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OPTIMAL PROCESSES IN IRREVERSIBLE MICROECONOMICS

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SUMMARY

In this paper we consider optimal trading processes in economic systems. The analysis is based on accounting for irreversibility factor using wealth function concept.

KEY WORDS

microeconomics, non-equilibrium thermodynamics, irreversibility

CLASSIFICATION

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INTRODUCTION

In the last decades macro system theory has been extended to economic systems, see [1 – 3]. The crucial role here is played by the concept of resource value for a subsystem and the concept of exchange kinetics that is based on the differential of resource's value estimate by two economic systems. This technique makes it possible to determine the optimal behaviour of an economic intermediary operating in an irreversible economic system. In this framework economic intermediary is similar to a heat engine in thermodynamics. It controls its intensive variables (prices it orders to buyers and sellers). A direct economic exchange is always irreversible. However an exchange via an intermediary can be reversible, if the price for a resource used by an intermediary is intuitively close to resource's value estimate by a subsystem. In this case the rate of exchange will be intuitively close to zero. It is worth noting that if an exchange with the given rate is carried out via an intermediary then its irreversibility is lower than irreversibility of a direct exchange. If the duration of exchange or its rate is constrained then the problem of finding what are the prices an intermediary has to order to buyers and sellers in order to obtain maximal profit.

We denote the cash holding of an intermediary as M and the value of its assets as F.

PROFITABILITY AND CONDITIONS OF MINIMAL DISSIPATION

We denote profitability as the maximal amount of cash that can be extracted from the system subject to given conditions. The system is denoted as a number (possibly one) of economic intermediaries and their environment. The constraints here play an important role. They can include the constraints on the final states of some of the economic agents, conditions for the intensive variables of the system, constraints on the exchange duration and others. These constraints reduce the profitability. If a system does not have an environment with constant estimates this definition of profitability will still be valid.

The problem of finding the profitability does not have a solution for some systems without constraints. For example, profitability for a system with more than one economic reservoir (market) is unlimited, because an intermediary operating between them can generate an infinite profit. Note that profitability for a system with one reservoir, given initial state of an intermediary and no constraints on the duration of exchange represents an economic analogy of exergy in thermodynamics, which is widely used in engineering.

Let us consider an economic system with k economic agents. Each agent has resource's inventory N_i (i = 1, 2, …., k) and cash holding M_i. Resource's value estimate p_i depends on N_i and M_i. Economic reservoir (system with constant resource value estimate p) can be one of subsystems.

We assume that the system is closed with respect to resources. When there is contact between i-th and j-th subsystem then resource and capital flows between them n_{ij} and m_{ij} occur. Resource flow is directed from the system where its estimate is lower to the system where this estimate is higher. The capital flows in the opposite direction.

If a system contains an intermediary (economic agent) then its objective is to organize resource exchange in such a way that it extracts capital M from the system. We will assume here that economic subsystems cannot exchange resources directly but only via an exchange though an intermediary. This intermediary regulated this exchange by setting up the prices for buyers and sellers. It controls this price setting to maximise M. The flows of buying and selling depend on the price c_ν, offered by the intermediary to the ν-th subsystem and on the resource i value estimate for this subsystem $p_{i\nu}$. Thus,

$$n_{i\nu} = n_{i\nu}(p_i,\ c_i),\ n_{i\nu} = 0,\ p_\nu = c_\nu,\ \mathrm{sign}(n_{i\nu}) = \mathrm{sign}(p_\nu\ \ c_\nu).$$

We denote the flow directed to the intermediary as positive and from it as negative. The flow is a monotonically increasing function of c_i. The intermediary does not produce anything, it just resells what it purchased earlier.

It is clear that the flow of capital

$$m_\nu(p_\nu, c_\nu) = -\sum_i c_{i\nu} n_{i\nu}(p_i, c_i) .$$

The evolution of resource and capital inventories in the ν-th subsystem is described by the equations

$$\dot{N}_{i\nu} = -n_{i\nu}(p_\nu, c_\nu),\ N_{i\nu}(0) = N_{i\nu 0}.$$
$$\dot{M}_\nu = \sum_i c_{i\nu} n_{i\nu}(p_i, c_i),\ M_\nu(0) = M_{\nu 0}.$$

As a rule, estimates $p_{i\nu}(N_\nu, M_\nu)$ monotonically decrease when N_i increases and M_ν is fixed. These estimates also are non-decreasing functions of M_ν for fixed N_ν.

Next we will calculate how much money can be extracted by an intermediary over an infinity period and over a finite period of time for a system that includes economic reservoir and which lacks it.

PROCESS DURATION IS NOT CONSTRAINED

System with one reservoir

The profit from reselling of a resource can be only extracted if initially the system is in a non-equilibrium state. That is, if vectors of resources' estimates $p_\nu(0)$ for different subsystems have different values. The trading stops in equilibrium when estimates for all subsystems become equal to reservoir's estimates

$$p_{i\nu}(\overline{M}_\nu, \overline{N}_\nu) = p_i^0,\ (\mathrm{i} = 1, \ldots, n \text{ and } \nu = 1, \ldots, m). \tag{1}$$

The maximum of the extracted profit corresponds to a minimum of the combined capital of the economic intermediary and reservoir

$$\sum_{\nu=0}^{m} \overline{M}_\nu \to \min . \tag{2}$$

To achieve that objective, the intermediary buys resource using the lowest prices (estimates) from economic subsystems with estimates of the i-th resource below p_i^0, and resells it using the highest prices (estimates to economic subsystems with estimates higher than p_i^0. Both buying and selling processes are reversible and the increment of the combined wealth function is equal zero. The initial stocks of capital of economic subsystem is given and the increment of the capital of economic reservoir is

$$\Delta M^0 = \sum_{\nu=1}^{m} \sum_{i=1}^{k} [\overline{N}_{i\nu} - N_{i\nu}(0)] p_i^0 .$$

We have $m \times n$ conditions (1), and m reversibility conditions to find the state of the system in equilibrium

$$S_\nu(\overline{M}_\nu, \overline{N}_\nu) = S_\nu[M_\nu(0), N_\nu(0)] = S_\nu,\ \nu = 1, \ldots, m. \tag{3}$$

Equations (1) and (3) determine the equilibrium stocks of resource and capital. The extracted capital is equal to the difference between combined final and initial capital of the system minus reservoir's capital increment

$$E_\infty = \sum_{\nu=1}^{m} [M_\nu(0) - \overline{M}_\nu] - \Delta M^0 . \tag{4}$$

System without reservoir

In this case the condition of equilibrium (1) still holds but the vector of equilibrium estimates p^0 is unknown. It is to be found form the condition that intermediary does not accumulate resource

$$\sum_{\nu=1}^{m}[\overline{N}_{i\nu} - N_{i\nu}(0)] = 0\,,\ \mathrm{i} = 1,\ \ldots,\ n. \tag{5}$$

The system (1), (3), (5) determines $(m + 1)\cdot n$ subsystem's state variable in equilibrium. Naturally, equilibrium in a system with an intermediary $\overline{N}_0$, $\overline{N}$ is different from equilibrium for direct exchange. The maximum of the extracted capital is

$$E_\infty = \sum_{\nu=1}^{m}[M_\nu(0) - \overline{M}_\nu]\,. \tag{6}$$

Example

We consider an economic system which consists of an economic reservoir, a passive economic subsystem with finite capacity and an intermediary. Subsystem's wealth has the following form

$$S = M^{1/3}N_1^{\ 1/2}N_2^{\ 1/6}.$$

The initial inventories of both resource and capital are

$$M(0) = 150,\ N_1(0) = 20,\ N_2(0) = 30.$$

and the equilibrium prices for economic reservoir are

$$p_1^{\ 0} = 5,\ p_2^{\ 0} = 2.$$

The equilibrium states of subsystem equilibrium estimates are found from (1) and (3). The maximal amount of capital extractable from the system is determined by (4) and is equal to $M = 20{,}95$. Suppose that reservoirs' prices are scaled with the coefficient k. The dependence of $E_\infty(k)$ is shown in Figure 1. Let us emphasise that unlike the case of direct contact, the equilibrium state does not depend on the exchange kinetics for an exchange via an intermediary.

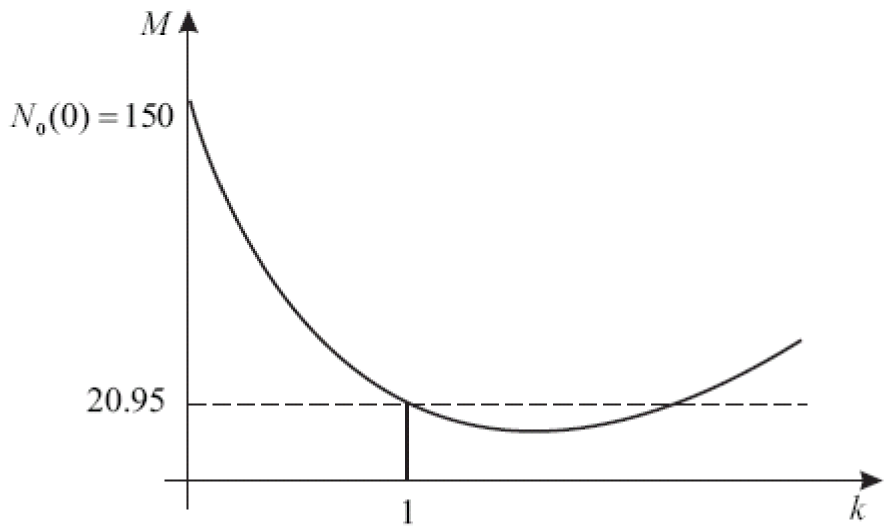

Figure 1. The dependence of intermediary's profit on the economic reservoir's price scale.

LIMITED DURATION OF EXCHANGE

We assume that the duration of exchange τ is given. In this case an intermediary is forced to increase the prices offered to sellers above the equilibrium estimates p_i. It is also forced to reduce the prices it offers to buyers below equilibrium estimates. This leads to an irreversible losses and reduces the amount of capital it is able to extract from the system. The maximal possible value of this capital E_∞ turns out to be lower than E_1. Their difference

$$\Delta E = (E_\infty - E_\tau) > 0,$$

describes the irreversibility of the trading.

We shall call the capital loss above the capital loss in equilibrium (reduction in system's profitability) the *capital dissipation*

$$\sigma = n(p, c)(p - c). \tag{7}$$

For scalar exchange the dissipative losses are determine as

$$\Delta S(\tau) = \int_0^\tau \sigma(t)\mathrm{d}t = \int_0^\tau n(p,c)(p-c)\mathrm{d}t . \tag{8}$$

Condition of optimality for trading

We consider an exchange between an intermediary and a finite-capacity economic subsystem. We want to find out how to control the price offered to buyers c in order to sell in given time τ the given amount of resource ΔN and to obtain the maximal price for it. It is clear that the optimal prices offered to the sellers would obey to the same conditions. In both case the capital of the finite-capacity subsystem $M(\tau)$ must be minimal possible.

The problem takes the following form

$$\overline{M} = M(\tau) \to \min_{c(t)} . \tag{9}$$

subject to

$$\overline{N} = N(\tau) = N_0 - \Delta N . \tag{10}$$

$$\frac{\mathrm{d}M}{\mathrm{d}N} = -c . \tag{11}$$

$$\int_0^\tau \mathrm{d}t = \int_{\overline{N}}^{N_0} \frac{\mathrm{d}N}{n[p(N,M),c]} = \tau . \tag{12}$$

We can substitute the independent variable dt with dN using the kinetic dependence

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -n(p, c),$$

on the interval (0, τ) with a non-zero flow n. In the problem (9) – (12) it is required to find such function $c^*(N)$ that the increment of the economic intermediaries capital is minimal.

The conditions of optimality of this problem are given by the Maximum principle. It is formulated in terms of the problem's optimization Hamiltonian

$$H = -\psi c + \lambda \frac{1}{n[p(N,M),c]} .$$

The maximum principle consists of the equations of motion, the equation for the adjoint variable

$$\frac{\mathrm{d}\psi}{\mathrm{d}N} = -\frac{\partial H}{\partial M} = -\lambda \frac{\partial n / \partial p(\partial p / \partial M)}{n^2[p(N,M),c]} ,\ \psi(\overline{N}) = 0 , \tag{13}$$

and the condition on maximum on c of the Hamiltonian (which is a convex and differentiable function)

$$\frac{\partial H}{\partial c} = -\psi + \lambda \frac{\partial n / \partial c}{n^2[p(N,M),c]} = 0 .$$

After eliminating ψ using (13) we obtain the condition of optimality for trading condition of minimal dissipation for resource exchange

$$\frac{d}{dN}\left[\frac{\partial n / \partial c}{n^2(p,c)}\right] = \frac{\partial n / \partial c(\partial p / \partial M)}{n^2[p(N,M),c]} = 0 . \qquad (14)$$

which determines $c(N, M)$ up to the constant. This constant is to be found from the equality (12).

If resource estimate p depends on its stock N only ($\partial p/\partial M = 0$), then the condition (14) becomes simpler

$$\frac{\partial n / \partial c}{n^2(p,c)} = \text{const} . \qquad (15)$$

Thus for

$$n(p, c) = \alpha(c - p), \qquad (16)$$

from (15) it follows that the optimal price for given finite time τ is

$$c_\tau^*(N,\overline{N}) = p(N) - \frac{\overline{N} - N}{\alpha\tau} . \qquad (17)$$

and the profit from trading is

$$E_\tau(\overline{N}) = E_\infty(\overline{N}) - \frac{(\overline{N} - N_0)^2}{\alpha\tau} . \qquad (18)$$

where E_∞ is the capital from trading for infinite long period $\tau \to \infty$ using equilibrium prices $c(N) = p(N)$. The function $E(\tau)$ is shown in Figure 2. For $\tau < \tau^0 = \Delta N^2/(\alpha E_\infty)$ the intermediary is forced to charge the seller less than it charges the buyer. For $\tau < 2\tau^0$ the average rate of profit $e(\tau) = E(\tau)/\tau$ is maximal and equal to

$$e^* = \frac{\alpha}{4}\left[\frac{E_\infty(\overline{N} - N_0)}{\overline{N} - N_0}\right]^2 . \qquad (19)$$

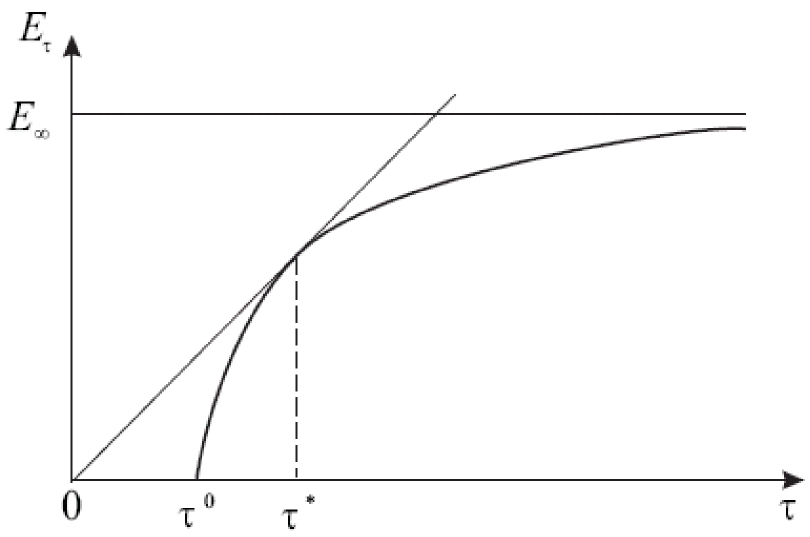

Figure 2. Dependence of trading profitability of its duration.

A given rate of trading flow corresponds to each τ. In particular the flow which corresponds to $\tau = \tau^0$ occur when trading is not profitable. For a liner dependence of flow of price estimate differential the rate of exchange flow that maximizes profit is two times higher than the rate of non-profitable, equilibrium trading.

For the above considered example the irreversible losses are

$$\Delta E(\tau,\overline{N}) = \int_0^{\tau} \alpha[p(N)-c(N)]^2\,\mathrm{d}t = \frac{(\overline{N}-N_0)^2}{\alpha\tau},$$

thus

$$E(\tau) = E_\infty(\overline{N}) - \Delta E(\tau,\overline{N}) = E_\infty(\overline{N}) - \int_0^{\tau} n(p,c)(p-c)\mathrm{d}t\,. \tag{20}$$

Equation (20) holds for an arbitrary $n(p, c)$. Indeed after substitution of dt with dN the integral in (8) can be rewritten as follows

$$\Delta E(\overline{N}) = E(\overline{N}) - E(N_0) = \int_{N_0}^{\overline{N}} [p(N) - c_\tau(N,\overline{N})]\mathrm{d}N\,.$$

The capital extracted is

$$E(\tau,\overline{N}) = \int_{N_0}^{\overline{N}} c_\tau(N,\overline{N})\mathrm{d}N\,,\ E_\infty(\overline{N}) = \int_{N_0}^{\overline{N}} p(N)\mathrm{d}N\,. \tag{21}$$

Equation (20) follows from comparing these two equations. Thus, the optimal trading processes are minimal dissipation processes and the condition (14) is the condition of minimal dissipation.

Extracting maximal profit in a system with a few intermediaries

We assume that an intermediary operates in a system that is closed with respect to resource. Intermediary facilitates the exchange between subsystems in order maximize the amount of capital it extracts from the system. The problem here can be decomposed into the problem of optimal trading by an intermediary with a single subsystem. The trading is optimal if the price c and the estimate p obey the conditions of minimal dissipation (14), (15) for any moment when trading takes place. The trading volumes ΔN_i for each of m subsystems are to be chosen optimally. The following condition holds

$$\sum_{i=1}^{m} \overline{N}_i = \sum_{i=1}^{m} N_{i0}\,. \tag{22}$$

We can view a reservoir as one of subsystems with the estimate p_- that is independent on its stocks of resource and capital. Therefore for any dependence $n(c, p_-)$ the optimal price c for trading on this market must be time independent.

Thus, the problem of extracting maximal capital from a closed microeconomic system in a given time is reduced to a two staged process. During the first stage m problems (9) – (12) about the optimal trading with each of the subsystems with given initial and final resource stocks (N_{i0} and N_i) are solved. During the second stage the optimal N_i are found from the condition

$$\sum_{i=1}^{m} E_i(\tau,\overline{N}_i) = \to \max_{\overline{N}_i}\,. \tag{23}$$

subject to (22). The optimality conditions for the problem (22) and (23) take the form

$$\frac{\partial E_i(\tau,\overline{N}_i)}{\partial \overline{N}_i} = \Lambda\,,\quad \mathrm{i} = 1, \ldots, m,$$

with Λ to be found from (22).

After taking into account (21) we obtain

$$\frac{\partial E_i(\tau,\overline{N}_i)}{\partial \overline{N}_i} = c_{i\tau}(\overline{N}_i,\overline{N}_i) + \int_{N_{i0}}^{\overline{N}_i} \frac{\partial c_{i\tau}(N_i,\overline{N}_i)}{\partial \overline{N}_i} \mathrm{d}N_i = \overline{c}_{i\tau}(\overline{N}_i)\,. \tag{24}$$

The first term in the right hand side is the optimal price at time τ. The second term is the correctional one. It is determined by the averaged sensitivity of the optimal price to the volume of trading. The condition of the optimal choice of trading volumes takes the form

$$\overline{c}_{i\tau}(\overline{N}_i) = \Lambda\,, \quad \mathrm{i} = 1,\, \ldots,\, m. \tag{25}$$

Example

Consider the system where for each subsystem

$$p_\mathrm{i} = h_\mathrm{i}/N_\mathrm{i}. \tag{26}$$

$$n_i(c,p) = \alpha_i(c_i - p_i)\,. \tag{27}$$

Suppose that trading time is not constrained. From (21) and (26) we get

$$E_{i\infty}(\overline{N}_i) = h_i \int_{N_{i0}}^{\overline{N}_i} \frac{\mathrm{d}N_i}{N_i} = h_i \ln\frac{\overline{N}_i}{N_{i0}}\,,$$

with i = 1, …, m in previous equations. For time-constrained exchange after integrating (21) we obtain

$$E_i(\tau,\overline{N}_i) = h_i \ln\frac{\overline{N}_i}{N_{i0}} - \frac{(\overline{N}_i - N_{i0})^2}{\alpha_i\tau}\,. \tag{28}$$

The condition (25) for the optimal choice of N_i takes the form

$$\overline{c}_{i\tau}(\overline{N}_i) = p_i(\overline{N}_i) - 2\frac{\overline{N}_i - N_{i0}}{\alpha_i\tau} = \Lambda\,. \tag{29}$$

The problem becomes much simpler when all subsystems have constant estimates p = const. The condition of optimality (29) then is reduced to the equations

$$p_i - 2\frac{\overline{N}_i - N_{i0}}{\alpha_i\tau} = \Lambda \to \Delta N_i = \frac{\alpha_i\tau}{2}(p_i - \Lambda)\,. \tag{30}$$

From (22) it follows that Λ is equal to the averaged value of resource estimate

$$\Lambda = \frac{\sum_{i=1}^{m}\alpha_i p_i}{\sum_{i=1}^{m}\alpha_i}$$

$$\overline{N}_i^* = \frac{\tau\alpha_i}{2}\left(p_i - \frac{\sum_{\nu=1}^{m}\alpha_\nu p_\nu}{\sum_{\nu=1}^{m}\alpha_\nu}\right) + N_{i0} \tag{31}$$

after substitution of $\overline{N}_i^*$ into (28) we obtain $E_\mathrm{i}(\tau,\ \overline{N}_i^*)$ – the maximal amount of capital that can be extracted from the system in given time τ. The profitability of the system is

$$E_\tau^* = \sum_{i=1}^{m}\left[p_i(\overline{N}_i^* - N_{i0}) - \frac{(\overline{N}_i^* - N_{i0})^2}{\alpha_i\tau}\right].$$

After taking into account (31) we obtain for estimates that are independent from resource's stocks

$$E_{\tau}^{*}=\frac{\tau}{4}(p_i^2-\Lambda^2).$$

In most of the cases the estimates are reduced when resource stock increases and profitability is a monotone convex function of trading (purchasing) time.

CLASSIFICATION OF RESOURCE-EXCHANGE LAWS BY THEIR MINIMAL DISSIPATION CONDITIONS

The law of resource exchange for an economic system is determined by the function $n(x, u)$. The conditions of minimal capital dissipation, (14) and (15), are expressed in terms of this function. In this section we will demonstrate that it is possible to single out classes of exchange kinetics that have similar optimal trading regimes.

CONDITION CONSTANT DISCOUNT IS OPTIMAL

This problem has the form

$$I=\int_0^{\tau} n(c,p)\mathrm{d}t \to \min_c .$$

subject to constraints

$$\begin{gathered}\int_0^{\tau} n(c,p)\mathrm{d}t=\Delta N, \qquad n>0,\\ \dot{N}=-n(c,p), \qquad N(0)=N_0.\end{gathered}.$$

The estimate $p(N)$ here is the given function. The conditions of optimality for this problem have been obtained above.

Let us find $n(p, c)$ for which the optimal trading price c for any moment of time t is equal to the estimate of this resource in the subsystem P up to a constant ($\varphi(p, c) = c - p = \text{const}$). From the above-derived conditions of minimal dissipation it follows that for an irreversible exchange

$$F=\frac{1}{n^2(p,c)}\frac{\partial n}{\partial c}=\text{const.}, \; n(c,p)=0, \; c=p. \tag{32}$$

It was shown that in order for the optimal discount to be constant it is necessary and sufficient that the following equation holds

$$\frac{F_c}{F_p}=\frac{nn_c c-2n_c^2}{nn_c p-2n_p n_c}=\frac{\varphi_c}{\varphi_p}. \tag{33}$$

This condition determines the exchange kinetics for each dependence φ of the optimal price on resource estimate.

From (33) it follows that constant optimal discount corresponds to such dependencies $n(p, c)$ that

$$n(c,p)=\frac{M(c-p)}{1+R(p)M(c-p)}. \tag{34}$$

Here $M(c - p)$ and $R(p)$ are arbitrary functions of p and $c - p$, and $M(0) = 0$. We denote this discount as $c - p = \delta$. The expression (34) takes the form $n(c, p) = \mu(\delta)/[1+R(p)\mu(\delta)]$. Because

$$\int_0^{\tau} n(c,p)\mathrm{d}t=\Delta N .$$

$$\mu(\delta) = \frac{\Delta N}{\int\limits_0^\tau \frac{dt}{1 + R[p(t)]\mu(\delta)}}. \tag{35}$$

This condition determines the optimal discount *δ*. The irreversibility of the trading is described the integral

$$\Delta E = \int\limits_0^\tau \delta n(c,p)\mathrm{d}t = \delta\Delta N.$$

The average dissipation (average trading costs) is

$$\bar{\sigma} = \frac{\Delta E}{\tau} = \frac{\delta\Delta N}{\tau}. \tag{36}$$

Equations (35) and (36) determine irreversibility of trading for any function $n(c,p)$ of the form (34).

CONDITION WHEN OPTIMAL FLOW IS CONSTANT

The condition of optimality (32) is reduced to the condition that the flow $n(p, c)$ on the optimal solution $c^*(p)$ is constant when the left hand side of (32) depends on n only

$$F(p,c) = \varphi[n(p,c)],$$

where $\varphi(\cdot)$ is an arbitrary function, or, which is the same, when $\partial n/\partial c$ is some function of n

$$\frac{\partial n}{\partial c} = n_c(p,c^*) = \varsigma[n(p,c^*)],\ \forall p. \tag{37}$$

The following statement holds: *the optimal trading is the constant flow trading if and only if the resource exchange can be represented in the following form*

$$n(c,p) = (c-p)M(c-p). \tag{38}$$

Here, $M(\cdot)$ is an arbitrary non-negative function of price differential. The optimal dependence $c^*(p)$ is determined by the condition

$$(c^* - p)M(c^* - p) = n^* = \frac{\Delta N}{\tau}. \tag{39}$$

EXAMPLE

We define

$$n(c,p) = \alpha\cdot\mathrm{arctg}(c-p),\ c > p.$$

Since (38) holds for this function, the optimal dependence of the price on time $c^*(t)$ is

$$c^*(t) = p(N^*) + \mathrm{tg}\frac{\Delta N}{\alpha\tau},$$

and the optimal profit

$$N^*(t) = N_0 - \frac{\Delta N}{\tau}t.$$

The optimal exchange flow is constant and equal to $\Delta N/\tau$. Proof: from (37) it follows that

$$\frac{n_{cc}}{n_{cp}} = \frac{n_c}{n_p} \Rightarrow \frac{\partial}{\partial c}\ln\left|\frac{n_c}{n_p}\right| = 0 \Rightarrow \frac{n_c}{n_p} = r(p),$$

where $r(\cdot)$ is an arbitrary function. We get the equation for n

$$n_c - r(p)n_p = 0,\quad n(p,c) = 0,\quad p = c. \tag{40}$$

The characteristics equation is

$$\dot{c} = 1\,,\ \dot{p} = -r(p)\,.$$

We get

$$c(t) = c_0 + t\,,\quad \mu(p) = t - t_0, \tag{41}$$

where $\mu(p)$ is an arbitrary differentiable function such that $d\mu/dp = -1/r(p)$. After elimination of t from (41), we get the first integral of the equation (40)

$$\mu(p) - c = t_0 - c_0 = \text{const.},$$

thus the common solution is

$$n(c, p) = M[\mu(p) - c].$$

After taking into account that $n(p, c) = 0$ for $c = p$ we obtain the class of resource exchange laws (38) for which it is optimal to trade using constant exchange flow.

TRADING WITH A NUMBER OF SUBSYSTEMS WITH PRICE DISCRIMINATION

In previous text we considered the problem when an intermediary was able to offer different prices to different buyers and sellers. In some cases it cannot do that but has to offer the single price to all buyers and another single price to all sellers. Naturally this reduces its profit.

We denote the prices offered to sellers and buyers as $c_1(t)$ and $c_2(t)$ correspondingly. The composition of trading partners for an intermediary is determined by exchange laws. All subsystems at any moment of time $t \in [0, \tau]$ can be divided into three category: sellers (from the intermediary) ($p_i(t) < c_1(t)$); buyers ($p_i(t) > c_2(t)$); and neutral. The intermediary does not contact them because it is not profitable for it ($c_1(t) \le p_i(t) \le c_2(t)$).

This problem becomes much simpler when subsystems are reservoirs (have constant estimates). In this case c_1, c_2 and p_i are constants that maximise the profit.

The dependence of the exchange flow on c_1 can be written as

$$n_+(c_1, p_i) = \sum_{\nu=1}^{j} n_\nu(c_1, p_\nu)\,, \tag{42}$$

where summing is done on all reservoirs with estimates lower than c_1. Similarly

$$n_-(c_2, p_i) = \sum_{\nu=1}^{n} n_\nu(p_\nu, c_2)\,. \tag{43}$$

Here p_i is the minimal estimate higher than c_2. The rate of profit must be maximal

$$s = [c_2 n_-(c_2, p) - c_1 n_+(c_1, p)] \to \max_{c_1, c_2}\,, \tag{44}$$

subject to non-accumulation of resource by the intermediary

$$n_+(c_1, p) = n_-(c_2, p) = n\,. \tag{45}$$

Equation (45) allows us to express c_1 and c_2 in terms of n. Substitution of these dependencies in (44) leads to unconditioned optimisation problem with respect to n. Substitution of its solution n^* back into c_1^* and c_2^* determine division of the reservoirs into buyers sellers and neutral non traders.

Let us specify the dependencies n_ν

$$n_\nu = \alpha_\nu(c - p_\nu)\,,$$

and rewrite (45) as two equalities

$$n_+ = \sum_{\nu=1}^{j} \alpha_\nu (c_1 - p_\nu) = n,$$

$$n_- = \sum_{\nu=i}^{n} \alpha_\nu (p_\nu - c_2) = n.$$

After denoting

$$M_1(j) = \sum_{\nu=1}^{j} \alpha_\nu p_\nu, \quad M_2(i) = \sum_{\nu=i}^{n} \alpha_\nu p_\nu,$$
$$A_1(j) = \sum_{\nu=1}^{j} \alpha_\nu, \quad A_2(i) = \sum_{\nu=i}^{n} \alpha_\nu,$$

we obtain

$$c_1(n,j) = \frac{n + M_1(j)}{A_1(j)}, \quad c_2(n,i) = \frac{M_2(i) - n}{A_2(i)}. \tag{46}$$

The objective (44) takes the form

$$s = n[c_2(n,i) - c_1(n,j)] \to \max. \tag{47}$$

For fixed n the values of i and j are to be found from the conditions of maximum of c_2 and minimum of c_1, respectively.

The condition of minimum of c_1 on j yields

$$p_{j+1} > \frac{n + M_1(j)}{A_1(j)} > p_j. \tag{48}$$

Similarly, for maximum of c_2 on i we get

$$p_i > \frac{M_2(i) - n}{A_2(i)} > p_{i-1}. \tag{49}$$

The maximum of (47) on n, subjected to (46), (48) and (49) determines the maximal rate of capital extraction in a system with common prices. For a convex s we obtain

$$c_2(n^*, i) - c_1(n^*, j) = n^* \left(\frac{\partial c_1}{\partial n} - \frac{\partial c_2}{\partial n} \right)_{n=n^*}. \tag{50}$$

For two economic agents (j = 1, i = 2) the problem becomes very simple. The optimal prices offered by an intermediary to buyers and sellers at any moment of time t obey the following equations

$$c_1 = \frac{2\alpha_1 p_1 + \alpha_2 (p_1 + p_2)}{2(\alpha_1 + \alpha_2)}, \; c_2 = \frac{2\alpha_2 p_2 + \alpha_1 (p_1 + p_2)}{2(\alpha_1 + \alpha_2)},$$

and the limiting rate of profit extraction is

$$s^*(t) = \frac{\alpha_1 \alpha_2 (p_2 - p_1)^2}{4(\alpha_1 + \alpha_2)}.$$

If resource estimates p_ν for both economic agents depend on time then the optimal solution $c_1^*(t)$ and $c_2^*(t)$ is determined by these equation for every moment of time t.

Trading in a competitive market: Let us consider the system where a number of economic intermediaries compete to trade with a single finite-capacity economic subsystem in a given time (Figure 3). We assume that resource estimate p depends only on its stock N and does not depend on the subsystem's capital. We assume that function $p(N)$ is known.

The expenses

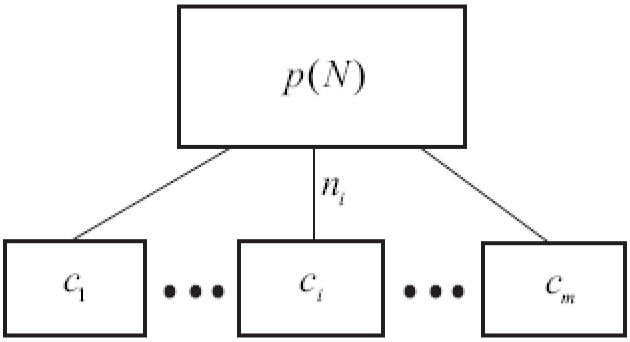

Figure 3. The structure of an economic system with a number of economic intermediaries and a finite-capacity subsystem.

$$\Delta S = \sum_{i=1}^{m} \int_0^{\tau} n_i(c_i, p)(c_i - p)dt \to \min_{c_i} . \tag{51}$$

are, for the given trading volume,

$$\int_0^{\tau} n_i(c_i, p)dt = \Delta N_i, \; i = 1, \ldots, \mathrm{m}. \tag{52}$$

The stock of resources is described by

$$\frac{dN}{dt} = -\sum_{i=1}^{m} n_i(c_i, p), \; N(0) = a > \sum_{i=1}^{m} \Delta N_i . \tag{53}$$

We denote

$$b = a - \sum_{i=1}^{m} \Delta N_i$$

and substitute the resource stock N as a new independent variable. The problem then takes the form

$$\int_b^a \frac{\sum_{i=1}^{m} (c_i - p)n_i}{\sum_{i=1}^{m} n_i} dN \to \min_{c_i} , \tag{54}$$

$$\int_b^a \frac{n_i}{\sum_{i=1}^{m} n_i} dN = \Delta N_i , \; \mathrm{i} = 1, \ldots, m, \tag{55}$$

$$\int_b^a \frac{1}{\sum_{i=1}^{m} n_i} dN = \tau . \tag{56}$$

The Lagrange function of the problem (54 – 56) is

$$L = \frac{1}{\sum_{i=1}^{m} n_i \left[\sum_{i=1}^{m} (c_i - p + \lambda_i)n_i - \zeta \right]} . \tag{57}$$

The necessary conditions of optimality become

$$\frac{\partial L}{\partial c_i}=0\Rightarrow-\frac{1}{\left(\sum_{i=1}^{m}n_i\right)^2}\frac{\partial n_i}{\partial c_i}\left[\sum_{i=1}^{m}(c_i-p+\lambda_i)n_i-\zeta\right]+\frac{1}{\sum_{i=1}^{m}n_i}\left[n_i+(c_i-p+\lambda_i)\frac{\partial n_i}{\partial c_i}\right]=0\,.$$

Thus, for all i the following conditions hold

$$\frac{n_i(c_i,p)}{\frac{\partial n_i}{\partial c_i}}+c_i+\lambda_i=p+\frac{\sum_{i=1}^{m}(c_i-p+\lambda_i)-\zeta}{\sum_{i=1}^{m}n_i(c_i,p)}\,,\ i=1,\ldots,\mathrm{m}, \tag{58}$$

For given dependence $p(N)$ these conditions, jointly with (55) and (56), determine the optimal solution of the problem $c(N)$.

INTERMEDIARY OPERATING BETWEEN TWO FINITE-CAPACITY SUBSYSTEMS

Until this point we considered trading in a closed economic system. We will now consider trading in an open system in a stationary or cyclic regime.

Economic reservoirs with constant prices

We consider the system with intermediary which maximizes its profit by trading with two economic reservoirs. It can establish contacts with reservoirs in turn by controlling not only prices it offers but also the timing of contact. Or intermediary can contact both reservoirs simultaneously and trade continuously. We denote the prices (estimates) of two reservoirs as p_1 and p_2, with $p_1 < p_2$. Exchange kinetics is given by

$$g(\bar{p},p)(\bar{p}-p). \tag{59}$$

The estimate p here can take to values, p_1 and p_2, and the prices offered by the intermediary for buying and selling p are the unknowns.

Maximal rate of profit

The objective of the intermediary is to achieve maximal rate of capital extraction per cycle.

Sequence of buying and selling

We now consider the case when intermediary buys and sells from each of the reservoirs (markets) in sequence. The rate of profit is

$$N=\frac{1}{T}\int_0^T pg(\bar{p},p)dt=\overline{pg(\bar{p},p)}\rightarrow\max\,. \tag{60}$$

The intermediary here sells on the second market everything it buys on the first market,

$$\frac{1}{T}\int_0^T g(\bar{p},p)dt=\overline{g(\bar{p},p)}=0\,. \tag{61}$$

This is an averaged nonlinear programming problem with one constraint (61). The Lagrange function of the corresponding non-averaged problem is

$$L=pg(\bar{p},p)-\lambda g(\bar{p},p). \tag{62}$$

We denote L as L^0 for $p = \mathrm{p}^0$. We require that each of L^0 attains maximum on p, and get

$$\frac{dg(p_v, p)}{dp}(p-\lambda)+g(p_v, p)=0\,,\ v=1,2. \tag{63}$$

These equations determine the basic values $p_v^*(\bar{p}_v,\ \lambda)$. Their substitution into L^0 yields $L_1^*(\bar{p}_1,\ \lambda)$ and $L_2^*(\bar{p}_2,\ \lambda)$. The optimal λ^* is determined by the condition

$$\max_v L_v^*\left(\overline{p_v\lambda} \to \min_\lambda\right). \tag{64}$$

Thus, the optimal prices for buying and selling are

$$p_1 = p^*\left(\lambda^*, \overline{p_1}\right),\ p_2 = p^*\left(\lambda^* \overline{p_2}\right).$$

Note that $p_1 > \overline{p_1}$ and $p_2 < \overline{p_2}$.

Suppose that α in (59) depends only on the reservoirs estimate. That is, for $\bar{p} = \bar{p}_1$, $\alpha = \alpha_1$ and for $\bar{p} = \bar{p}_2$, $\alpha = \alpha_2$. Then L takes the form

$$L = \alpha(\bar{p})(\bar{p}-p)(p-\lambda).$$

The conditions (63) become

$$-\alpha_v(p-\lambda)+\left(\overline{p_v}-p\right)=0\,,\ v=1,2,$$

and

$$p_v^* = \frac{\overline{p_v}+\lambda}{2},\ v=1,2. \tag{65}$$

Substitution of these p into L gives

$$L_1 = \alpha_1\left(\overline{p_1}-\frac{\overline{p_1}+\lambda}{2}\right)\left(\frac{\overline{p_1}+\lambda}{2}-\lambda\right)=\alpha_1\left(\frac{\overline{p_1}-\lambda}{2}\right)^2,$$
$$L_2 = \alpha_2(\bar{p}_2-\lambda)/2. \tag{66}$$

Since $p_1^* > \bar{p}_1$, $p_2^* < \bar{p}_2$, the function L_2^* decreases when λ increases. At the same time L_1^* increases. The minimum on λ of the maximum of these two function is attained at the point where $L_1^* = L_2^*$:

$$L_1^*(\lambda)=L_2^*(\lambda)\Rightarrow\sqrt{\alpha_1}\left(\overline{p_1}-\lambda\right)=-\sqrt{\alpha_2}\left(\overline{p_2}-\lambda\right).$$

The functions L_1^* and L_2^* are shown in Figure 4. Their maximum is denoted with bold line. The minimum of $\max_i[L_1^*(\lambda)]$ on λ is achieved at λ^*. Therefore,

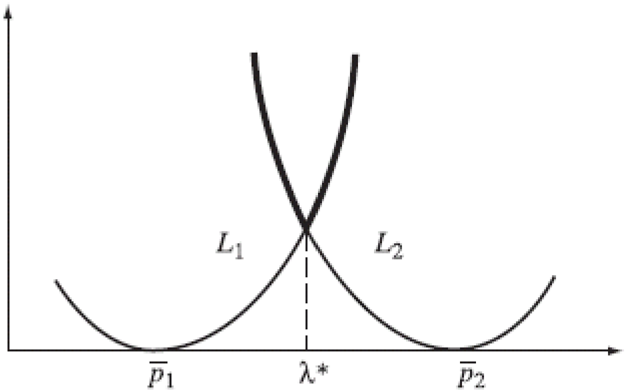

Figure 4. Characteristic dependence of maximum on p_1 and p_2 of the Lagrange function.

$$\lambda^* = \frac{\overline{p_1}\sqrt{\alpha_1} + \overline{p_2}\sqrt{\alpha_2}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}}. \tag{67}$$

Substitution of (65) (67) into (61) gives

$$\gamma_1\alpha_1\left(\overline{p_1} - p_1^*\right) + \gamma_2\alpha_2\left(\overline{p_2} - p_2^*\right) = \frac{\gamma_1\alpha_1}{2}\frac{\sqrt{\alpha_2}\left(\overline{p_1} - \overline{p_2}\right)}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} + \frac{\gamma_2\alpha_2}{2}\frac{\sqrt{\alpha_1}\left(\overline{p_2} - \overline{p_1}\right)}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} = 0.$$

Thus

$$\frac{\gamma_1}{\gamma_2} = \sqrt{\frac{\alpha_2}{\alpha_1}},\ \gamma_1 = \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}},\ \gamma_2 = \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}}, \tag{68}$$

for $\alpha_1 = \alpha_2$, $\lambda^* = (\bar{p}_1 + \bar{p}_2)/2$. The optimal prices for buying and selling are obtained after substitution of λ^* into (65). For $\alpha_1 = \alpha_2$

$$p_1^* = \frac{3\overline{p_1} + \overline{p_2}}{4},\ p_2^* = \frac{3\overline{p_2} + \overline{p_1}}{4}.$$

The rate of profit here is

$$\eta = \frac{p_2^*}{p_1^*} - 1 = \frac{3\overline{p_2} + \overline{p_1}}{3\overline{p_1} + \overline{p_2}} - 1.$$

This is lower than $\eta_0 = \bar{p}_2 / \bar{p}_1 - 1$, for reversible buying and selling at reservoir prices.

The limiting rate of capital extraction is

$$N^* = \frac{\sqrt{\alpha_1\alpha_2}}{4\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)}\left[\sqrt{\alpha_2}\left(\overline{p_2^2} - \lambda^{*2}\right) + \sqrt{\alpha_1}\left(\overline{p_1^2} - \lambda^{*2}\right)\right], \tag{69}$$

with λ^* to be found form (67).

One of possible constraints is the average over the cycle flow of capital spent by the intermediary to buy resource. This flow is given by the formula

$$U = p_1 g\left(\overline{p_1}, p_1\right)\gamma_1. \tag{70}$$

For (59)

$$U^* = \alpha_1\frac{\lambda^{*2} - \overline{p_1^2}}{4}\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}}. \tag{71}$$

If the flow of capital is constrained, eg. $U \le U^{\max} < U^*$, then the intermediary has to add the equality (70) into its optimization problem as an additional constraint. Its profit N here will be lower than N^*. If $U^{\max} \ge U^*$ then it does not make sense for an intermediary to spend all its capital and it will use only its fraction equal to U^*. Maximum of N here corresponds to the maximal rate of profit.

Simultaneous buying and selling

For continuous trading the intermediary has to select the prices it offers to buyers and sellers p_1 and p_2 in such a way that it rate of profit

$$\overline{N} = p_2 g_2\left(\overline{p_2}, p_2\right) + p_1 g_1\left(\overline{p_1}, p_1\right) \tag{72}$$

is maximal subject to selling everything it buys

$$g_1\left(\overline{p_1}, p_1\right) + g_2\left(\overline{p_2}, p_2\right) = 0. \tag{73}$$

This is a standard nonlinear programming problem. Its solution gives the conditions for optimal prices

$$\frac{g_1\left(\overline{p_1}, p_1\right)}{\frac{\partial g_1}{\partial p_1}} + p_1 = \frac{g_2\left(\overline{p_2} p_2\right)}{\frac{\partial g_2}{\partial g_2}} + p_2 . \tag{74}$$

If $g_\nu = \alpha_\nu(\overline{p}_\nu - p_\nu)$ ($\nu = 1, 2$), then the condition (74) takes the form

$$2p_1 - \overline{p_1} = 2p_2 - \overline{p_2} .$$

This condition jointly with (73)

$$\alpha_1\left(\overline{p_1} - p_1\right) = -\alpha_2\left(\overline{p_2} - p_2\right),$$

allows us to obtain the optimal buying and selling prices

$$p_1^* = \frac{\alpha_1 \overline{p_1}}{\alpha_1 + \alpha_2} + \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{\overline{p_2} + \overline{p_1}}{2}, \tag{75}$$

$$p_2^* = \frac{\alpha_2 \overline{p_2}}{\alpha_1 + \alpha_2} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\overline{p_2} + \overline{p_1}}{2}. \tag{76}$$

The rate of profit is

$$\overline{N^*} = \frac{\alpha_1 \alpha_2 \left(\overline{p_2} - \overline{p_1}\right)^2}{4\left(\alpha_1 + \alpha_2\right)}. \tag{77}$$

It is easy to see that if α_ν are equal then $\overline{N}^*$ is two times higher than N^* in (69). This is natural because the exchange flows are the same and selling takes place during half of the cycle. If $\alpha_1 \neq \alpha_2$ then $\overline{N}^*/N^* < 2$.

Vector exchange

When intermediary contacts reservoirs in turn and there is no constraint of the rate of spending the problem (60), (61) takes the form

$$N = \sum_{i=1}^{n} \overline{p_i g_i\left(\overline{p}, p_i\right)} \to \max_{p, \overline{p}} , \ \overline{g_i\left(\overline{p}, p\right)} = 0 , \ \text{i} = 1, \ldots, \text{n}.$$

This is an average nonlinear programming problem with n constraints. The number of its basic solutions does not exceed $n + 1$. In order to solve it we first solve an auxiliary problem

$$L = \sum_{i=1}^{n} \left[g_i\left(\overline{p}, p\right)\left(p_i - \lambda_i\right)\right] \to \max_{p, \overline{p}} \min_{\lambda} , \ \overline{p} = \left(\overline{p_1}, \overline{p_2}\right).$$

This problem becomes much simpler if L turns out to be strictly convex on p. The prices offered by the intermediary here can be found by solving equations

$$\sum_{i=1}^{n} \frac{\partial g_i}{\partial p_\nu}\left(p_i - \lambda_i\right) + g_\nu\left(\overline{p_j}, p\right) = 0 , \quad \text{j} = 1, 2, \quad \nu = 1,\ldots,n.$$

If $g_i = g_i(\overline{p}_i, p_i)$, the problem with vector resource can be decomposed into n independent sub-problems similarly to (60) and (61). All the results derived above hold. Note that (67) holds here for every λ_i, and the expression (69) for N^* contains sum on i. Each term in this sum is positive. The profit rate can be found here using the formula

$$\eta = \frac{N^*}{\gamma_1 \sum_{i=1}^{n} g_i\left(\overline{p_{i1}}, p_i\right) p_i} .$$

The same is true for the problem where intermediary contacts reservoirs simultaneously.

The characteristic features of vector exchange become important when the rate of capital spending is constrained below the value which corresponds to N^*. This case is considered in the next paragraph.

Intermediary operating between two finite-capacity economic subsystems.

In intermediary trades with finite-capacity subsystems instead of markets-reservoirs then the prices it offers must depend on time. The problem of obtaining the maximal profit here in a given time is decomposed into three problems: the problem of optimal trading (optimal buying and optimal selling) with a single subsystem, and the problem of adjusting of the optimal buying and optimal selling by selecting optimally their common parameters.

It is important here that buying and selling here obeys the condition of minimal capital dissipation.

We consider a cycle where resource is first bought by the intermediary and is then sold by it. We denote the capacities of two subsystems with which the intermediary trades as C_1 and C_2. These parameters link the stock of resource with its estimate by the subsystem $p_i(t)$ (i = 1, 2). The estimates are defined as $(dp_i/dt)C_i = - dN_i/dt$. The price of the intermediary is denoted as $p(t)$. Furthermore, τ_i (i = 1, 2) are the durations of contact between intermediary and two sub-systems. It is required here to find such τ_1 and τ_2 that the total duration of the trading cycle was equal to the given τ. The other control variable here is the volume of trading ΔN. Intermediary maximises its profit per cycle

$$I = \int_0^{\tau_2} p(t) g(p_2, p) dt - \int_0^{\tau_1} p(t) g_1(p, p_1) dt \to \max . \tag{78}$$

subject to constraint

$$\tau_1 + \tau_2 = \tau . \tag{79}$$

$$\frac{dp_1}{dt} = \frac{g_1(p, p_1)}{C_1},\ \frac{dp_2}{dt} = -\frac{g_2(p_2, p)}{C_2},\ p_i(0) = \overline{p_i}\ , \tag{80}$$

$$\int_0^{\tau_1} g_1(p, p_1) dt = \int_0^{\tau_2} g_2(p_2, p) dt = \Delta N . \tag{81}$$

Suppose the exchange kinetics is linear

$$\begin{aligned} g_2(p_2, p) &= \alpha_2(p_2 - p), \\ g_1(p, p_1) &= \alpha_1(p - p_1) \end{aligned} . \tag{82}$$

Here from the conditions of minimal dissipation it follows that for each of the half-cycles it is optimal to maintain the constant flow of resource. As the result the equilibrium prices for buying and selling are

$$p_1^*(t) = \overline{p_1} + \frac{\Delta N}{\tau_1 C_1} t\ ,\ t \in [0, \tau_1], \tag{83}$$

$$p_2^*(t) = \overline{p_2} - \frac{\Delta N}{\tau_2 C_2} t\ ,\ t \in [0, \tau_2]. \tag{84}$$

The price offered exceeds $p_1^*(t)$ by $\delta_1 = \Delta N/(\tau_1\alpha_1)$ and is $\delta_2 = \Delta N/(\tau_2\alpha_2)$ below $p_2^*(t)$. Thus, the prices offered by the intermediary for buying and selling are

$$p^*(t) = \overline{p_1} + \frac{\Delta N}{\tau_1}\left(\frac{t}{C_1} - \frac{1}{\alpha_1}\right),$$
$$p^*(t) = \overline{p_2} - \frac{\Delta N}{\tau_2}\left(\frac{t}{C_2} + \frac{1}{\alpha_2}\right). \tag{85}$$

Substitution of the dependencies (85) into the objective (78) allows calculating the dependence of profit on τ_1, τ_2 and ΔN, and to find its maximum on these variables subject to (79). We get

$$I = \Delta N\left(\overline{p_2} - \overline{p_1}\right) - \Delta N^2\left(\frac{1}{\tau_1\alpha_1} + \frac{1}{\tau_2\alpha_2} + \frac{1}{2C_1} + \frac{1}{2C_2}\right). \tag{86}$$

The first term corresponds to the profit from equilibrium exchange. The second term describes losses due to finite time and capacities.

Quantities τ_1 and τ_2 are found by solving the problem

$$\left(\frac{1}{\tau_1\alpha_1} + \frac{1}{\tau_2\alpha_2}\right) \to \min / \tau_1 + \tau_2 = \tau. \tag{87}$$

Its solution can be reduced to the condition

$$\tau_1^* = \tau\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}}, \quad \tau_2^* = \tau\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}},$$

which gives

$$\frac{1}{\alpha_1\tau_1^*} + \frac{1}{\alpha_2\tau_2^*} = \frac{\sqrt{\alpha_1} + \sqrt{\alpha_2}}{\tau}\left(\frac{1}{\alpha_1\sqrt{\alpha_2}} + \frac{1}{\alpha_2\sqrt{\alpha_1}}\right). \tag{88}$$

The optimal trading volumes ΔN is found by maximizing I on ΔN subject to (88)

$$\Delta N^* = \frac{\left(\overline{p_2} - \overline{p_1}\right)\tau}{\tau\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + 2\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)\left(\frac{1}{\alpha_1\sqrt{\alpha_2}} + \frac{1}{\alpha_2\sqrt{\alpha_1}}\right)}. \tag{89}$$

The maximal profit is

$$I^* = \frac{\left(\overline{p_2} - \overline{p_1}\right)^2\tau}{4\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)\left(\frac{1}{\alpha_1\sqrt{\alpha_2}} + \frac{1}{\alpha_2\sqrt{\alpha_1}}\right) + 2\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\tau}. \tag{90}$$

The longer is the cycle the lower is the rate of profit $N^* = I^*/\tau$.

For simultaneous buying and selling τ_1 and τ_2 should be chosen in such a way that the following condition holds for any $\theta \in [0, \tau]$

$$\int_0^\theta g_1(p_1, p_1)dt \geq \int_0^\theta g_2(p_2, p_2)dt. \tag{91}$$

When τ_1 and τ_2 increase, the flow of resource in the optimal process decreases. Since the total amount of traded resource ΔN during its buying and selling is the same, as a rule $\tau_1 \leq \tau_2$. In particular, for linear exchange kinetics, (91) becomes an equality $\tau_1 = \tau_2 = \tau$. The prices offered to sellers and buyers obey (85), and the profit is given by (86) and (85) after substituting τ_1 and τ_2 with τ. The optimal profit is

$$\Delta N^* = \frac{\overline{p_2} - \overline{p_1}}{\frac{2}{\tau}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)}, \tag{92}$$

$$I^* = \frac{\left(\overline{p_2} - \overline{p_1}\right)^2}{\frac{4}{\tau}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) + 2\left(\frac{1}{C_1} + \frac{1}{C_2}\right)}. \tag{93}$$

Limiting rate of profit

If profit M is below I^*, then the intermediary may want to achieve this profit using minimal amount of its capital. The problem becomes

$$\begin{gathered} I_1 = \int_0^{\tau_1} pg(p_1, p)dt \to \min, \\ \int_0^{\tau_1} p_2 g(p_2, p)dt - \int_{\tau_2}^{0} pg(p_1^p)dt = M. \end{gathered} \tag{94}$$

This problem is very similar to the previous problem. The same conditions of minimal dissipation must hold here for half-cycles of buying and selling.

The problem of adjusting these half-cycles takes the form

$$I_1^*(\tau_1, \Delta N) \to \min_{\Delta N, \tau_1, \tau_2}, / \, I_2^*(\tau_1, \Delta N) - I_1^*(\tau_2, \Delta N) = M . \tag{95}$$

This gives the same expressions for the optimal τ_1^* and τ_2^* as above in the problem (87). The optimal volume of buying is

$$\Delta N^*(\lambda) = \frac{\lambda\left(\overline{p_1} - \overline{p_2}\right) - \overline{p_1}}{\lambda\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{2}{\alpha_1\tau_1^*} + \frac{2}{\alpha_2\tau_2^*}\right) + \left(\frac{1}{C_1} + \frac{2}{\alpha_2\tau_2^*}\right)}.$$

Substitution of $\Delta N^*(\lambda)$ into (95) gives equation for λ^*.

Constraints on the rate of spending

If intermediary is constraint on how much it can spend buying his stock then it is optimal for it to maximize its profit N subject to the given expenses. Since the profit rate is defined as

$$\eta = \frac{N}{U},$$

the problem is equivalent to maximization of η subject to fixed U or N.

Consider a cycle when intermediary-monopolist contacts in turns with two markets and the exchange kinetic is given by (82). The problem of optimal trading here takes the form

$$N = \left[\alpha_2\gamma_2\left(\overline{p_2} - p_2\right)p_2 - \alpha_1\gamma_1\left(p_1\overline{p_1}\right)p_1\right] \to \max , \tag{96}$$

with constrained expenses

$$U = \alpha_1\gamma_1\left(p_1 - \overline{p_1}\right)p_1 \le U^{\max} \tag{97}$$

and non-accumulation of the resource by the intermediary

$$\alpha_1\gamma_1\left(p_1 - \overline{p_1}\right) = \alpha_2\gamma_2\left(\overline{p_2} - p_2\right), \tag{98}$$

with $\gamma_1 + \gamma_2 = 1$, $\gamma_i \ge 0$, i = 1, 2.

The condition (97) can be rewritten in the following form

$$\alpha_1\left(p_1-\overline{p_1}\right)^2\left(2p_2-\overline{p_2}\right)=\alpha_2\left(\overline{p_2}-p_2\right)^2\left(2p_1-\overline{p_1}\right). \qquad (99)$$

This condition jointly with (97), (98) determines p_1, p_2, γ_1 and γ_2.

If buying and selling occur simultaneously then the cost constraint uniquely determines p_1 and p_2 and the optimal profit N. For (82) we get

$$p_1=0,5\overline{p_1}+\sqrt{0,25\overline{p_1}^2+\frac{U^{\max}}{\alpha_1}},\ p_2=\overline{p_2}-\frac{U^{\max}}{p_1\alpha_2}, \qquad (100)$$

$$N=U^{\max}\frac{p_2}{p_1}=U^{\max}\left(\frac{\overline{p_2}}{p_1}-\frac{U^{\max}}{p_1^2\alpha_2}\right). \qquad (101)$$

Optimal choice of trading composition

If an intermediary can trade in a number or resources (vector exchange) then it can optimise its performance by controlling both prices offered and composition of its trading

$$N=\sum_{i=1}^{n}N_i\left(U_i\right)\to\max \Big/ \sum_i U_i=U^{\max}, U_i\geq 0. \qquad (102)$$

Here U_i are the expenses from purchasing i-th resource. The dependence of N_i on U_i is determined by (100) and (101) after substitution of U_{max} into them.

If $N_i(U_i)$ are strictly convex functions and their derivatives at the coordinate origin are infinitely large, the optimal U_i are positive and obey equations

$$\frac{dN_i}{dU_i}=\lambda,\ \mathrm{i}=1,\ldots,n.$$

After taking into account (100) and (101) they take the form

$$\frac{\overline{p}_{2i}}{p_{1i}}-\frac{2U_i}{p_{1i}^2\alpha_{2i}}+\frac{U_i}{2\alpha_{1i}}\left(\frac{2U_i}{\alpha_{2i}p_{1i}^3}-\frac{\overline{p}_{2i}}{p_{1i}^2}\right)\frac{1}{\sqrt{0,25\overline{p}_{i1}^2+U_i/\alpha_{1i}}}=\lambda, \qquad (103)$$

with $i=1,...n$, $p_{1i}=0,5\overline{p}_{1i}+\sqrt{0,25\overline{p}_{1i}^2+U_i/\alpha_{1i}}$ and $\sum_{i=1}^{n}U_i=U^{\max}$.

Solution of the system (103) determines the optimal composition of the resources for buying and their optimal prices.

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OPTIMALNI PROCESI U IREVERZIBILNOJ MIKROEKONOMIJI

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SAŽETAK

U radu se razmatra proces optimalnog trgovanja u ekonomskim sustavima. Pristup se temelji na uzimanju u obzir faktora ireverzibilnosti putem koncepta funkcije bogatstva.

KLJUČNE RIJEČI

mikroekonomija, neravnotežna termodinamika, ireverzibilnost