and are very small in comparison to other terms of the spectrum) can cause very large error in evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution. Moreover, frequency ranges inside which the procedure of evaluation of eigenvalues of the monodromy matrix does not converge with increasing the number of harmonics included in the supposed approximate solution are detected.

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# ODZIV I DINAMIČKA STABILNOST VIBRACIJSKIH SUSTAVA S PREKINUTOM ILI STRMOM DERIVACIJOM POVRATNE KARAKTERISTIKE

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## SAŽETAK

U radu su razmatrani odziv i dinamička stabilnost vibracijskih sustava koji imaju prekinutu ili strmu prvu derivaciju povratne karakteristike. U tu svrhu je analiziran jednostavni vibracijski sustav s jednim stupnjem slobode gibanja i karakteristikom krutosti koja se sastoji od linearnih segmenata koji je uzbuđen s harmonijskom silom. Odziv sustava je u vremenskoj domeni dobiven s metodom povezivanja egzaktnih rješenja po segmentima (MPES), a u frekvencijskoj domeni s inkrementalnom metodom harmonijske ravnoteže (IHBM). Procjena stabilnosti periodičnih rješenja dobivenih u frekvencijskoj domeni korištenjem IHBM izvršena je primjenom Floquet-Lyapunovog teorema. Dobiveni graf funkcije povećanja je vrlo složen i sadrži višefrekvencijske odzive uzrokovane jednofrekvencijskom uzbudom, tzv. skokove amplitude, te višestruka i neperiodična rješenja. Određivanje grafa funkcije povećanja u vremenskoj domeni s MPES izuzetno je dugotrajno a to je najizraženije u područjima frekvencija u kojima postoje višestruka stabilna rješenja. IHBM, s kojom se odziv sustava određuje u frekvencijskoj domeni vrlo je efikasna i dobro prilagođena metoda za određivanje cjelovitog grafa funkcije povećanja, kao i za parametarsku i bifurkacijsku analizu. S druge strane, zanemarivanje vrlo malih harmonika (koji neznatno utječu na srednju vrijednost i efektivnu amplitudu odziva i koji su vrlo mali u odnosu na ostale harmonike u spektru) može uzrokovati vrlo velike pogreške u određivanju vlastitih vrijednosti prijenosne matrice i tako dovesti do pogrešne procjene dinamičke stabilnosti rješenja. Štoviše, uočena su frekvencijska područja unutar kojih postupak određivanja vlastitih vrijednosti prijenosne matrice ne konvergira s povećanjem broja harmonika uključenih u pretpostavljeno približno rješenje.

## KLJUČNE RIJEČI

dinamička stabilnost, graf funkcije povećanja, nelinearne vibracije, sustav linearan po segmentima