Original scientific paper
Accepted 19. 12. 2011.

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Universal Hyperbolic Geometry III: First Steps in Projective Triangle Geometry

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ABSTRACT

We initiate a triangle geometry in the projective metrical setting, based on the purely algebraic approach of universal geometry, and yielding in particular a new form of hyperbolic triangle geometry. There are three main strands: the Orthocenter, Incenter and Circumcenter hierarchies, with the last two dual. Formulas using ortholinear coordinates are a main objective. Prominent are five particular points, the b, z, x, h and s points, all lying on the Orthoaxis A. A rich kaleidoscopic aspect colours the subject.

Key words: universal hyperbolic geometry, triangle geometry, projective geometry, bilinear form, ortholinear coordinates, incenter, circumcenter, orthoaxis

MSC 2010: 51M10, 14N99, 51E99

Univerzalna hiperbolička geometrija III: Prvi koraci u projektivnoj geometriji trokuta

SAŽETAK

Na temelju algebarskog pristupa univerzalne geometrije, uvodimo geometriju trokuta u projektivno-metrički okvir. To rezultira jednim novim oblikom hiperboličke geometrije trokuta. Tri su glavne okosnice: hijerarhije ortocentara, središta upisanih i središta opisanih kružnica, od kojih su posljednje dvije dualne. Primjena ortolinearnih koordinata u formulama ima bitnu ulogu. Istaknuto je pet posebnih točaka (b, z, x, h i s) koje leže na ortogonalnoj osi A. Bogato, kaleidoskopsko gledište karakterizira obradu teme.

Ključne riječi: univerzalna hiperbolička geometrije, geometrija trokuta, projektivna geometrija, bilinearna forma, ortolinearne koordinate, središte upisane kružnice, središte opisane kružnice, ortogonalna os

1 Introduction

Recently there has been a revival of interest in classical geometry and in particular the study of triangles ([6], [7], [9], [10], [12], [13], [14]). This paper introduces triangle geometry into the framework of *Universal Hyperbolic Geometry (UHG)* ([18], [19]) and beyond; in the context of a general metrical structure on the projective plane. The basic measurements of *quadrance* and *spread* replace the usual notions of *distance* and *angle*, and these depend on a general bilinear form. Hyperbolic geometry provides the motivation and is used for the illustrations. The approach is purely algebraic and works over any field not of characteristic two; the reader may easily keep the fundamental example of the *rational number field* foremost in mind. Ultimately this theory is a natural consequence of Rational Trigonometry ([15], [16], [17]).

Triangle geometry in this setting has features that resemble and also contrast with classical hyperbolic geometry, studied and described in [1], [2], [3], [4], [5], [11] and [21]. The *Orthocenter hierarchy*, involving *Altitudes, Orthic triangles,* the *Orthic axis*, the *Double triangle,* and the *Orthoaxis*, on which the important s,h,x,b and z *points* are to be found, is primary. The *Incenter* and *Circumcenter hierarchies* are precisely dual, and their existences depend on number theoretic conditions, unlike the usual Euclidean situation. The former contains the *Incenters*, *Bilines* (analogs of *vertex* or *angle bisectors*), *Bipoints, Apollonius points*, *Centrian lines*, *Sight lines, Contact points, Gergonne points* and *Nagel points* etc. The latter contains *Circumlines, Midpoints, Midlines* (analogs of *perpendicular bisectors*), *Medians, Centroids*, *Sound points*, *Tangent lines*, *Jay lines* and *Wren lines* etc. *Duality* pervades the subject; interchanging points and lines, sides and vertices, and quadrance and spread.

This paper is largely self-contained; we start with a general introduction to universal metrical projective geometry. When we study a triangle $\overline{a_1a_2a_3}$, it will prove convenient to use a linear transformation to change coordinates, so that we may assume that $a_1 = [1:0:0]$, $a_2 = [0:1:0]$ and $a_3 = [0:0:1]$, with the *orthocenter* represented by $h = [1:1:1]$. With these *ortholinear coordinates* the bilinear form is given by a pair of inverse symmetric projective matrices:

$$\mathbf{B} = \begin{bmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{bmatrix}, \quad \mathbf{A} = \mathbf{B}^{-1} = \begin{bmatrix} 1-bc & c-1 & b-1 \\ c-1 & 1-ac & a-1 \\ b-1 & a-1 & 1-ab \end{bmatrix}. \qquad (1)$$

This shifts projective triangle geometry from the study of a general triangle under a particular bilinear form to the study of a particular triangle under a general bilinear form, giving a simpler and more general theory.

Formulas will be our main aims; most of these depend on the three parameters a,b,c occurring in (1), and hopefully will provide a solid platform for further investigations. They also suggest a possible alternative to trilinear coordinates in affine/Euclidean triangle geometry. This paper introduces a rich theory which has many additional relationships and remarkable aspects which will be further studied in the coming years.

1.1 Projective linear algebra and Universal geometry

In this section we introduce the main objects: *(projective) points* and *lines*, via projective linear algebra. This is linear algebra with vectors and matrices defined only up to non-zero scalar multiples. We write the usual vectors and matrices with round brackets, while projective vectors and projective matrices, in square brackets, are by definition unchanged if we multiply all coordinates simultaneously by a non-zero number. So while $\overrightarrow{v} \equiv (3,1,2) \equiv \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$ represents a usual row vector (or 1×3 matrix), the corresponding projective row vector is $a = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$. By definition a is also equal to $\begin{bmatrix} -3 & -1 & -2 \end{bmatrix}$ or to $\begin{bmatrix} 6 & 2 & 4 \end{bmatrix}$.

We will generally use bold labels to represent projective matrices: while

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -1 & -11 \\ 0 & 2 & -2 \\ 0 & 0 & 6 \end{pmatrix}$$

denote ordinary matrices, the corresponding *projective matrices are*

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 12 \\ 0 & 9 & 3 \\ 0 & 0 & 3 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 3 & -1 & -11 \\ 0 & 2 & -2 \\ 0 & 0 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & -1 & -11 \\ 0 & 2 & -2 \\ 0 & 0 & 6 \end{bmatrix}.$$

Inverses are easier to compute in the projective setting, since determinants in the denominator can be dispensed with: for example $\mathbf{A}^{-1} = \mathbf{B}$, so that integer arithmetic only is required. While in general projective matrices cannot be added, they can be multiplied!

We now introduce additional notation and terminology that allows us to work consistently with both row and column vectors horizontally. A *non-zero* projective row vector a will be written in either of the following forms:

$$a \equiv \begin{bmatrix} x & y & z \end{bmatrix} \equiv [x:y:z]$$

and will be called a **(projective) point**. A *non-zero* projective column vector L will be written as

$$L \equiv \begin{bmatrix} l \\ m \\ n \end{bmatrix} \equiv \langle l:m:n \rangle$$

and will be called a **(projective) line**. The point $a \equiv [x:y:z]$ and the line $L \equiv \langle l:m:n\rangle$ are **incident** precisely when $lx+my+nz=0$; equivalently a **lies on** L, or L **passes through** a. The corresponding matrix equation is

$$aL \equiv \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = [x:y:z]\langle l:m:n\rangle = 0. \qquad (2)$$

Three or more points are **collinear** precisely when they all lie on a line L, and three or more lines are **concurrent** precisely when they all pass through a point a.

The **join** a_1a_2 of distinct points $a_1 \equiv [x_1:y_1:z_1]$ and $a_2 \equiv [x_2:y_2:z_2]$ is the line

$$\begin{aligned} a_1a_2 &\equiv [x_1:y_1:z_1] \times [x_2:y_2:z_2] \\ &\equiv \langle y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 \rangle. \end{aligned}$$

The **meet** L_1L_2 of distinct lines $L_1 \equiv \langle l_1:m_1:n_1\rangle$ and $L_2 \equiv \langle l_2:m_2:n_2\rangle$ is the point

$$\begin{aligned} L_1L_2 &\equiv \langle l_1:m_1:n_1\rangle \times \langle l_2:m_2:n_2\rangle \\ &\equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1]. \end{aligned}$$

These operations, using the usual Euclidean cross product, are well-defined, and will be used repeatedly in this paper. The symbol $\times$ in the linear algebra context avoids confusion with matrix multiplication.

Then *a_1a_2 is the unique line incident with both a_1 and a_2, and L_1L_2 is the unique point incident with both L_1 and L_2.* A complete symmetry or *duality* between points and lines is a key feature of this subject.

We also recall a few more definitions from [18] and [19]. A **side** $\overline{a_1a_2} \equiv \{a_1,a_2\}$ is a set of two points. A **vertex** $\overline{L_1L_2} \equiv \{L_1,L_2\}$ is a set of two lines. A **triangle** $\overline{a_1a_2a_3} \equiv \{a_1,a_2,a_3\}$ is a set of three non-collinear points, and a **trilateral** $\overline{L_1L_2L_3} \equiv \{L_1,L_2,L_3\}$ is a set of three non-concurrent lines.

A triangle $\overline{a_1a_2a_3}$ determines an **associated trilateral** $\overline{L_1L_2L_3}$, where $L_1 \equiv a_2a_3$, $L_2 \equiv a_1a_3$ and $L_3 \equiv a_1a_2$. Symmetrically a trilateral $\overline{L_1L_2L_3}$ determines an associated triangle $\overline{a_1a_2a_3}$, where $a_1 \equiv L_2L_3$, $a_2 \equiv L_1L_3$ and $a_3 \equiv L_1L_2$. The triangle $\overline{a_1a_2a_3}$ has three sides, namely $\overline{a_1a_2}$, $\overline{a_2a_3}$ and $\overline{a_1a_3}$, as well as three vertices, namely $\overline{L_1L_2}$, $\overline{L_2L_3}$ and $\overline{L_1L_3}$. In this paper we concentrate on triangles.

1.2 Projective bilinear forms

We now introduce a metrical structure on our three-dimensional vector space; this will be done via a symmetric bilinear form $\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 \equiv \overrightarrow{v}_1 A \overrightarrow{v}_2^T$ given by an invertible symmetric 3×3 matrix A, where $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are ordinary row vectors, and T denotes transpose. We wish to transfer this bilinear form to projective points and lines: let's start with perpendicularity. Recall that vectors $\overrightarrow{v}_1, \overrightarrow{v}_2$ are perpendicular precisely when $\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = 0$.

Denote by $\mathbf{A}$ and $\mathbf{B}$ the projective matrices associated to A and its inverse matrix B respectively. Points a_1 and a_2 are **perpendicular** precisely when $a_1 \mathbf{A} a_2^T = 0$, and in this case we write $a_1 \perp a_2$. This is a symmetric relation. Dually, lines L_1 and L_2 are **perpendicular** precisely when $L_1^T \mathbf{B} L_2 = 0$; we write $L_1 \perp L_2$. It is useful to restate these relations by introducing a formal notion of duality: the projective point a and the projective line L are **dual** precisely when

$$L = a^{\perp} \equiv \mathbf{A} a^T \quad \text{or equivalently} \quad a = L^{\perp} \equiv L^T \mathbf{B}.$$

So two points, or two lines, are perpendicular precisely when *one is incident with the dual of the other*. It now follows that $a_1 \perp a_2$ precisely when $a_1^{\perp} \perp a_2^{\perp}$, since the latter condition is

$$\begin{aligned} 0 &= \left(\mathbf{A} a_1^T\right)^T \mathbf{B} \left(\mathbf{A} a_2^T\right) = \left(a_1 \mathbf{A}^T\right) \mathbf{B} \left(\mathbf{A} a_2^T\right) \\ &= a_1 (\mathbf{AB}) \left(\mathbf{A} a_2^T\right) = a_1 \mathbf{A} a_2^T. \end{aligned} \tag{3}$$

A point a is **null** precisely when it is perpendicular to itself, that is, when $a \mathbf{A} a^T = 0$. Dually a line L is **null** precisely when it is perpendicular to itself, that is, when $L^T \mathbf{B} L = 0$.

Our main interests are *hyperbolic* and *elliptic geometries*, which arise respectively from the special cases

$$\mathbf{A} = \mathbf{J} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{B}, \quad \mathbf{A} = \mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}. \tag{4}$$

But other possibilities are also of interest, and *for triangle geometry also important*, as we shall soon see.

1.3 Visualization

The Figures in this paper all come from hyperbolic geometry: we represent the point $a \equiv [x : y : z]$ by the affine point $[X, Y] \equiv [x/z, y/z]$, and the line $L \equiv \langle l : m : n \rangle$ by the linear equation $lX + mY + n = 0$, which would be the hyperbolic line $(l : m : -n)$ in [18]. Null points are those a for which $x^2 + y^2 - z^2 = 0$; the corresponding affine points lie on the **null circle** $X^2 + Y^2 = 1$, always in blue. Null lines are tangent to this null circle. The duality becomes exactly the *projective polarity* between points and lines associated with the null circle.

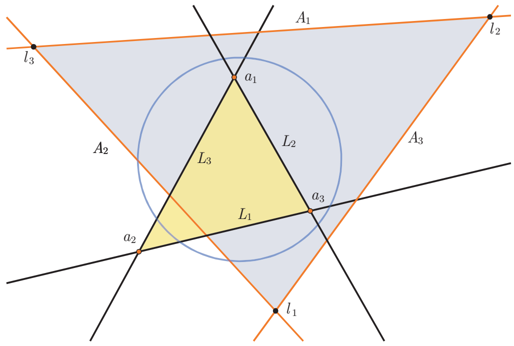

Figure 1: *A Triangle $\overline{a_1 a_2 a_3}$ and its Dual triangle $\overline{l_1 l_2 l_3}$*

We will adopt the general convention that triangle geometry constructs associated to a particular triangle are Capitalized (a familiar idea for German readers). So Figure 1 shows a Triangle $\overline{a_1 a_2 a_3}$, in yellow, with the notation we will consistently use: the **Points** of the triangle are a_1, a_2, a_3, the **Lines** are $L_1 \equiv a_2 a_3$, $L_2 \equiv a_1 a_3, L_3 \equiv a_1 a_2$, the **Dual points** are $l_1 \equiv L_1^{\perp}$, $l_2 \equiv L_2^{\perp}, l_3 \equiv L_3^{\perp}$, the **Dual lines** are $A_1 \equiv a_1^{\perp}$, $A_2 \equiv a_2^{\perp}, A_3 \equiv a_3^{\perp}$, and the **Dual triangle** is $\overline{l_1 l_2 l_3}$, in light blue. Points and their dual lines are generally pictured with the same colour.

1.4 Quadrance and spread

An inverse pair of symmetric projective matrices $\mathbf{A}$ and $\mathbf{B}$ give us more than perpendicularity: they allow the *introduction of metrical quantities into algebraic geometry*. This has been a blind spot in the history of the subject!

The **quadrance** $q(a_1, a_2)$ between points a_1 and a_2, and the **spread** $S(L_1, L_2)$ between lines L_1 and L_2, are the respective numbers

$$\begin{aligned} q(a_1, a_2) &\equiv 1 - \frac{\left(a_1 \mathbf{A} a_2^T\right)^2}{\left(a_1 \mathbf{A} a_1^T\right)\left(a_2 \mathbf{A} a_2^T\right)} \quad \text{and} \\ S(L_1, L_2) &\equiv 1 - \frac{\left(L_1^T \mathbf{B} L_2\right)^2}{\left(L_1^T \mathbf{B} L_1\right)\left(L_2^T \mathbf{B} L_2\right)}. \end{aligned} \tag{5}$$

While the numerators and denominators of these expressions depend on choices of representative vectors and matrices for $a_1, a_2, \mathbf{A}, L_1, L_2$ and $\mathbf{B}$, the *quotients are independent of scaling*, so the overall expressions are indeed well-defined projectively.

Clearly $q(a,a) = 0$ and $S(L,L) = 0$, while $q(a_1, a_2) = 1$ precisely when $a_1 \perp a_2$, and dually $S(L_1, L_2) = 1$ precisely when $L_1 \perp L_2$. An argument similar to (3) shows that for points a_1 and a_2,

$$S\left(a_1^{\perp}, a_2^{\perp}\right) = q(a_1, a_2). \tag{6}$$

Quadrance and spread are undefined if one or both of the points or lines involved is null. We will adopt the zero denominator convention: *statements involving a fraction with zero in the denominator are empty*, and a variant: *statements involving a proportion with all entries zero are empty*.

Example 1 *In the hyperbolic case, the quadrance between* $a_1 \equiv [x_1 : y_1 : z_1]$ *and* $a_2 \equiv [x_2 : y_2 : z_2]$ *is*

$$\begin{aligned} q(a_1,a_2) &\equiv 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)} \\ &= -\frac{(y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2-(x_1y_2-y_1x_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)} \end{aligned} \tag{7}$$

and the spread between $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ *and* $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ *is*

$$\begin{aligned} S(L_1,L_2) &\equiv 1 - \frac{(l_1l_2+m_1m_2-n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)} \\ &= -\frac{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2-(l_1m_2-l_2m_1)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}. \end{aligned} \tag{8}$$

Example 2 *In the elliptic case, the quadrance between* $a_1 \equiv [x_1 : y_1 : z_1]$ *and* $a_2 \equiv [x_2 : y_2 : z_2]$ *is*

$$\begin{aligned} q(a_1,a_2) &\equiv 1 - \frac{(x_1x_2+y_1y_2+z_1z_2)^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)} \\ &= \frac{(y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2+(x_1y_2-y_1x_2)^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)} \end{aligned} \tag{9}$$

and the spread between $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ *and* $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ *is*

$$\begin{aligned} S(L_1,L_2) &\equiv 1 - \frac{(l_1l_2+m_1m_2+n_1n_2)^2}{(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)} \\ &= \frac{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}{(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)}. \end{aligned} \tag{10}$$

Theorem 1 (Null quadrance/spread) *If* a_1 *and* a_2 *are distinct points, then* $q(a_1,a_2) = 0$ *precisely when* a_1a_2 *is a null line. If* L_1 *and* L_2 *are distinct lines, then* $S(L_1,L_2) = 0$ *precisely when* L_1L_2 *is a null point.*

Proof. We prove the first statement, the second follows by duality. Suppose that A is a 3×3 invertible symmetric matrix with B the adjugate matrix (the inverse of A up to a scalar), so we may write

$$A \equiv \begin{pmatrix} a & b & c \\ b & d & f \\ c & f & g \end{pmatrix}, \quad B \equiv \begin{pmatrix} dg-f^2 & cf-bg & bf-cd \\ cf-bg & ag-c^2 & bc-af \\ bf-cd & bc-af & ad-b^2 \end{pmatrix}.$$

Since $L \equiv a_1a_2$ is a null line precisely when $L^TBL = 0$, the theorem is a consequence of the following remarkable identity in the various variables, involving only vectors and the usual linear algebra:

$$\begin{aligned} &\left((x_1,y_1,z_1)A(x_1,y_1,z_1)^T\right)\left((x_2,y_2,z_2)A(x_2,y_2,z_2)^T\right) - \\ &\quad - \left((x_1,y_1,z_1)A(x_2,y_2,z_2)^T\right)^2 \\ &= (y_1z_2-y_2z_1, z_1x_2-z_2x_1, x_1y_2-x_2y_1)\cdot B \cdot \\ &\quad \cdot (y_1z_2-y_2z_1, z_1x_2-z_2x_1, x_1y_2-x_2y_1)^T. \qquad \square \end{aligned}$$

In the paper [17] we show that this general projective metrical geometry *obeys exactly the same main trigonometric laws as those of Universal Hyperbolic Geometry* as set out in the paper [18], independent of the quadratic form. In particular *the laws of trigonometry for hyperbolic and elliptic geometries, which are both projective theories, are exactly identical*. This is indeed Universal Geometry.

1.5 Linear transformations and the Fundamental theorem of projective geometry

A bilinear form $\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = \overrightarrow{v}_1 A \overrightarrow{v}_2^T$ is transformed when we change coordinates. Suppose we have an invertible linear transformation $T(\overrightarrow{v}) \equiv \overrightarrow{v}M = \overrightarrow{w}$ on three-dimensional space, acting on row vectors via right multiplication by an invertible 3×3 matrix M, with inverse matrix N, so that $\overrightarrow{w}N = \overrightarrow{v}$. Define a new bilinear form $\odot$ by

$$\begin{aligned} \overrightarrow{w}_1 \odot \overrightarrow{w}_2 \equiv (\overrightarrow{w}_1N)\cdot(\overrightarrow{w}_2N) &= (\overrightarrow{w}_1N)A(\overrightarrow{w}_2N)^T \\ &= \overrightarrow{w}_1\left(NAN^T\right)\overrightarrow{w}_2^T. \end{aligned}$$

So the matrix A for the original bilinear form $\cdot$ becomes the matrix NAN^T for the new bilinear form $\odot$.

The linear transformation T acting on row vectors induces a projective transformation $\mathbf{T}$ on one-dimensional subspaces, which are essentially (projective) points, as well as two-dimensional subspaces, which are essentially (projective) lines. Let $\mathbf{M}$ and $\mathbf{N}$ be the projective matrices associated to M and N. On points, we define $\mathbf{T}(a) = a\mathbf{M}$. To see how $\mathbf{T}$ acts on lines, we use duality; the point a is incident with the line L precisely when $aL = 0$, which is precisely when $(a\mathbf{M})(\mathbf{N}L) = 0$, so we require that $\mathbf{T}(L) \equiv \mathbf{N}L$. In this way incidence is preserved when we apply a linear transformation to both points and lines.

The notion of perpendicularity is also modified: the points a_1 and a_2 are **$\odot$-perpendicular** precisely when $a_1\mathbf{N}$ and $a_2\mathbf{N}$ are perpendicular, in other words precisely when

$a_1\left(\mathbf{NAN}^T\right)a_2^T=0$, while the lines L_1 and L_2 are $\odot$-**perpendicular** precisely when $L_1^T\left(\mathbf{M}^T\mathbf{BM}\right)L_2=0$. The inverse pair of symmetric projective matrices

$$\widetilde{\mathbf{A}}=\mathbf{NAN}^T \quad \text{and} \quad \widetilde{\mathbf{B}}=\mathbf{M}^T\mathbf{BM}$$

determine new notions of duality: $a^{\perp}=\widetilde{\mathbf{A}}a^T$ and $L^{\perp}=L^T\widetilde{\mathbf{B}}$ as well as new quadrances and spreads:

$$\widetilde{q}(a_1,a_2)\equiv 1-\frac{\left(a_1\widetilde{\mathbf{A}}a_2^T\right)^2}{\left(a_1\widetilde{\mathbf{A}}a_1^T\right)\left(a_2\widetilde{\mathbf{A}}a_2^T\right)} \quad \text{and}$$

$$\widetilde{S}(L_1,L_2)\equiv 1-\frac{\left(L_1^T\widetilde{\mathbf{B}}L_2\right)^2}{\left(L_1^T\widetilde{\mathbf{B}}L_1\right)\left(L_2^T\widetilde{\mathbf{B}}L_2\right)}. \tag{11}$$

Recall that the *Fundamental theorem of projective geometry* in this setting is really basic linear algebra: a general linear transformation of three-dimensional space maps any three linearly independent vectors $\overrightarrow{v}_1,\overrightarrow{v}_2,\overrightarrow{v}_3$ to any other three vectors. If in addition we are given a fourth vector $\overrightarrow{v}_4=\lambda_1\overrightarrow{v}_1+\lambda_2\overrightarrow{v}_2+\lambda_3\overrightarrow{v}_3$ with none of $\lambda_1,\lambda_2,\lambda_3$ zero, then we can send $\overrightarrow{v}_1,\overrightarrow{v}_2,\overrightarrow{v}_3$ respectively to $(1/\lambda_1,0,0)$, $(0,1/\lambda_2,0)$, $(0,0,1/\lambda_3)$, so that $\overrightarrow{v}_4$ is sent to $(1,1,1)$. When we view this projectively, we have essentially a proof of the Fundamental theorem: we can construct a projective linear transformation that sends four generic projective points a_1,a_2,a_3 and a_4 (no three collinear) respectively to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$.

1.6 An example with the basic Triangle

We illustrate these abstractions in a concrete example. Our basic Triangle shown in Figure 2 comes from the hyperbolic plane where the points originally have the *approximate values*:

$$a_1\approx[-0.4:0.4:1], a_2\approx[-0.7:-0.4:1], a_3\approx[0.1:0.1:1]$$

corresponding to the affine points $A_1\approx[-0.4,0.4]$, $A_2\approx[-0.7,-0.4]$, $A_3\approx[0.1,0.1]$. The following calculations are subject to round-off and approximation.

The Orthocenter, using formulas for hyperbolic geometry altitudes, is $h\approx[-0.286886:0.217349:1]$. Now

$$(x,y,z)\begin{pmatrix}-0.4 & 0.4 & 1\\ -0.7 & -0.4 & 1\\ 0.1 & 0.1 & 1\end{pmatrix}=(-0.2869,0.2173,1)$$

has the solution $(x,y,z)\approx(0.586371,0.117125,0.296503)$. We conclude that the transformation $T(v)=vN$ where

$$N\equiv\begin{pmatrix}0.586371 & 0 & 0\\ 0 & 0.117125 & 0\\ 0 & 0 & 0.296503\end{pmatrix}\begin{pmatrix}-0.4 & 0.4 & 1\\ -0.7 & -0.4 & 1\\ 0.1 & 0.1 & 1\end{pmatrix}$$
$$=\begin{pmatrix}-0.2345484 & 0.2345484 & 0.586371\\ -0.0819875 & -0.04685 & 0.117125\\ 0.0296503 & 0.0296503 & 0.296503\end{pmatrix}$$

sends $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ to multiples of $(-0.4,0.4,1)$, $(-0.7,-0.4,1)$, $(0.1,0.1,1)$ respectively, and also $(1,1,1)$ to $(-0.2869,0.2173,1)$.

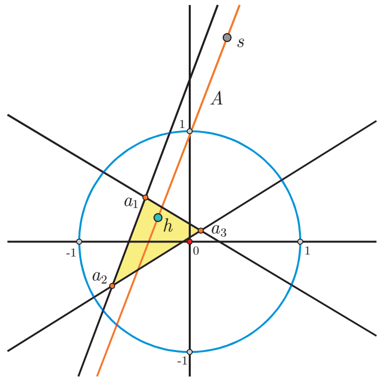

Figure 2: *Basic triangle $\overline{a_1a_2a_3}$ with Orthocenter h Orthostar s, and Orthoaxis A*

The inverse projective matrix $\mathbf{N}^{-1}=\mathbf{M}$ projectively sends the points a_1,a_2,a_3 to $[1:0:0],[0:1:0],[0:0:1]$ and h to $[1:1:1]$. Recalling the definition of $\mathbf{J}$ in (4), the bilinear form in the new standard coordinates is given (approximately) by the pair of projective inverse matrices $\mathbf{A}=\mathbf{NJN}^T$ and $\mathbf{B}=\mathbf{M}^T\mathbf{JM}$, which are

$$\mathbf{A}\approx\begin{bmatrix}-0.2338 & -0.0604 & -0.1739\\ -0.0604 & -0.00480 & -0.0385\\ -0.1739 & -0.0385 & -0.0862\end{bmatrix} \quad \text{and}$$

$$\mathbf{B}\approx\begin{bmatrix}-0.7173 & 1 & 1\\ 1 & -6.745 & 1\\ 1 & 1 & -1.692\end{bmatrix}$$

so that

$$a\approx-0.7173, \quad b\approx-6.745, \quad c\approx-1.692.$$

As an application, let's look at an important point associated to the Triangle $\overline{a_1a_2a_3}$ called the *Orthostar* $s=[a+2:b+2:c+2]$. In our example this would be the point $[1.2827:-4.745:0.308]$, and to convert that back into the original projective or hyperbolic coordinates, we would multiply by $\mathbf{N}$ to get

$$[1.2827:-4.745:0.308]\mathbf{N}\approx[0.0973:0.5322:0.2877]\approx$$
$$\approx[0.34:1.85:1]$$

which agrees approximately with the affine value for s of $[0.34,1.85]$ in Figure 2. In the same spirit, the *Orthoaxis* $A\equiv hs$ would have standard coordinates

$$[1:1:1]\times[a+2:b+2:c+2]=\langle c-b:a-c:b-a\rangle\approx$$
$$\approx\langle 5.053:0.9747:-6.0277\rangle.$$

Since this is a line, to convert back to the original coordinates we would multiply by M on the left:

$$M\begin{bmatrix}5.053\\0.9747\\-6.0277\end{bmatrix}\approx\begin{bmatrix}-47.08\\18.02\\-17.42\end{bmatrix}\approx\begin{bmatrix}2.702\\-1.03\\1.0\end{bmatrix}$$

giving the line $2.702X-1.03Y+1=0$ with projective coordinates $\langle 2.702:-1.03:1\rangle$ or hyperbolic coordinates $(-2.702:1.03:1)$. The Orthoaxis A appears in Figure 2 as the orange line.

1.7 Midpoints, midlines, bilines and bipoints

There are four more important metrical concepts that play a big role in projective triangle geometry. A side $\overline{ab}$ has a **midpoint** m precisely when m is a point lying on ab which satisfies $q(a,m)=q(m,b)$, and it has a **midline** M precisely when M is a line passing through a midpoint, perpendicular to the corresponding line ab of the side. Midlines are called *perpendicular bisectors* in Euclidean geometry; we prefer the more compact terminology, which emphasizes the duality between midpoints and midlines. Figure 3 shows our standard Triangle $\overline{a_1a_2a_3}$ that we will be using throughout this paper, together with its six Midpoints m and six Midlines M.

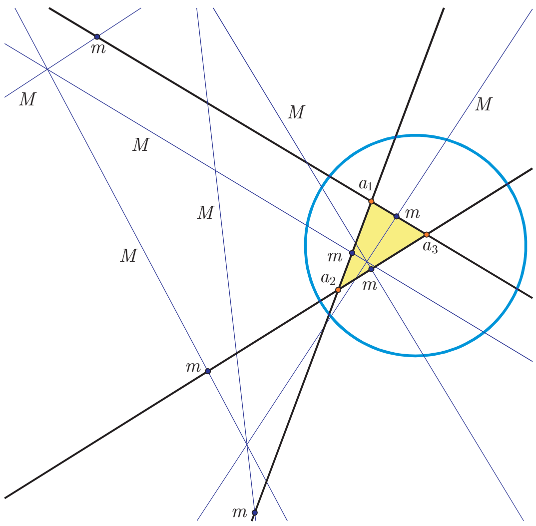

Figure 3: *Midpoints m and Midlines M of the Triangle $\overline{a_1a_2a_3}$*

Dually a vertex $\overline{KL}$ has a **biline** B precisely when B is a line passing through KL which satisfies $S(K,B)=S(B,L)$, and it has a **bipoint** b precisely when b is a point lying on a biline, perpendicular to the corresponding point KL of the vertex. Bilines are called *angle* or *vertex bisectors* in Euclidean geometry. Bipoints have no Euclidean analogs.

Figure 4 shows the six Bilines B and four of the six Bipoints b of our standard Triangle. Both Figures 3 and 4 have interesting collinearities and concurrences that the reader might like to observe; we will explore these later.

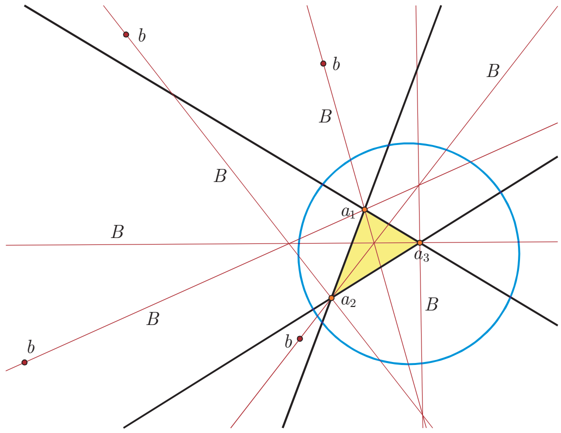

Figure 4: *Bilines B and Bipoints b*

We will see that the existence of midpoints and bilines depends on certain quadratic equations having solutions, with the consequence that sides and vertices generally have zero or two midpoints, or bilines. In a general triangle there are then several possibilities about which sides and vertices have midpoints or bilines. In future work we will explore interesting variants to these concepts which partially replace them when they do not exist.

2 Ortholinear coordinates

2.1 The Orthocenter theorem

Here is a main theorem which will be pivotal in our approach to triangle geometry in this general projective setting. There has recently been renewed interest in the Orthocenter in hyperbolic geometry ([8]); deservedly so.

Theorem 2 (Orthocenter theorem) *Suppose that $\overline{a_1a_2a_3}$ is a triangle which is not a right triangle, so that no two of the three lines $L_1\equiv a_2a_3$, $L_2\equiv a_1a_3$ and $L_3\equiv a_1a_2$ are perpendicular. Then the **altitude lines** (or just **altitudes**) $N_1\equiv a_1L_1^{\perp}$, $N_2\equiv a_2L_2^{\perp}$ and $N_3\equiv a_3L_3^{\perp}$ are defined and concurrent. Their common meet, the **Orthocenter** h, does not lie on L_1,L_2 or L_3.*

Proof. If $\overline{a_1a_2a_3}$ is not a right triangle, then none of the points a_1,a_2,a_3 are dual to the opposite lines L_1,L_2,L_3, so the three altitudes $N_1\equiv a_1L_1^{\perp}$, $N_2\equiv a_2L_2^{\perp}$ and $N_3\equiv a_3L_3^{\perp}$ are well-defined. Set $h\equiv N_1N_2$, with the idea of proving that N_3 is also incident with h. Now h does not lie on any of the lines L_1, L_2 or L_3, since otherwise $\overline{a_1a_2a_3}$ would be a right triangle, contrary to our assumption. From the Fundamental theorem of projective geometry, we can apply a linear transformation to change coordinates so that

$$a_1 = [1:0:0], a_2 = [0:1:0], a_3 = [0:0:1], h = [1:1:1].$$

It follows that

$$\begin{aligned} L_1 &= a_2a_3 = [0:1:0]\times[0:0:1] = \langle 1:0:0\rangle, \\ L_2 &= a_1a_3 = [1:0:0]\times[0:0:1] = \langle 0:1:0\rangle, \\ L_3 &= a_1a_2 = [1:0:0]\times[0:1:0] = \langle 0:0:1\rangle, \end{aligned} \tag{12}$$

and

$$\begin{aligned} N_1 &= a_1h = [1:0:0]\times[1:1:1] = \langle 0:1:-1\rangle, \\ N_2 &= a_2h = [0:1:0]\times[1:1:1] = \langle 1:0:-1\rangle. \end{aligned}$$

Suppose that the inverse projective matrix $\mathbf{B}$ for the quadratic form in these new coordinates is

$$\mathbf{B} \equiv \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

Then since $L_1 \perp N_1$

$$\langle 1:0:0\rangle^T \mathbf{B} \langle 0:1:-1\rangle = [d-e] = 0$$

and since $L_2 \perp N_2$

$$\langle 0:1:0\rangle^T \mathbf{B} \langle 1:0:-1\rangle = [d-f] = 0.$$

From these two equations we deduce that $e = f$, so that also

$$\langle 0:0:1\rangle^T \mathbf{B} \langle 1:-1:0\rangle = [e-f] = 0,$$

which implies that $a_3h = \langle 1:-1:0\rangle$ is indeed perpendicular to L_3. So $N_3 = a_3L_3^{\perp} = a_3h$ passes through h, which does not lie on L_1, L_2 or L_3. □

Theorem 3 (Ortholinear forms) *If $a_1 = [1:0:0]$, $a_2 = [0:1:0]$, $a_3 = [0:0:1]$ and $h = [1:1:1]$ is the orthocenter of $\overline{a_1a_2a_3}$, then either*

$$\mathbf{B} = \begin{bmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{bmatrix} \quad or \quad \mathbf{B} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

*The second possibility occurs precisely when $\overline{a_1a_2a_3}$ is a **fully right** triangle: any two of its lines are perpendicular.*

Proof. This follows from the proof of the previous theorem: the orthocenter being h implies that $d = e = f$. So up to a re-scaling, the possibilities are either $d = e = f = 1$ or $d = e = f = 0$.

Let us now consider the second alternative: where

$$\mathbf{B} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

and each of a, b, c is non-zero by assumption. This then yields the dual points of the triangle to be $\langle 1:0:0\rangle^T \mathbf{B} = [1:0:0]$, and also $[0:1:0]$ and $[0:0:1]$. The dual points are then exactly the same as the original points, so this is a fully right triangle: all three points and lines are mutually perpendicular. □

To summarize, we state the following result.

Theorem 4 (Ortholinear coordinates) *If the triangle $\overline{a_1a_2a_3}$ is not a right triangle, then we may change coordinates so that the bilinear form is given by the pair of projective matrices*

$$\mathbf{B} = \begin{bmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{bmatrix}, \quad \mathbf{A} = \mathbf{B}^{-1} = \begin{bmatrix} 1-bc & c-1 & b-1 \\ c-1 & 1-ac & a-1 \\ b-1 & a-1 & 1-ab \end{bmatrix} \tag{13}$$

which depend only on the three numbers a, b, c, and so that a_1, a_2, a_3 and the orthocenter h have the forms

$$a_1 \equiv [1:0:0], \ a_2 \equiv [0:1:0], \ a_3 \equiv [0:0:1], h \equiv [1:1:1].$$

We say refer to this as the **standard bilinear form**, and that $\overline{a_1a_2a_3}$ is the **standard triangle,** or just the **Triangle**. The coordinates of this framework are called **ortholinear coordinates**. *We will henceforth assume that we have made this choice of coordinates.*

The duals of the Altitudes $N_1 = \langle 0:1:-1\rangle$, $N_2 = \langle 1:0:-1\rangle, N_3 = \langle 1:-1:0\rangle$ are the **Altitude points**

$$\begin{aligned} n_1 &= N_1^T\mathbf{B} = [0:b-1:1-c], \\ n_2 &= N_2^T\mathbf{B} = [a-1:0:1-c], \\ n_3 &= N_3^T\mathbf{B} = [a-1:1-b:0]. \end{aligned} \tag{14}$$

The dual of the Orthocenter $h = [1:1:1]$ is the **Ortholine**

$$H = \mathbf{A}[1:1:1]^T = \langle b+c-bc-1 : a+c-ac-1 : a+b-ab-1\rangle.$$

Theorem 5 (Null points/lines) *The point $p \equiv [x:y:z]$ in Ortholinear coordinates is a null point precisely when*

$$(1-bc)x^2 + (1-ac)y^2 + (1-ab)z^2 + \\ + 2(c-1)xy + 2(b-1)xz + 2(a-1)yz = 0.$$

The line $L \equiv \langle l:m:n\rangle$ is a null line precisely when

$$al^2 + bm^2 + cn^2 + 2lm + 2ln + 2mn = 0.$$

Proof. These follow by using (13) to expand the respective conditions

$$[x:y:z]\,\mathbf{A}\,[x:y:z]^T = 0 \quad \text{and}$$

$$\langle l:m:n\rangle^T\,\mathbf{B}\,\langle l:m:n\rangle = 0. \qquad \square$$

Corollary 1 *Using ortholinear coordinates, the Points $a_1 \equiv [1:0:0]$, $a_2 \equiv [0:1:0]$ and $a_3 \equiv [0:0:1]$ are null points precisely when $bc = 1, ac = 1$ and $ab = 1$ respectively, and the Lines $L_1 \equiv \langle 1:0:0\rangle$, $L_2 \equiv \langle 0:1:0\rangle$ and $L_3 \equiv \langle 0:0:1\rangle$ are null lines precisely when $a = 0, b = 0$ and $c = 0$ respectively.*

Define

$$D \equiv abc - a - b - c + 2. \tag{15}$$

Then it is straightforward to check that

$$\det B = \det\begin{pmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix} = D \quad \text{and}$$

$$\det A = \det\begin{pmatrix} 1-bc & c-1 & b-1 \\ c-1 & 1-ac & a-1 \\ b-1 & a-1 & 1-ab \end{pmatrix} = -D^2.$$

Theorem 6 (Triangle quadrances and spreads) *Using Ortholinear coordinates, the quadrances $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$, $q_3 \equiv q(a_1,a_2)$ and spreads $S_1 \equiv S(L_2,L_3)$, $S_2 \equiv S(L_1,L_3)$, $S_3 \equiv S(L_1,L_2)$ of the standard Triangle $\overline{a_1a_2a_3}$ are*

$$q_1 = \frac{-Da}{(ab-1)(ac-1)},\ q_2 = \frac{-Db}{(ab-1)(bc-1)},$$
$$q_3 = \frac{-Dc}{(ac-1)(bc-1)}$$

and

$$S_1 = \frac{bc-1}{bc},\ S_2 = \frac{ac-1}{ac},\ S_3 = \frac{ab-1}{ab}.$$

These numbers also satisfy

$$1-q_1 = \frac{(a-1)^2}{(ab-1)(ac-1)},$$
$$1-q_2 = \frac{(b-1)^2}{(ab-1)(bc-1)},$$
$$1-q_3 = \frac{(c-1)^2}{(ac-1)(bc-1)} \tag{16}$$

and

$$1-S_1 = \frac{1}{bc},\ 1-S_2 = \frac{1}{ac},\ 1-S_3 = \frac{1}{ab}. \tag{17}$$

Proof. These are straightforward computations. $\square$

Although it will play only a small role in this paper, we also introduce the most important number associated to the Triangle, and a formula for it in terms of a,b,c.

Theorem 7 (Triangle quadrea) *The **quadrea** $\mathcal{A}$ of the triangle $\overline{a_1a_2a_3}$ is*

$$\mathcal{A} \equiv q_2q_3S_1 = q_1q_3S_2 = q_1q_2S_3 = \frac{D^2}{(ab-1)(ac-1)(bc-1)}.$$

Proof. This follows directly from the formulas of the previous theorem. $\square$

We cannot help but point out an important trigonometric formula that follows from this: the *Extended Spread law* asserts that

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3} = \frac{\mathcal{A}}{q_1q_2q_3}.$$

2.2 Cevians, traces, Desargues theorem and Canonical lines

Consider a variable point $p \equiv [x:y:z]$ distinct from the Points a_1,a_2,a_3 of the Triangle $\overline{a_1a_2a_3}$. The lines a_1p, a_2p, a_3p are the **Cevian lines**, or just **Cevians,** of p. These are

$$a_1p = [x:y:z]\times[1:0:0] = \langle 0:z:-y\rangle,$$
$$a_2p = [x:y:z]\times[0:1:0] = \langle z:0:-x\rangle,$$
$$a_3p = [x:y:z]\times[0:0:1] = \langle y:-x:0\rangle.$$

The points $t_1 \equiv (a_1p)L_1, t_2 \equiv (a_2p)L_2, t_3 \equiv (a_3p)L_1$ are the **trace points**, or just **traces**, of p. These are

$$t_1 = \langle 0:z:-y\rangle \times \langle 1:0:0\rangle = [0:y:z],$$
$$t_2 = \langle z:0:-x\rangle \times \langle 0:1:0\rangle = [x:0:z],$$
$$t_3 = \langle y:-x:0\rangle \times \langle 0:0:1\rangle = [x:y:0].$$

Theorem 8 (Desargues theorem) *Suppose that $p \equiv [x:y:z]$ is a point that does not lie on any of the Lines of the triangle, with traces t_1,t_2,t_3. Then the points $g_1 \equiv (t_2t_3)L_1, g_2 \equiv (t_1t_3)L_2, g_3 \equiv (t_1t_2)L_3$ are collinear, and their join is the line $S(p) \equiv \langle yz:xz:xy\rangle$.*

Proof. Using the formulas above for the traces, we compute

$$\begin{aligned} g_1 \equiv (t_2t_3)L_1 &= \langle -yz:xz:xy\rangle \times \langle 1:0:0\rangle \\ &= [0:xy:-xz] = [0:y:-z], \\ g_2 \equiv (t_1t_3)L_2 &= \langle yz:-xz:xy\rangle \times \langle 0:1:0\rangle \\ &= [xy:0:-yz] = [x:0:-z], \\ g_3 \equiv (t_1t_2)L_3 &= \langle yz:xz:-xy\rangle \times \langle 0:0:1\rangle \\ &= [xz:-yz:0] = [x:-y:0]. \end{aligned}$$

We have used the fact that x,y,z are all non-zero, by assumption, to cancel these common factors as they occur. The points g_1,g_2,g_3 are collinear since

$$\det\begin{pmatrix} 0 & y & -z \\ x & 0 & -z \\ x & -y & 0 \end{pmatrix} = 0$$

and their join is

$$[0:y:-z]\times[x:0:-z] = \langle yz:xz:xy\rangle \equiv S(p). \qquad \square$$

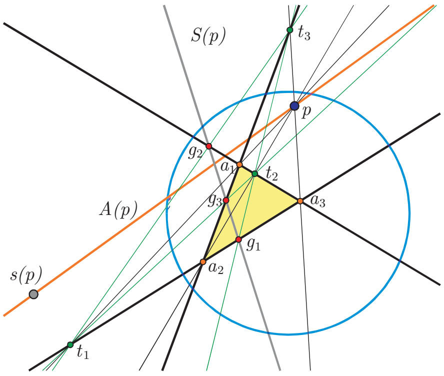

Figure 5: *Cevians, traces and the lines $S(p)$ and $A(p)$*

Associated to p is the dual of the line $S(p)$:

$$\begin{aligned} s(p) &\equiv S(p)^{\perp} = \langle yz:xz:xy\rangle^{T}\mathbf{B} \\ &= [xy+xz+ayz : xy+yz+bxz : xz+yz+cxy]. \end{aligned}$$

Furthermore, the join of p and $s(p)$

$$\begin{aligned} A(p) \equiv ps(p) = \langle & y^2z-yz^2+cxy^2-bxz^2 : \\ & xz^2-x^2z+ayz^2-cyx^2 : \\ & x^2y-xy^2+bzx^2-azy^2\rangle \end{aligned}$$

is the **canonical line** of the generic point p. This is an interesting and important construction that is not available in Euclidean geometry, and it has many applications. In the special case when $p=h$, the Orthocenter of $\overline{a_1a_2a_3}$, the canonical line $A\equiv A(h)$ will be called the *Orthoaxis* of the triangle, and will be seen to be the most important line in triangle geometry.

There is also a dual formulation: consider a line M distinct from the Lines L_1,L_2,L_3. The points L_1M,L_2M,L_3M are the **Menelaus points** of M. If $M\equiv\langle l:m:n\rangle$ then the Menelaus points are

$$L_1M=[0:n:-m],\ L_2M=[n:0:-l],\ L_3M=[m:-l:0].$$

The lines $T_1\equiv(L_1M)a_1, T_2\equiv(L_2M)a_2, T_3\equiv(L_3M)a_3$ are the **trace lines** of M, these are

$$T_1=\langle 0:m:n\rangle,\ T_2=\langle l:0:n\rangle,\ T_3=\langle l:m:0\rangle.$$

Theorem 9 (Desargues dual theorem) *Suppose that $M=\langle l:m:n\rangle$ is a line that does not pass through any of the Points of the triangle, with trace lines T_1,T_2,T_3. Then the lines $(T_2T_3)a_1,(T_1T_3)a_2,(T_1T_2)a_3$ are concurrent, and they pass through the point $[mn:ln:lm]$.*

Proof. This is dual to the previous theorem. $\square$

Note that the transforms implicit in both these theorems are of the form $x:y:z\to x^{-1}:y^{-1}:z^{-1}$ which makes it clear that they are inverses of each other.

2.3 Existence of midpoints and bilines

Theorem 10 (Side midpoints) *Suppose that p_1 and p_2 are non-null, non-perpendicular points, forming a non-null side $\overline{p_1p_2}$. Then $\overline{p_1p_2}$ has a non-null midpoint m precisely when $1-q(p_1,p_2)$ is a square, and in this case there are exactly two perpendicular midpoints m.*

Proof. We suppose without loss of generality that $p_1=a_1\equiv[1:0:0]$ and $p_2=a_2\equiv[0:1:0]$ so that by the Triangle quadrances and spreads theorem

$$1-q(p_1,p_2)=\frac{(c-1)^2}{(bc-1)(ac-1)}.$$

By assumption each of $c-1, bc-1$ and $ac-1$ are nonzero. An arbitrary point m on $ab=\langle 0:0:1\rangle$ has the form $m=[x:y:0]$, which is null precisely when $(bc-1)x^2+(ac-1)y^2+2(1-c)xy=0$, by the Null point theorem.

Assuming that m is non-null, we compute that

$$\begin{aligned} q(p_1,m) &= \frac{Dcy^2}{(bc-1)((bc-1)x^2+(ac-1)y^2+2(1-c)xy)} \\ q(p_2,m) &= \frac{Dcx^2}{(ac-1)((bc-1)x^2+(ac-1)y^2+2(1-c)xy)}. \end{aligned}$$

By assumption $\overline{p_1p_2}$ is non-null, so by the Corollary to the Null points/lines theorem, $c\neq 0$, and so the above expressions are equal precisely when $x^2(bc-1)=y^2(ac-1)$ has a solution, which occurs precisely when $1-q(p_1,p_2)$ is a square. In fact if

$$\frac{1}{(bc-1)(ac-1)}=r^2 \tag{18}$$

then the two midpoints are $m=[(ac-1)r:\pm1:0]$, and they are perpendicular, since

$$\begin{aligned} & [(ac-1)r:1:0]\,\mathbf{A}\,[(ac-1)r:-1:0]^T \\ &= (ac-1)\left(1-(bc-1)(ac-1)r^2\right)=0. \end{aligned} \qquad \square$$

We refer to the pair of midpoints m of a side as **opposites**. It follows that the dual midline M of a midpoint m passes through the opposite midpoint. While the next theorem is dual to the previous one, we give a direct proof.

Theorem 11 (Vertex bilines) *Suppose that L_1 and L_2 are non-null non-perpendicular lines forming a non-null vertex $\overline{L_1L_2}$. Then $\overline{L_1L_2}$ has a non-null biline B precisely when $1-S(L_1,L_2)$ is a square, and in this case there are exactly two perpendicular bilines B.*

Proof. We suppose without loss of generality that $L_1 = \langle 1:0:0\rangle$ and $L_2 = \langle 0:1:0\rangle$, so that from the Triangle quadrances/spreads theorem

$$1-S(L_1,L_2) = \frac{1}{ab}.$$

An arbitrary line through $L_1L_2 = [0:0:1]$ has the form $B = \langle l:m:0\rangle$, which by the Null spread theorem is null precisely when $al^2+bm^2+2lm = 0$, and then

$$S(L_1,B) = \frac{(ab-1)m^2}{a(al^2+bm^2+2lm)} \quad \text{and}$$
$$S(L_2,B) = \frac{(ab-1)l^2}{b(al^2+bm^2+2lm)}.$$

By assumption $\overline{L_1L_2}$ is non-null, so by the Corollary to the Null points/lines theorem, $ab-1 \neq 0$, and so the above expressions are equal precisely when $l^2a = m^2b$ has a solution, which occurs precisely when $1-S(L_1,L_2)$ is a square. In fact if

$$\frac{1}{ab} = w^2 \tag{19}$$

then the two bilines are $B = \langle l:m:0\rangle = \langle bw:\pm 1:0\rangle$ and they are perpendicular since

$$\langle bw:1:0\rangle^T \mathbf{B} \langle bw:-1:0\rangle = b\left(abw^2-1\right) = 0. \qquad \square$$

We refer to the pair of bilines B of a vertex as **opposites**. It follows that the dual bipoint b of a biline B lies on the opposite biline.

3 Orthocenter Hierarchy

We now initiate our study of triangle geometry constructions involving *perpendicularity*. The focus is on the *Orthocenter h* and various other key points that are related to the most important line in the subject: the *Orthoaxis A*. The computations are based on ortholinear coordinates; finding meets and joins, which essentially amount to taking cross products; and finding duals, either by multiplying transposes of points by $\mathbf{A}$ (on the left) or transposes of lines by $\mathbf{B}$ (on the right). Our goal is to establish formulas for important points and lines to facilitate the understanding of relationships between them: the reader is encouraged to follow along and check our computations, which are mostly elementary.

3.1 Triangle lines, dual points, dual lines

We start with a review of the basic Triangle $\overline{a_1a_2a_3}$, whose Points a and Lines L are

$$a_1 = [1:0:0],\ a_2 = [0:1:0],\ a_3 = [0:0:1] \quad \text{and}$$
$$L_1 = \langle 1:0:0\rangle,\ L_2 = \langle 0:1:0\rangle,\ L_3 = \langle 0:0:1\rangle.$$

The **Dual points** $l_1 \equiv L_1^\perp, l_2 \equiv L_2^\perp, l_3 \equiv L_3^\perp$ are the duals of the Lines L, and the **Dual lines** $A_1 \equiv a_1^\perp, A_2 \equiv a_2^\perp, A_3 \equiv a_3^\perp$ are the duals of the Points a. These are

$$l_1 = [a:1:1],\ l_2 = [1:b:1],\ l_3 = [1:1:c] \quad \text{and}$$
$$A_1 = \langle 1-bc : c-1 : b-1\rangle,$$
$$A_2 = \langle c-1 : 1-ac : a-1\rangle,$$
$$A_3 = \langle b-1 : a-1 : 1-ab\rangle.$$

The **Altitudes** are $N_1 \equiv a_1l_1, N_2 \equiv a_2l_2, N_3 \equiv a_3l_3$, and the **Altitude dual points** are $n_1 \equiv A_1L_1, n_2 \equiv A_2L_2, n_3 \equiv A_3L_3$. These are, as established previously,

$$N_1 = \langle 0:1:-1\rangle,\ N_2 = \langle 1:0:-1\rangle,\ N_3 = \langle 1:-1:0\rangle \ \text{and}$$
$$n_1 = [0:b-1:1-c],\ n_2 = [a-1:0:1-c],$$
$$n_3 = [a-1:1-b:0].$$

The dual of the Orthocenter $h = [1:1:1]$ is the Ortholine

$$H = \langle b+c-bc-1 : a+c-ac-1 : a+b-ab-1\rangle.$$

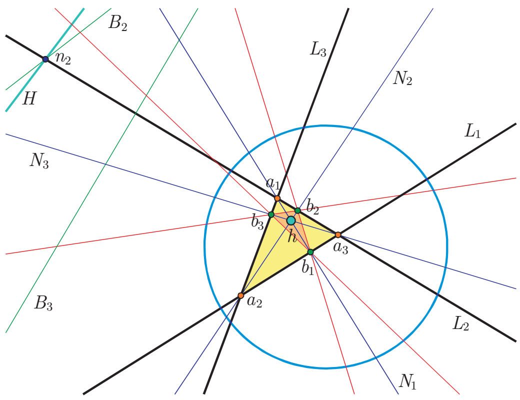

Figure 6: *Altitudes, Base points, Orthocenter h, Orthline H and Orthic triangle*

The **Base points** $b_1 \equiv N_1L_1, b_2 \equiv N_2L_2, b_3 \equiv N_3L_3$ are the meets of corresponding Altitudes N and Lines L, and the **Base lines** $B_1 \equiv n_1l_1, B_2 \equiv n_2l_2, B_3 \equiv n_3l_3$ are their duals. These are

$$b_1 = [0:1:1],\ b_2 = [1:0:1],\ b_3 = [1:1:0] \quad \text{and}$$
$$B_1 = \langle b+c-2 : a(1-c) : a(1-b)\rangle,$$
$$B_2 = \langle b(1-c) : a+c-2 : b(1-a)\rangle,$$
$$B_3 = \langle c(1-b) : c(1-a) : a+b-2\rangle.$$

The **Orthic lines** $C_1 \equiv b_2b_3, C_2 \equiv b_1b_3, C_3 \equiv b_1b_2$ are the joins of Base points b, and the **Orthic points** $c_1 \equiv B_2B_3$, $c_2 \equiv B_1B_3, c_3 \equiv B_1B_2$ are the meets of Base lines B. These are

$$C_1 = \langle -1:1:1\rangle,\ C_2 = \langle 1:-1:1\rangle,\ C_3 = \langle 1:1:-1\rangle,$$
$$c_1 = [2-a:b:c],\ c_2 = [a:2-b:c],\ c_3 = [a:b:2-c].$$

The **Orthic triangle** $\overline{b_1b_2b_3}$ is perspective with the Triangle $\overline{a_1a_2a_3}$, with center of perspectivity the Orthocenter h, since the Altitudes are the lines of perspectivity.

Theorem 12 (Triangle Base center) *The **Orthic dual triangle** $\overline{c_1c_2c_3}$ is perspective with the Triangle $\overline{a_1a_2a_3}$, and the center of perspectivity is the **Base center** $b = [a:b:c]$.*

Proof. We compute the lines

$$a_1c_1 = [1:0:0] \times [2-a:b:c] = \langle 0:-c:b\rangle,$$
$$a_2c_2 = [0:1:0] \times [a:2-b:c] = \langle c:0:-a\rangle,$$
$$a_3c_3 = [0:0:1] \times [a:b:2-c] = \langle -b:a:0\rangle$$

and check that these are all incident with $b \equiv [a:b:c]$. □

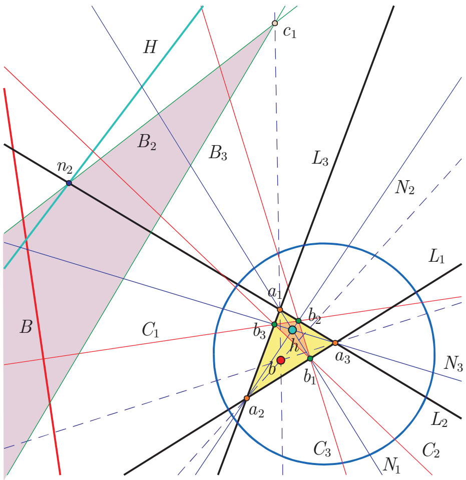

Figure 7: *Orthic dual triangle $\overline{c_1c_2c_3}$ and Base center b*

Note that there is a bit of duplication of symbols here, the letter b being used in the same formula with two different meanings, hopefully without undue confusion. The Base center is an important triangle point, as we shall see; its dual is the **Base axis**

$$B \equiv \langle a-b-c+2bc-abc: \\ -a+b-c+2ac-abc: \\ -a-b+c+2ab-abc\rangle.$$

Figure 7 shows the three Orthic lines C_1, C_2, C_3 but only one of the Orthic points, namely c_1, since the other points are off the screen. The mathematical symmetry between points and lines is not respected by our biology; lines tend to be more visible, while points are simpler.

3.2 Orthic axis and Orthoaxis (they are different!)

The **Desargues points** $g_1 \equiv C_1L_1$, $g_2 \equiv C_2L_2$, $g_3 \equiv C_3L_3$ are the meets of corresponding Orthic lines C and Lines L, and the **Desargues lines** $G_1 \equiv c_1l_1$, $G_2 \equiv c_2l_2$, $G_3 \equiv c_3l_3$ are the joins of corresponding Orthic points and Dual points. These are

$$g_1 = [0:1:-1],\ g_2 = [1:0:-1],\ g_3 = [1:-1:0] \quad \text{and}$$
$$G_1 = \langle b-c:a+ac-2:2-a-ab\rangle,$$
$$G_2 = \langle 2-ba-b:c-a:b+ab-2\rangle,$$
$$G_3 = \langle c+bc-2:2-ac-c:a-b\rangle.$$

Theorem 13 (Triangle orthic axis) *The Desargues points g_1, g_2, g_3 are collinear, and lie on the **Orthic axis** $S \equiv \langle 1:1:1\rangle$. The Desargues lines G_1, G_2, G_3 are concurrent, and pass through the **Orthostar** $s \equiv [a+2:b+2:c+2]$. The Orthic axis and Orthostar are dual.*

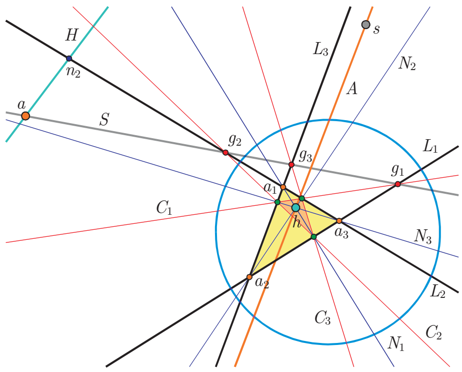

Figure 8: *Desargues points g, Orthic axis S, Orthoaxis A*

Proof. The Desargues points g_1, g_2, g_3 are collinear either by Desargues theorem applied to the Cevian triangle $\overline{b_1b_2b_3}$ of the Orthocenter h, or directly since

$$\det \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = 0.$$

Their join, the Orthic axis, is

$$S \equiv g_1g_2 = [0:1:-1] \times [1:0:-1] = \langle -1:-1:-1\rangle \\ = \langle 1:1:1\rangle.$$

Dually the Desargues lines G_1, G_2, G_3 are concurrent, which we can check by evaluating the corresponding determinant. The common point through which they pass is the Orthostar

$$s \equiv S^{\perp} = \langle 1:1:1\rangle^T B = [a+2:b+2:c+2].$$

This is clearly dual to the Orthic axis. □

We now come to the feature attraction of this paper: the **Orthoaxis** A is the join of the Orthocenter h and the Orthostar s, or equivalently the canonical line of h. It is

$$A \equiv hs = [1:1:1] \times [a+2:b+2:c+2] \\ = \langle c-b : a-c : b-a \rangle .$$

Note that the Orthoaxis A is perpendicular to the Orthic axis S, since the Orthostar s lies on the Orthoaxis. *The Orthoaxis is the most important line in projective triangle geometry.*

The **Orthoaxis point** $a \equiv HS$ is the dual of the Orthoaxis; it is

$$a = A^{\perp} = \langle c-b : a-c : b-a \rangle^{T} \mathbf{B} \\ = [(a-1)(b-c) : (b-1)(a-c) : (c-1)(a-b)] .$$

Theorem 14 (Base center on Orthoaxis) *The Base center b lies on the Orthoaxis A.*

Proof. We check incidence between the Base center $b = [a:b:c]$ and the Orthoaxis $A = \langle c-b : a-c : b-a \rangle$:

$$bA = [a:b:c] \langle c-b : a-c : b-a \rangle \\ = [a(c-b) + b(a-c) + c(b-a)] = 0. \quad \square$$

The **AntiOrthic lines** $T_1 \equiv a_1g_1, T_2 \equiv a_2g_2, T_3 \equiv a_3g_3$ are the joins of corresponding Points a and Desargues points g, and the **AntiOrthic points** $t_1 \equiv A_1G_1, t_2 \equiv A_2G_2, t_3 \equiv A_3G_3$ are the meets of corresponding Dual lines A and Desargues lines G. They have the form

$$T_1 = \langle 0:1:1 \rangle, \ T_2 = \langle 1:0:1 \rangle, \ T_3 = \langle 1:1:0 \rangle \quad \text{and}$$
$$t_1 = [2 : b+1 : c+1],$$
$$t_2 = [a+1 : 2 : c+1],$$
$$t_3 = [a+1 : b+1 : 2].$$

The **AntiBase points** $e_1 \equiv T_2T_3, e_2 \equiv T_1T_3, e_3 \equiv T_1T_2$ are the meets of AntiOrthic lines T, and the **AntiBase lines** $E_1 \equiv t_2t_3, E_2 \equiv t_1t_3, E_3 \equiv t_1t_2$ are the joins of AntiOrthic points t. They have the form

$$e_1 = [-1:1:1], \ e_2 = [1:-1:1], \ e_3 = [1:1:-1]$$

and

$$E_1 = \langle b+c+bc-3 : (1-c)(a+1) : (1-b)(a+1) \rangle,$$
$$E_2 = \langle (1-c)(b+1) : (a+c+ac-3) : (1-a)(b+1) \rangle,$$
$$E_3 = \langle (1-b)(c+1) : (1-a)(c+1) : (a+b+ab-3) \rangle.$$

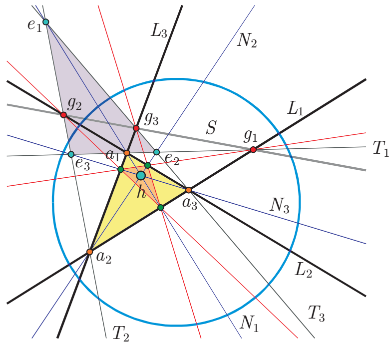

Figure 9: *AntiBase points and the AntiOrthic triangle* $\overline{e_1e_2e_3}$

Theorem 15 (AntiOrthic perspectivity) *The* ***AntiOrthic triangle*** $\overline{e_1e_2e_3}$ *and the Triangle* $\overline{a_1a_2a_3}$ *are perspective from the Orthocenter h.*

Proof. This is equivalent to the statement that the Altitudes N pass through the corresponding AntiBase points e. For example N_1 is incident with e_1, since $e_1N_1 = [-1:1:1]\langle 0:1:-1 \rangle = 0$, and similarly N_2 is incident with e_2, and N_3 is incident with e_3. $\square$

It is perhaps worth mentioning that one can work out formulas for these various constructs directly in hyperbolic geometry in terms of the coordinates of a general triangle. However this proves rather taxing; even the Orthocenter involves for each coefficient a homogeneous polynomial of degree six with 24 terms. The system presented here punches far above its weight, as the relative simplicity of the formulas so far confirms.

3.3 Parallels and the Double triangle

In universal hyperbolic geometry, the notion of parallel is more specialized than in classical hyperbolic geometry. We do not refer to two lines (or two points) as being parallel. Rather we refer to **a line P through a point a being parallel to a line L**: it means that P is perpendicular to the altitude from a to L. In Euclidean geometry this is like defining parallel lines to be "perpendicular to a perpendicular": a local definition rather than a global one. This motivates the important construction of the Double triangle of a Triangle.

The **Parallel lines** $P_1 \equiv a_1n_1$, $P_2 \equiv a_2n_2$, $P_3 \equiv a_3n_3$ are the joins of corresponding Points a and Altitude points n, and the **Parallel points** $p_1 \equiv A_1N_1, p_2 \equiv A_2N_2, p_3 \equiv A_3N_3$ are their duals: meets of corresponding Dual lines A and

Altitudes N. These are

$$P_1 = \langle 0 : c-1 : b-1 \rangle, \; P_2 = \langle c-1 : 0 : a-1 \rangle,$$
$$P_3 = \langle b-1 : a-1 : 0 \rangle \quad \text{and}$$
$$p_1 = [2-b-c : 1-bc : 1-bc],$$
$$p_2 = [1-ac : 2-a-c : 1-ac],$$
$$p_3 = [1-ab : 1-ab : 2-a-b].$$

The **Double points** $d_1 \equiv P_2P_3$, $d_2 \equiv P_1P_3$, $d_3 \equiv P_1P_2$ are the meets of Parallel lines P, and the **Double lines** $D_1 \equiv p_2p_3$, $D_2 \equiv p_1p_3$, $D_3 \equiv p_1p_2$ are the joins of Parallel points p. These are

$$d_1 = [1-a : b-1 : c-1], \; d_2 = [a-1 : 1-b : c-1],$$
$$d_3 = [a-1 : b-1 : 1-c] \quad \text{and}$$
$$D_1 = \langle a+2b+2c-bc-abc-3 : (ac-1)(b-1) : (ab-1)(c-1) \rangle,$$
$$D_2 = \langle (bc-1)(a-1) : 2a+b+2c-ac-abc-3 : (ab-1)(c-1) \rangle,$$
$$D_3 = \langle (bc-1)(a-1) : (ac-1)(b-1) : 2a+2b+c-ab-abc-3 \rangle.$$

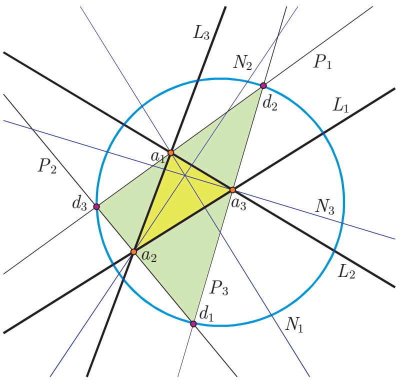

Figure 10: *Parallel lines and the Double triangle $\overline{d_1d_2d_3}$*

The triangle $\overline{d_1d_2d_3}$ is the **Double triangle** of the Triangle $\overline{a_1a_2a_3}$. The following theorems seem remarkable.

Theorem 16 (Double triangle midpoint) *The Points a_1, a_2, a_3 are midpoints of the Double triangle $\overline{d_1d_2d_3}$.*

Proof. Using the expressions above, we compute that

$$q(a_3, d_1) = \frac{D(a+b-2)}{4a+4b+4c-ab-ac-bc-a^2-b^2-c^2+abc^2+ab^2c+a^2bc-4abc-5}$$
$$= q(a_3, d_2)$$

where D is the determinant defined in (15), and where a common factor of $(ab-1)$ in the numerator and denominator has been cancelled provided that $ab \neq 1$. So a_3 is a midpoint of $\overline{d_1d_2}$. Similarly a_1 is a midpoint of $\overline{d_2d_3}$, and a_2 is a midpoint of $\overline{d_1d_3}$. □

Theorem 17 (Double triangle null points) *The Double triangle $\overline{d_1d_2d_3}$ has a null point precisely when all of its points are null points, and this occurs precisely when the quadrea $\mathcal{A}$ of the Triangle $\overline{a_1a_2a_3}$ is equal to* 1.

Proof. Using the Null point theorem, $d_1 = [1-a : b-1 : c-1]$ is null precisely when

$$0 = (1-bc)(1-a)^2 + (1-ac)(b-1)^2 + (1-ab)(c-1)^2$$
$$+ (2c-2)(1-a)(b-1) + (2b-2)(1-a)(c-1) +$$
$$+ (2a-2)(b-1)(c-1).$$

After expanding and simplifying, the right hand side is the symmetric expression

$$5-4a-4b-4c+ab+ac+bc+a^2+b^2+c^2+$$
$$+4abc-abc^2-ab^2c-a^2bc.$$

This same expression arises for the nullity of $d_2 = [a-1 : 1-b : c-1]$ and $d_3 = [a-1 : b-1 : 1-c]$, so if one point of the Double triangle is null, so are the other two.

Using the Triangle quadrea theorem, the difference between the quadrea $\mathcal{A}$ of the Triangle and 1 is

$$\frac{(abc-b-c-a+2)^2}{(ab-1)(ac-1)(bc-1)} - 1 =$$
$$\frac{5-4a-4b-4c+ab+ac+bc+a^2+b^2+c^2+4abc-abc^2-ab^2c-a^2bc}{(bc-1)(ac-1)(ab-1)}.$$

So the Double triangle has null points precisely when the quadrea $\mathcal{A}$ is equal to 1. □

In our standard example, the Triangle has approximate quadrea $\mathcal{A} \approx 1.04$, which explains why the points of the Double triangle in Figure 10 appear close to being null points.

Theorem 18 (Double triangle perspectivity) *The Double triangle $\overline{d_1d_2d_3}$ and the Triangle $\overline{a_1a_2a_3}$ are perspective from a point, the **Double point,** or x **point,** which is $x \equiv [a-1 : b-1 : c-1]$. The x point lies on the Orthoaxis A.*

Proof. We compute the lines

$$a_1d_1 = [1:0:0] \times [1-a : b-1 : c-1] = \langle 0 : 1-c : b-1 \rangle,$$
$$a_2d_2 = [0:1:0] \times [a-1 : 1-b : c-1] = \langle c-1 : 0 : 1-a \rangle,$$
$$a_3d_3 = [0:0:1] \times [a-1 : b-1 : 1-c] = \langle 1-b : a-1 : 0 \rangle.$$

These lines are concurrent (compute a determinant), and the common meet is

$$x = \langle 0 : 1-c : b-1 \rangle \times \langle c-1 : 0 : 1-a \rangle$$
$$= \left[(a-1)(c-1) : (b-1)(c-1) : (c-1)^2 \right]$$
$$= [a-1 : b-1 : c-1].$$

We have cancelled a common factor $c-1$; should this be zero, then the three lines are concurrent since two of them are equal. The x point lies on the Orthoaxis $A \equiv hs$ since $[a-1:b-1:c-1]$ is a (projective) linear combination of the Orthocenter $h \equiv [1:1:1]$ and the Orthostar $s \equiv [a+2:b+2:c+2]$. $\square$

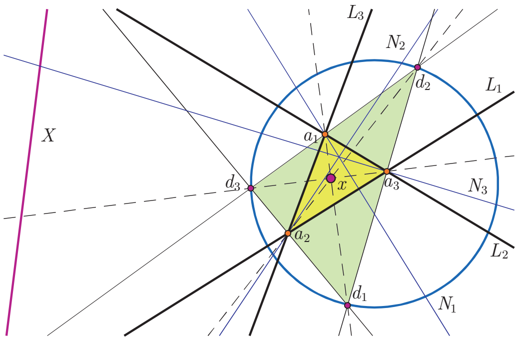

Figure 11: *The Double point, or x point*

The dual of the x point is the X **line**

$$\begin{aligned} X = \langle & a-2b-2c+3bc-abc+1: \\ & -2a+b-2c+3ac-abc+1: \\ & -2a-2b+c+3ab-abc+1\rangle. \end{aligned}$$

Theorem 19 (Double dual triangle perspectivity) *The Double triangle $\overline{d_1d_2d_3}$ and the Dual triangle $\overline{l_1l_2l_3}$ are perspective from a point, the* ***Double dual point,*** *or z* ***point****, which is $z \equiv [a+1:b+1:c+1]$. The z point lies on the Orthoaxis.*

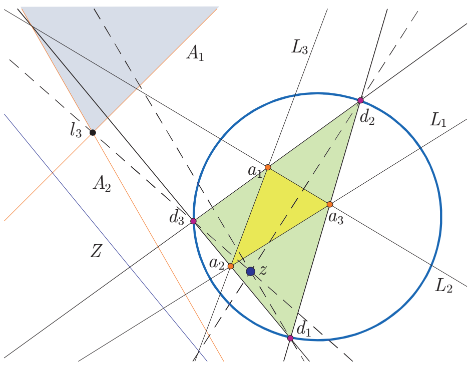

Figure 12: *The Double dual point, or z point*

Proof. We compute the lines

$$\begin{aligned} l_1d_1 &= [a:1:1] \times [1-a:b-1:c-1] \\ &= \langle c-b:1-ac:ab-1\rangle, \\ l_2d_2 &= [1:b:1] \times [a-1:1-b:c-1] \\ &= \langle bc-1:a-c:1-ab\rangle, \\ l_3d_3 &= [1:1:c] \times [a-1:b-1:1-c] \\ &= \langle 1-bc:ac-1:b-a\rangle. \end{aligned}$$

These lines are concurrent and their common meet is $z \equiv [a+1:b+1:c+1]$. This point lies on the Orthoaxis since it is a (projective) linear combination of the Orthocenter $h \equiv [1:1:1]$ and the Orthostar $s \equiv [a+2:b+2:c+2]$. $\square$

The dual of the z point is the Z **line**

$$Z \equiv \langle a+bc-abc-1:b+ac-abc-1:c+ab-abc-1\rangle.$$

The **AltDual lines** $K_1 \equiv a_1n_1$, $K_2 \equiv a_2N_2$, $K_3 \equiv a_3n_3$ are the joins of corresponding Points a and Altitude points n, and the **AltDual points** $k_1 \equiv A_1N_1$, $k_2 \equiv A_2N_2$, $k_3 \equiv A_3N_3$ are meets of corresponding Dual lines A and Altitudes N. These are

$$\begin{aligned} & K_1 = \langle 0:c-1:b-1\rangle, \; K_2 = \langle c-1:0:a-1\rangle, \\ & K_3 = \langle b-1:a-1:0\rangle \quad \text{and} \\ & k_1 = [2-b-c:1-bc:1-bc], \\ & k_2 = [1-ac:2-a-c:1-ac], \\ & k_3 = [1-ab:1-ab:2-a-b]. \end{aligned}$$

Clearly the **AltDual triangle** $\overline{k_1k_2k_3}$ is perspective to the Triangle, since the Altitudes pass through both Points and AltDual points, but in addition, as Figure 13 suggests, it is also in perspective with the Double triangle. The center of perspectivity, the **AltDual point** k, does not generally lie on the Orthoaxis. We leave this result to the reader, as well as the following exercise investigating other perspective centers of various secondary triangles associated to our basic Triangle.

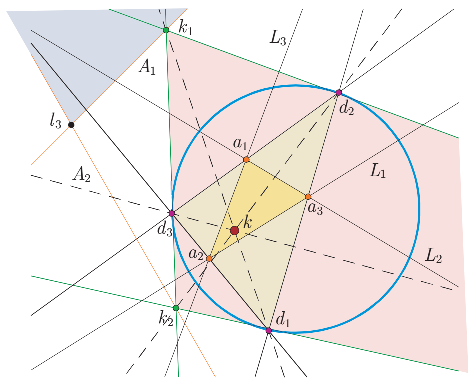

Figure 13: *AltDual triangle $\overline{k_1k_2k_3}$ and the AltDual point k*

Exercise 1 *Show that the following pairs of triangles are perspective, and find the centers of perspectivity: i) $\overline{c_1c_2c_3}$ and $\overline{p_1p_2p_3}$, ii) $\overline{b_1b_2b_3}$ and $\overline{d_1d_2d_3}$, iii) $\overline{b_1b_2b_3}$ and $\overline{l_1l_2l_3}$, iv) $\overline{c_1c_2c_3}$ and $\overline{d_1d_2d_3}$, v) $\overline{c_1c_2c_3}$ and $\overline{l_1l_2l_3}$, vi) $\overline{k_1k_2k_3}$ and $\overline{d_1d_2d_3}$. Can you find more?*

3.4 Special points on the Orthoaxis

There are five interesting, and related, points on the Orthoaxis $A = \langle c-b : a-c : b-a \rangle$, namely the points $z = [a+1 : b+1 : c+1]$, $b = [a:b:c]$, $x = [a-1 : b-1 : c-1]$, $h = [1:1:1]$ and $s = [a+2 : b+2 : c+2]$. Of course there very well may be more! In Figure 14 we see them in this particular order.

Theorem 20 (Orthoaxis harmonic ranges) *The points z,b,x,h form a harmonic range. The points z,b,h,s also form a harmonic range.*

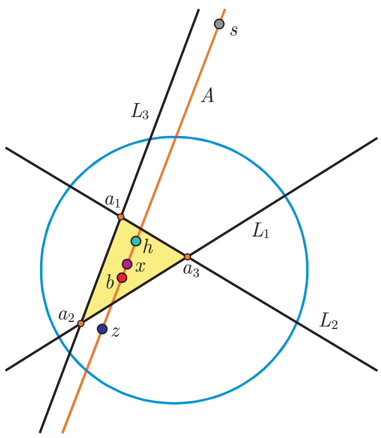

Figure 14: *Orthoaxis A and points z,b,x,h and s*

Proof. Recall that the points z,b,x,h form a harmonic range precisely when the cross ratio $R(z,x:b,h) = -1$. Define the following vectors which represent each of the five points:

$$\overrightarrow{z} = (a+1, b+1, c+1),$$
$$\overrightarrow{b} = (a,b,c),$$
$$\overrightarrow{x} = (a-1, b-1, c-1),$$
$$\overrightarrow{h} = (1,1,1),$$
$$\overrightarrow{s} = (a+2, b+2, c+2).$$

Then $\overrightarrow{z} + \overrightarrow{x} = 2\overrightarrow{b}$ and $\overrightarrow{z} - \overrightarrow{x} = 2\overrightarrow{h}$ so that b and h are harmonic conjugates with respect to z and x, so that $R(z,x:b,h) = -1$. Similarly $\overrightarrow{z} + \overrightarrow{h} = \overrightarrow{s}$ and $\overrightarrow{z} - \overrightarrow{h} = \overrightarrow{b}$ so that b and s are harmonic conjugates with respect to z and h, so that $R(z,h:b,s) = -1$. □

There are three more cross ratios naturally determined by the five points. We will leave it to the reader to check that in addition

$$R(z,x:b,s) = -3, R(b,h:x,s) = -1/2, R(z,h:x,s) = -2.$$

Theorem 21 (Second double triangle perspectivity) *The Triangle $\overline{a_1a_2a_3}$ and the double triangle of the Double triangle $\overline{d_1d_2d_3}$ are perspective from the **Second double point**, or y **point.***

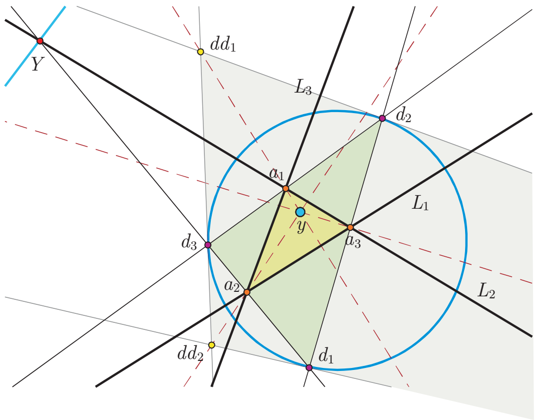

Figure 15: *The double of the Double triangle and the y point*

Proof. We leave the proof to the reader. The formula for y is somewhat lengthy to write out. □

It is worth pointing out that in general the y point does *not* lie on the Orthoaxis A, although it often, as in our example, gets very close! It is also worth noting that the obvious pattern does not appear to continue; the double of the double of the Double triangle is *not* in general perspective with the original Triangle.

4 Incenter Hierarchy

Although the Incenter and Circumcenter hierarchies are exactly dual, we treat first the former, which is closer to the Euclidean situation and so more familiar. It is also simpler, on account of the difference between (18) and (19).

4.1 Bilines, Incenters and Apollonians

From the Vertex bilines theorem, Bilines of the Triangle exist precisely when the spreads S_1, S_2, S_3 have the property that $1-S_1, 1-S_2, 1-S_3$ are all squares. In Figure 16, where we are working approximately, this amounts to each of these quantities being positive, which they are, and so there are six Bilines B, two opposite ones for each vertex, and six dual Bipoints b, two opposite ones lying on each Dual line.

From the Triangle quadrances and spreads theorem, Bilines exist precisely when we can find u,v,w satisfying the **quadratic relations**

$$\frac{1}{bc} = u^2, \; \frac{1}{ac} = v^2, \; \frac{1}{ab} = w^2. \qquad (20)$$

But there is also a **cubic relation**

$$\frac{1}{abc} = uvw \tag{21}$$

which we are able to impose, by taking the product of the three quadratic relations (20), and possibly changing the sign of any or all of u, v, w (they must all be non-zero).

These relations (20) and (21) will play an essential role in what follows; any triple $\{u, v, w\}$ satisfying them gives rise to three more such triples: namely $\{u, -v, -w\}, \{-u, v, -w\}$ and $\{-u, -v, w\}$. So there is a *fourfold Klein-type symmetry* occurring here. Another implication is that we also have the relations

$$u = avw, \ v = buw, \ w = cuv. \tag{22}$$

We saw at the end of the proof of the Vertex bilines theorem, that Bilines for the vertex $\overline{L_1L_2}$ are $B = \langle bw : \pm 1 : 0 \rangle$. We see now from (22) that these can be rewritten as $\langle v : u : 0 \rangle$ and $\langle v : -u : 0 \rangle$.

So the Bilines B are

$$\begin{aligned} &\langle 0 : w : v \rangle, \langle 0 : w : -v \rangle \quad \text{through} \quad a_1, \\ &\langle w : 0 : u \rangle, \langle w : 0 : -u \rangle \quad \text{through} \quad a_2, \\ &\langle v : u : 0 \rangle, \langle v : -u : 0 \rangle \quad \text{through} \quad a_3. \end{aligned}$$

The Bipoints b are dual, and are

$$\begin{aligned} &[v+w : v+bw : w+cv], \ [v-w : v-bw : -w+cv] \\ &\qquad \text{on} \quad A_1, \\ &[u+aw : u+w : w+cu], \ [u-aw : u-w : -w+cu] \\ &\qquad \text{on} \quad A_2, \\ &[u+av : v+bu : u+v], \ [u-av : v-bu : -u+v] \\ &\qquad \text{on} \quad A_3. \end{aligned}$$

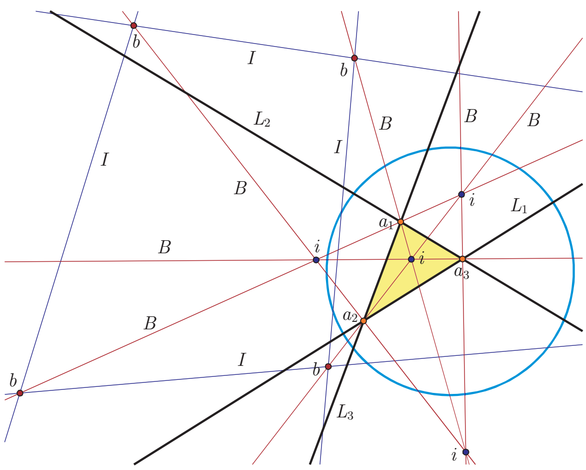

Figure 16: *Bilines, Bipoints, Incenters and Inlines*

Theorem 22 (Incenters) *Bilines B are concurrent in threes, meeting at four **Incenters** i_0, i_1, i_2 and i_3. Bipoints b are collinear in threes, joining on four **Inlines** I_0, I_1, I_2 and I_3.*

Proof. The following triples of Bilines B are concurrent:

$$\begin{aligned} &\langle 0 : w : -v \rangle, \ \langle w : 0 : -u \rangle, \ \langle v : -u : 0 \rangle \\ &\qquad \text{through} \quad i_0 \equiv [u : v : w], \\ &\langle 0 : w : -v \rangle, \ \langle w : 0 : u \rangle, \ \langle v : u : 0 \rangle \\ &\qquad \text{through} \quad i_1 \equiv [-u : v : w], \\ &\langle 0 : w : v \rangle, \ \langle w : 0 : -u \rangle, \ \langle v : u : 0 \rangle \\ &\qquad \text{through} \quad i_2 \equiv [u : -v : w], \\ &\langle 0 : w : v \rangle, \ \langle w : 0 : u \rangle, \ \langle v : -u : 0 \rangle \\ &\qquad \text{through} \quad i_3 \equiv [u : v : -w]. \end{aligned}$$

We check this by computing

$$\begin{aligned} &\det \begin{pmatrix} 0 & w & -v \\ w & 0 & -u \\ v & -u & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & w & -v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & w & v \\ w & 0 & -u \\ v & u & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & -u & 0 \end{pmatrix} = 0. \end{aligned}$$

The corresponding meets are $\langle 0 : w : -v \rangle \times \langle w : 0 : -u \rangle = \left[-uw : -vw : -w^2\right] = [u : v : w] \equiv i_0$ and similarly for the other Incenters. The situation with Bipoints b is dual. $\square$

At this point, it appears that there is no intrinsic reason to prefer one Incenter over the others; our notation seems somewhat arbitrary. However it is possible that this symmetry may eventually be broken.

Apollonian points a are meets of Bilines B and corresponding Lines L. There are six; they have the form

$$\begin{aligned} &[0 : v : w], \ [0 : v : -w], \\ &[u : 0 : w], \ [u : 0 : -w], \\ &[u : v : 0], \ [u : -v : 0] \end{aligned} \tag{23}$$

on L_1, L_2, L_3 respectively. The **Apollonian lines** A are joins of Bipoints b and corresponding Dual points l.

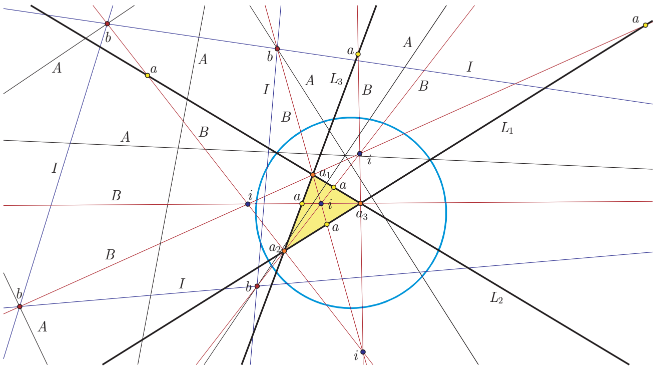

Figure 17: *Apollonian points a and lines A*

Theorem 23 (Apollonian harmonic conjugates) *The two Apollonian points on a side of the Triangle are harmonic conjugates with respect to the two Points of that side.*

Proof. Consider the side $\overline{a_2a_3}$ with Apollonian points $[0:w:v]$ and $[0:w:-v]$. Then it is a standard fact that the projective points determined by the vectors $(0,v,w)$ and $(0,v,-w)$ are harmonic conjugates with respect to those determined by the vectors $(0,v,0)$ and $(0,0,w)$. □

We leave the dual result concerning Apollonian lines to the reader.

A famous property of the Apollonian points in the Euclidean case is that the three circles built from pairs of these as diameters meet at the two *Isodynamic points*. This property is modified in the projective setting, by introducing an *important variant of a circle*. Given two points a and b, the **Thaloid** of the side $\overline{ab}$ is the locus of a point p satisfying the property that $pa \perp pb$, or equivalently

$$S(pa,pb) = 1. \tag{24}$$

It is straightforward that this is a conic. It is not generally a (metrical) circle, but shares some of its properties.

An **Apollonian Thaloid** is a Thaloid of a side consisting of two Apollonian points, both on a Line of the triangle. There are three Apollonian Thaloids, one for each side of the Triangle.

Theorem 24 (Isodynamic points) *If two Apollonian Thaloids meet at a point s, then the third does too.*

Proof. The equations of the three Thaloids are obtained directly from the forms of the Apollonians in (23) and the defining relation (24):

$$bcx^2 - acy^2 - 2bzx + 2azy = 0,$$
$$acy^2 - abz^2 - 2cxy + 2bxz = 0,$$
$$abz^2 - bcx^2 - 2ayz + 2cxy = 0.$$

If we add these three equations we get zero on both sides, so they are dependent. So if two Thaloids have a common point s, then this is shared by the third. □

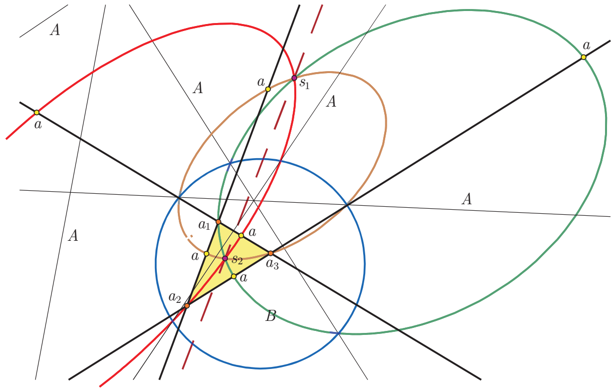

Figure 18: *Apollonian Thaloids and Isodynamic points s_1, s_2*

Such a common point of the three Apollonian Thaloids is an **Isodynamic point**; Figure 18 shows two such: s_1 and s_2, together with the line through them. In the Euclidean case this is the **Brocard line**, which also passes through the orthocenter—here in the projective situation that is not generally the case, even though it appears to in this example. In the Euclidean case the centers of the three Apollonian circles are collinear, falling on the **Lemoine line**: something analogous happens here but it requires additional ideas which we leave for another occasion.

4.2 Centrians, InCentrians, Contact points and Incircles

Theorem 25 (Centrian lines) *The Apollonian points a are collinear in threes, joining on four **Centrian lines** J. The Apollonian lines A are concurrent in threes, meeting at four **Centrian points** j.*

Proof. The following triples of Apollonian points are collinear:

$$[0:v:-w],\ [u:0:-w],\ [u:-v:0] \text{ on } J_0 \equiv \langle vw:uw:uv\rangle,$$
$$[0:v:-w],\ [u:0:w],\ [u:v:0] \text{ on } J_1 \equiv \langle -vw:uw:uv\rangle,$$
$$[0:v:w],\ [u:0:-w],\ [u:v:0] \text{ on } J_2 \equiv \langle vw:-uw:uv\rangle,$$
$$[0:v:w],\ [u:0:w],\ [u:-v:0] \text{ on } J_3 \equiv \langle vw:uw:-uv\rangle.$$

The collinearities may easily be checked by computing determinants. The corresponding meets are $[0:v:-w]\times[u:0:-w]=\langle vw:uw:uv\rangle\equiv J_0$ and similarly for the other Centrian lines. The situation with the Apollonian lines A is dual; here are the formulas for the Centrian points:

$$\begin{aligned}
j_0 &\equiv [uv+uw+avw : uv+vw+buw : uw+vw+cuv],\\
j_1 &\equiv [uv+uw-avw : uv-vw+buw : uw-vw+cuv],\\
j_2 &\equiv [uv-uw+avw : uv+vw-buw : vw-uw+cuv],\\
j_3 &\equiv [uw-uv+avw : vw-uv+buw : uw+vw-cuv]. \quad \square
\end{aligned}$$

The four Incenters and the four Centrian points are corresponding, since the three Bilines that meet in an Incenter also give rise to the three Apollonian points lying on a particular Centrian line, which is dual to a particular Centrian point. The **InCentrian lines** are joins of corresponding Incenters i and Centrian points j. These have the form

$$\begin{aligned}
i_0 j_0 &= \langle cv-bw : aw-cu : bu-av\rangle,\\
i_1 j_1 &= \langle cv-bw : aw+cu : -bu-av\rangle,\\
i_2 j_2 &= \langle -cv-bw : aw-cu : bu+av\rangle,\\
i_3 j_3 &= \langle cv+bw : -aw-cu : bu-av\rangle.
\end{aligned}$$

While it is easy to check that the Incenters lie on these InCentrian lines, showing that the Centrian points do so requires the quadratic relations. (The reader is encouraged to check this).

The **InCentrian points** are meets of corresponding Inlines I and Centrian lines J, and are dual to the InCentrian lines.

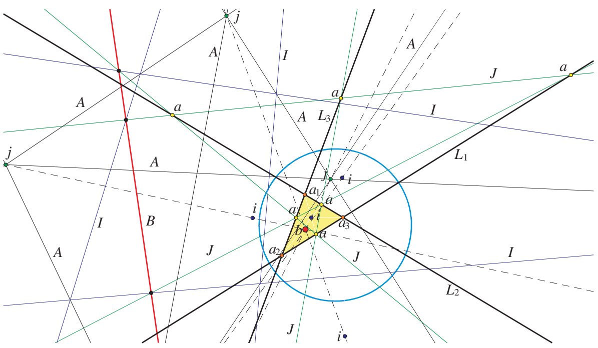

Figure 19: *Centrians, Incentrians and Base center b*

Theorem 26 (InCentrian center) *The four InCentrian lines are concurrent, and meet at the Base center b. The four InCentrian points are collinear, and join on the Base axis B.*

Proof. The InCentrian line $i_0 j_0=\langle cv-bw : aw-cu : bu-av\rangle$ passes through $b=[a:b:c]$ since

$$\begin{aligned}
&[a:b:c]\,\langle cv-bw : aw-cu : bu-av\rangle\\
&\quad=[(cv-bw)\,a+(aw-cu)\,b+(bu-av)\,c]=0.
\end{aligned}$$

Similarly $j_1 i_1$, $j_2 i_2$, $j_3 i_3$ also pass through b. The second statement follows by duality. $\square$

The **InDual** lines are joins of Incenters i and Dual points l. They may also be described as altitudes from Incenters to the Lines, and there are 12. The InDual lines associated to the Incenters are:

$$\begin{aligned}
&\langle v-w : aw-u : u-av\rangle,\ \langle v-bw : w-u : bu-v\rangle,\\
&\quad \langle cv-w : w-cu : u-v\rangle \quad \text{to} \quad i_0=[u:v:w],\\
&\langle v-w : aw+u : -u-av\rangle,\ \langle v-bw : w+u : -bu-v\rangle,\\
&\quad \langle cv-w : w+cu : -u-v\rangle \quad \text{to} \quad i_1=[-u:v:w],\\
&\langle -v-w : aw-u : u+av\rangle,\ \langle -v-bw : w-u : bu+v\rangle,\\
&\quad \langle -cv-w : w-cu : u+v\rangle \quad \text{to} \quad i_2=[u:-v:w],\\
&\langle v+w : -aw-u : u-av\rangle,\ \langle v+bw : -w-u : bu-v\rangle,\\
&\quad \langle cv=w : -w-cu : u-v\rangle \quad \text{to} \quad i_3=[u:v:-w].
\end{aligned}$$

The **InDual points** are meets of Inlines I and Lines L, and are dual to InDual lines.

The **Contact points** are meets of corresponding InDual lines and Lines L; there are a total of 12. They may also be described as the bases of the altitudes from the Incenters to the Lines. The Contact points associated to the Incenters are:

$$\begin{aligned}
&[0:u-av:u-aw],\ [v-bu:0:v-bw],\\
&\quad [w-cu:w-cv:0] \quad \text{to} \quad i_0=[u:v:w],\\
&[0:u+av:u+aw],\ [v+bu:0:v-bw],\\
&\quad [w+cu:w-cv:0] \quad \text{to} \quad i_1=[-u:v:w],\\
&[0:u+av:u-aw],\ [v+bu:0:v+bw],\\
&\quad [w-cu:w+cv:0] \quad \text{to} \quad i_2=[u:-v:w],\\
&[0:u-av:u+aw],\ [v-bu:0:v+bw],\\
&\quad [w+cu:w+cv:0] \quad \text{to} \quad i_3=[u:v:-w].
\end{aligned}$$

The **Contact lines** are joins of corresponding InDual points and Dual points l. Figure 20 shows the InDual lines and Contact points. The latter are intimately connected with important conics associated to the Triangle—the Incircles.

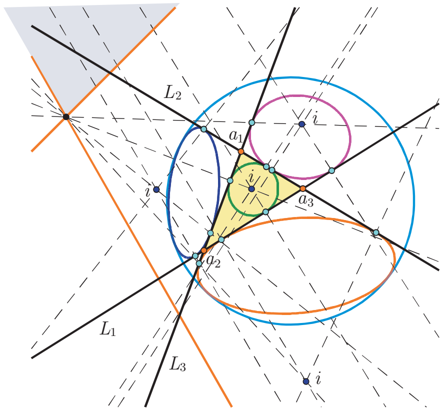

Figure 20: *InDual lines, Contact points and Incircles*

A **circle** is the locus of a point p satisfying $q(p,c) = K$ for some fixed point c and some fixed number K: this is a conic. Circles centered at the Incenters and passing through the associated Contact points are tangent to the Lines at these points, and are called **Incircles**; there are in this case four, and these are also shown in Figure 20.

Note that in this particular case the two Incircles whose centers are outside the null circle have a quite different character from the two with interior centers. The former are often called 'curves of constant width' in the classical literature, and are tangent to the null circle at the points where the dual of the center (in this case an Inline) meets it. Circles can take on different forms, appearing in our affine view also as hyperbolas outside the null circle, tangent to it at these same points. See [18] for some pictures; also the video *UnivHypGeom25: Geometer's Sketchpad and Visualizing circles in Universal Hyperbolic Geometry* in the YouTube playlist [20].

4.3 Sight lines, Gergonne and Nagel points

A **Sight line** is the join of a Contact point with the Point a opposite to the Line that it lies on. A **Sight point** is the dual of a Sight line. There are 12 Sight lines; three associated to each Incenter, and four incident with each Point. The Sight lines associated to the Incenters are:

$$\begin{aligned}
&\langle 0 : u-aw : -u+av\rangle,\ \langle v-bw : 0 : -v+bu\rangle,\\
&\qquad \langle w-cv : -w+cu, 0\rangle \quad \text{to} \quad i_0 = [u:v:w],\\
&\langle 0 : -u-aw : u+av\rangle,\ \langle v-bw : 0 : -v-bu\rangle,\\
&\qquad \langle w-cv : -w-cu, 0\rangle \quad \text{to} \quad i_1 = [-u:v:w],\\
&\langle 0 : u-aw : -u-av\rangle,\ \langle -v-bw : 0 : v+bu\rangle,\\
&\qquad \langle w+cv : -w+cu, 0\rangle \quad \text{to} \quad i_2 = [u:-v:w],\\
&\langle 0 : u+aw : -u+av\rangle,\ \langle v+bw : 0 : -v+bu\rangle,\\
&\qquad \langle -w-cv : w+cu, 0\rangle \quad \text{to} \quad i_3 = [u:v:-w].
\end{aligned}$$

Theorem 27 (Gergonne points) *The three Sight lines associated to an Incenter meet at a **Gergonne point** g. These are:*

$$\begin{aligned}
g_0 &= [-abcu+acv+abw-1 : bcu-abcv+abw-1 :\\
&\qquad bcu+acv-abcw-1],\\
g_1 &= [abcu+acv+abw+1 : -bcu-abcv+abw+1 :\\
&\qquad -bcu+acv-abcw+1],\\
g_2 &= [-abcu-acv+abw+1 : bcu+abcv+abw+1 :\\
&\qquad bcu-acv-abcw+1],\\
g_3 &= [-abcu+acv-abw+1 : bcu-abcv-abw+1 :\\
&\qquad bcu+acv+abcw+1].
\end{aligned}$$

Proof. We check that g_0 as defined is incident with the Sight line $\langle 0 : u-aw : -u+av\rangle$ by computing

$$\begin{aligned}
&[-abcu+acv+abw-1 : bcu-abcv+abw-1 :\\
&\quad bcu+acv-abcw-1]\,\langle 0 : u-aw : -u+av\rangle\\
&= \left[-a\left(v-w+abw^2-acv^2-buw+cuv\right)\right] = 0
\end{aligned}$$

using the quadratic relations and (22). The computations for the other Sight lines and g_1, g_2, g_3 are similar. □

An **InGergonne line** is the meet of a corresponding Incenter i and Gergonne point g. An **InGergonne point** is the join of a corresponding Inline and Gergonne line G. The four InGergonne lines are

$$\begin{aligned}
g_0 i_0 &= \langle v-w-bcuv+bcuw : w-u+acuv-acvw :\\
&\qquad u-v-abuw+abvw\rangle,\\
g_1 i_1 &= \langle w-v+bcuv-bcuw : -w-u-acuv-acvw :\\
&\qquad u+v+abuw+abvw\rangle,\\
g_2 i_2 &= \langle v+w+bcuv+bcuw : u-w-acuv+acvw :\\
&\qquad -v-u-abuw-abvw\rangle,\\
g_3 i_3 &= \langle v+w+bcuv+bcuw : -u-w-acuv-acvw :\\
&\qquad u-v-abuw+abvw\rangle.
\end{aligned}$$

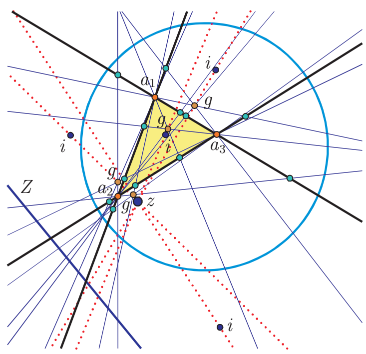

Figure 21: *Sight lines, Gergonne points g, InGergonne lines and the z point*

Theorem 28 (InGergonne center) *The four InGergonne lines are concurrent, and meet at the z point.*

Proof. We check that $g_0 i_0$ passes through $z = [a+1 : b+1 : c+1]$ by computing

$$\begin{aligned}&[a+1:b+1:c+1]\cdot\\&\langle v-w-bcuv+bcuw:w-u+acuv-acvw:u-v-abuw+abvw\rangle\\&=[av-bu-aw+cu+bw-cv-abuw+acuv+abvw\\&\quad -bcuv-acvw+bcuw]\\&=(a-b)(cuv-w)+(c-a)(buw-v)+(b-c)(avw-u)=0\end{aligned}$$

where we have used the relations (22). Similarly $g_1 i_1$, $g_2 i_2$, $g_3 i_3$ also pass through the z point. □

Theorem 29 (Nagel points) *The following triples of Sight lines are concurrent. Each triple involves one Sight line associated to each of the Incenters, and so is associated to the Incenter with which it does not share a Sight line:*

$$\begin{aligned}&\langle 0:-u-aw:u+av\rangle,\quad \langle -v-bw:0:v+bu\rangle,\\&\quad\langle -w-cv:w+cu:0\rangle \quad\text{to}\quad i_0=[u:v:w],\\&\langle 0:u-aw:-u+av\rangle,\quad \langle v+bw:0:-v+bu\rangle,\\&\quad\langle w+cv:-w+cu:0\rangle \quad\text{to}\quad i_1=[-u:v:w],\\&\langle 0:u+aw:-u+av\rangle,\quad \langle v-bw:0:-v+bu\rangle,\\&\quad\langle w-cv:-w-cu:0\rangle \quad\text{to}\quad i_2=[u:-v:w],\\&\langle 0:u-aw:-u-av\rangle,\quad \langle v-bw:0:-v-bu\rangle,\\&\quad\langle w-cv:-w+cu:0\rangle \quad\text{to}\quad i_3=[u:v:-w].\end{aligned}$$

The points where these triples meet are the ***Nagel points***

$$\begin{aligned}n_0&=[abcu+acv+abw+1:bcu+abcv+abw+1:\\&\quad bcu+acv+abcw+1],\\n_1&=[abcu-acv-abw+1:bcu-abcv-abw+1:\\&\quad bcu-acv-abcw+1],\\n_2&=[-abcu+acv-abw+1:-bcu+abcv-abw+1:\\&\quad -bcu+acv-abcw+1],\\n_3&=[-abcu-acv+abw+1:-bcu-abcv+abw+1:\\&\quad -bcu-acv+abcw+1].\end{aligned}$$

Proof. We check that n_0 as defined is incident with $\langle 0:-u-aw:u+av\rangle$ by computing

$$\begin{aligned}&[abcu+acv+abw+1:bcu+abcv+abw+1:\\&\quad bcu+acv+abcw+1]\,\langle 0:-u-aw:u+av\rangle\\&=\left[a\left(v-w-abw^2+acv^2-buw+cuv\right)\right]=0\end{aligned}$$

using the quadratic relations and (22). The computations for the other Sight lines and n_1, n_2, n_3 are similar. □

The joins of Incenters i and corresponding Nagel points n are the **InNagel lines**. They are

$$\begin{aligned}n_0 i_0&=\langle w-v-bcuv+bcuw:u-w+acuv-acvw:\\&\quad v-u-abuw+abvw\rangle,\\n_1 i_1&=\langle v-w+bcuv-bcuw:w+u-acuv-acvw:\\&\quad -u-v+abuw+abvw\rangle,\\n_2 i_2&=\langle -v-w+bcuv+bcuw:-u+w-acuv+acvw:\\&\quad v+u-abuw-abvw\rangle,\\n_3 i_3&=\langle -v-w+bcuv+bcuw:u+w-acuv-acvw:\\&\quad -u+v-abuw+abvw\rangle.\end{aligned}$$

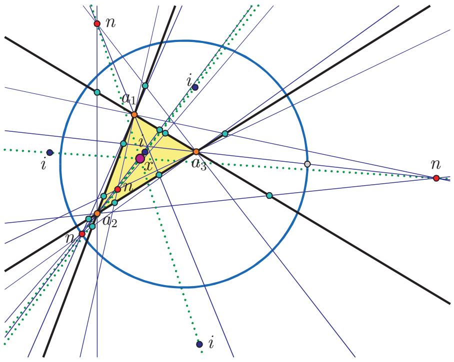

Figure 22: *Nagel points n, InNagel lines and the x point*

Theorem 30 (InNagel center) *The four InNagel lines are concurrent, and meet at the x point.*

Proof. We check that $n_0 i_0$ passes through $x = [a-1 : b-1 : c-1]$ by computing

$$\begin{aligned}&[a-1:b-1:c-1]\,\langle w-v-bcuv+bcuw:\\&\quad u-w+acuv-acvw:v-u-abuw+abvw\rangle\\&=[bu-av+aw-cu-bw+cv+abuw-acuv-abvw+\\&\quad +bcuv+acvw-bcuw]=0\end{aligned}$$

as in the proof of the InGergonne center theorem. The other incidences are similar. □

The joins of corresponding Gergonne points and Nagel points are the **Gergonne-Nagel lines**. They are

$$\begin{aligned}g_0 n_0&=\langle a(bc-1)(bw-cv):b(ac-1)(cu-aw):\\&\quad c(ab-1)(av-bu)\rangle,\\g_1 n_1&=\langle -a(bc-1)(bw-cv):b(ac-1)(cu+aw):\\&\quad -c(ab-1)(av+bu)\rangle,\\g_2 n_2&=\langle -a(bc-1)(bw+cv):-b(ac-1)(cu-aw):\\&\quad c(ab-1)(av+bu)\rangle,\\g_3 n_3&=\langle a(bc-1)(bw+cv):-b(ac-1)(cu+aw):\\&\quad -c(ab-1)(av-bu)\rangle.\end{aligned}$$

Theorem 31 (Gergonne-Nagel center) *The four Gergonne-Nagel lines are concurrent, and meet at the* ***Gergonne-Nagel center****, or* u ***point,*** *which is*

$$u = [(ac-1)(ab-1) : (bc-1)(ab-1) : (bc-1)(ac-1)].$$

Proof. We may check directly that the u point defined as above does indeed lie on each Gergonne-Nagel line: this does not require use of the quadratic or cubic relations. $\square$

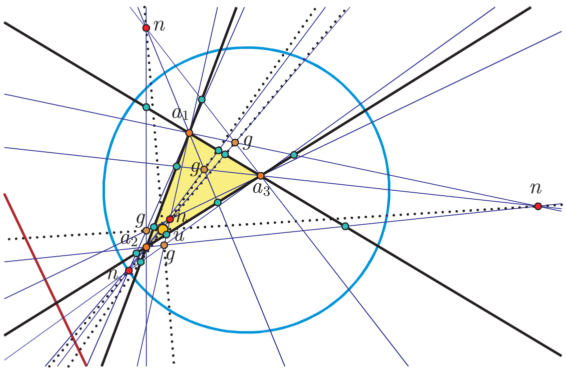

Figure 23: *Gergonne-Nagel lines and the Gergonne-Nagel center: the u-point*

We leave it as an exercise for the reader to establish that the u point lies on the Orthoaxis precisely when the numbers a, b, c are not distinct.

5 Circumcenter Hierarchy

There is a fundamental duality between the Incenter and Circumcenter hierarchies, since from (6) a point m is a midpoint of a side $\overline{ab}$ precisely when its dual line $M = m^{\perp}$ is a biline of the dual vertex $\overline{a^{\perp}b^{\perp}}$. So by dualizing we can transform all known facts about the Incenter hierarchy of a triangle to the Circumcenter hierarchy of the dual triangle, and vice versa. This is a striking difference between projective Triangle geometry and the more familiar Euclidean version, and sheds also some light on the latter.

All the results of this section are consequences of the dual results established in the previous section, after some book-keeping between the two hierarchies.

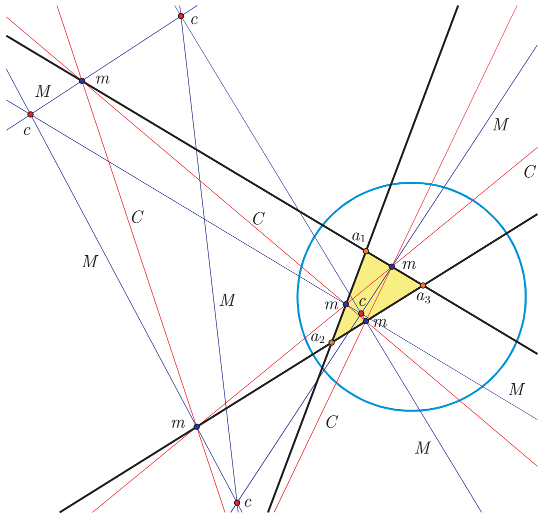

Figure 24: *Midpoints m, Midlines M, Circumcenters c, Circumlines C*

We will now assume that the triangle $\overline{a_1a_2a_3}$ has **Midpoints** m, and so also **Midlines** M. This occurs precisely when $1-q_1, 1-q_2, 1-q_3$ are all squares, and in this case there are six Midpoints, two on each side. This is equivalent to the Dual triangle $\overline{l_1l_2l_3}$ having bilines. Each Midline M passes through a Midpoint m, since any two Midpoints of a side are perpendicular.

Theorem 32 (Circumlines) *Midpoints m are collinear in threes, joining on four* ***Circumlines*** *C. Midlines M are concurrent in threes, meeting at four* ***Circumcenters*** *c.*

Median lines (or just **medians**) D are joins of corresponding Midpoints m and Points a. There are six Medians, two passing through each Point. The duals are the **Median points** d, the meets of corresponding Midlines M and Dual lines A.

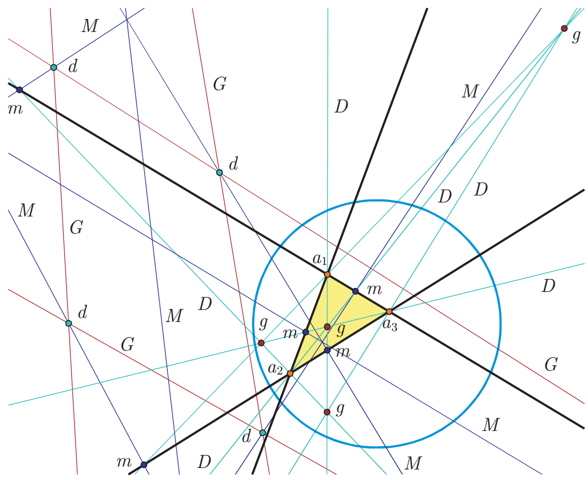

Figure 25: *Medians D, Median points d, Centroids g, Centroid lines G*

Theorem 33 (Median harmonic conjugates) *The two Median lines through a vertex of the Triangle are harmonic conjugates with respect to the two Lines of that vertex.*

Theorem 34 (Centroids) *The Median lines D are concurrent in threes, meeting at four* ***Centroid points*** *g. The Median points d are collinear in threes, joining on four* ***Centroid lines*** *G.*

A **Median Thaloid** is a Thaloid of a side consisting of two Median points, both on a Dual line of the Triangle. There are three Median Thaloids.

Theorem 35 (Isostatic points) *If two Median Thaloids meet at a point r, then the third does too.*

Such a common point is an **Isostatic point**; Figure 26 shows the three Median Thaloids as well as two Isostatic points: r_1 and r_2, and their join.

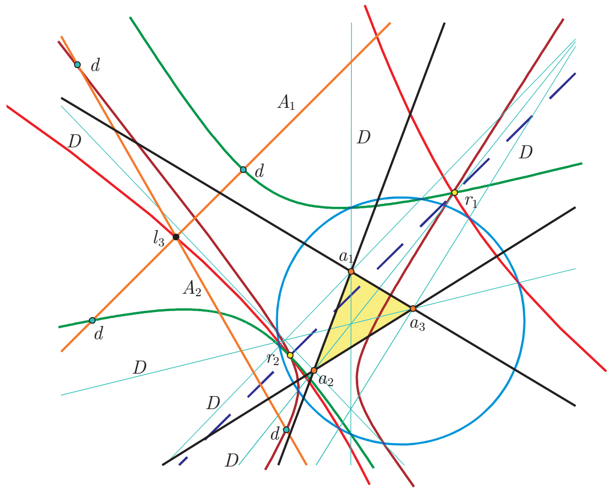

Figure 26: *Median Thaloids and Isostatic points r_1 and r_2*

The four Circumlines and the four Centroid lines are corresponding, since the three Midpoints that join in a Circumline also give rise to the three Median lines passing through a particular Centroid line, which is dual to a particular Centroid line. **CircumCentroid points** are meets of corresponding Circumlines and Centroid lines. **CircumCentroid lines** are partially analogous to *Euler lines*, being joins of Circumcenters and Centroids. The next result shows that the z point might also be called the **Euler center**!

Theorem 36 (CircumCentroid axis) *The four CircumCentroid points are collinear, and join on the Z line. The four CircumCentroid lines are concurrent, and meet at the z point.*

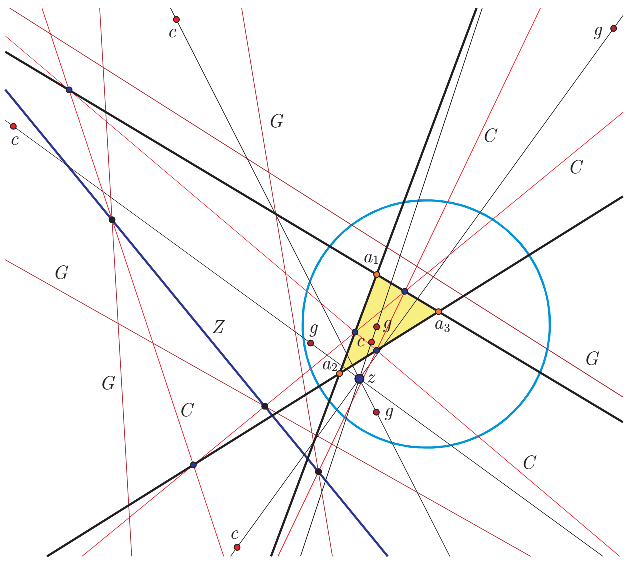

Figure 27: *CircumCentroids, the Z line and the z point*

CircumDual points are meets of Circumlines and Dual lines. There are twelve CircumDual points, four on each Dual line, three on each Circumline. **CircumDual lines** are duals of CircumDual points. **Tangent lines** are joins of corresponding CircumDual points and Points; there are twelve. Tangent points are duals of Tangent lines.

A **Circumcircle** is a circle centered at a Circumcenter passing through one, hence all of the Points. These are shown in Figure 28, with CircumDual points and Tangent lines, which really are tangent to the Circumcircles.

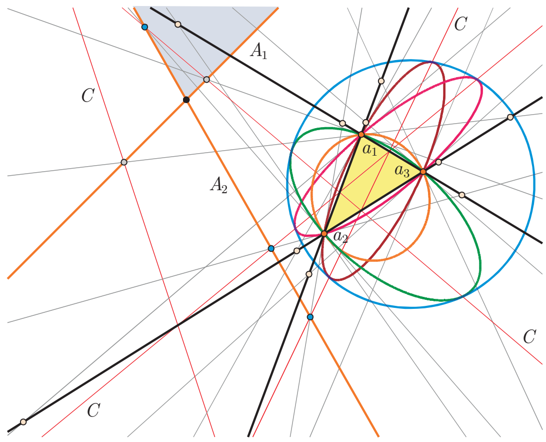

Figure 28: *CircumDuals points, Tangent lines, Circumcircles and Sound points*

Sound points are meets of Tangent lines and Lines; there are twelve, three associated to each Circumline. They are also shown in Figure 28. **Sound lines** are the duals of Sound points.

Theorem 37 (Jay lines) *The three Sound points associated to a particular Circumline C join on a **Jay line** J.*

There are four Jay lines, and **Jay points** j are their duals. **CircumJay points** are meets of Circumlines and associated Jay lines; there are four, and **CircumJay lines** are their duals; joins of Circumcenters and Jay points.

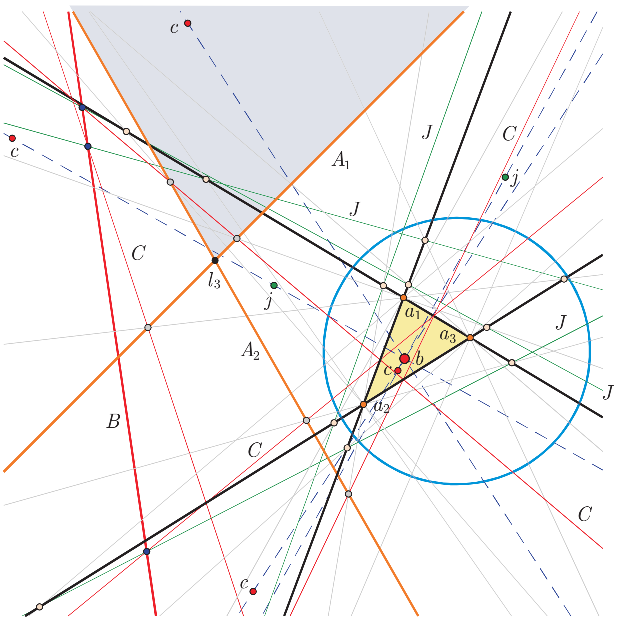

Figure 29: *Jay points and lines, CircumJay points and lines, and the Base center b*

Theorem 38 (CircumJay center) *The four CircumJay points join on the Base axis B. The four CircumJay lines meet at the Base center b.*

Theorem 39 (Wren lines) *Sound points associated to different Circumlines are collinear in threes, and join on four **Wren lines** W.*

The duals of the Wren lines are **Wren points** w. To each Wren line we associate the Circumline not associated to the Sound points on it. **CircumWren points** are the meets of Circumlines and associated Wren lines, and **CircumWren lines** are their duals.

Theorem 40 (CircumWren center) *The four CircumWren points join on the Orthic axis S. The four CircumWren lines meet at the Orthostar s.*

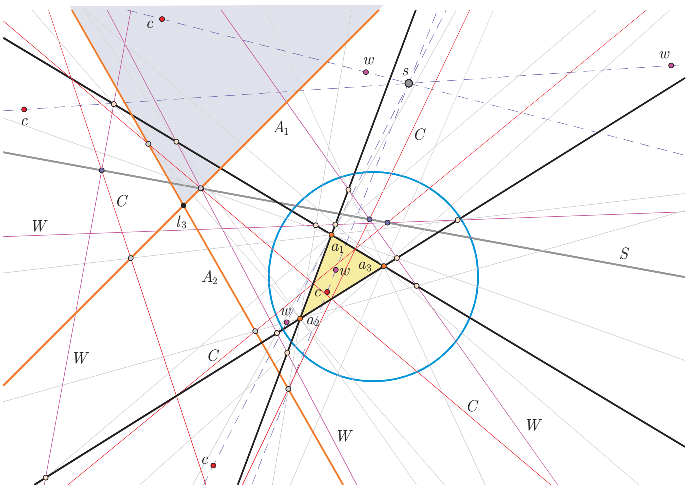

Figure 30: *Wren points and lines, CircumWren points and lines, and the Orthostar s*

A **JayWren point** is the meet of associated Jay lines and Wren lines; there are four. The duals are the **JayWren lines**.

Theorem 41 (JayWren center) *The four JayWren points join on the **JayWren axis**, or the V **line**. The four JayWren lines meet at the **JayWren center**, or v **point**.*

This is a good point to remark that in the projective situation there are remarkable additional constructions, that are available at times when midpoints and bilines may not exist, which allow a wide extension of many of the theorems in this paper. This topic will be developed elsewhere.

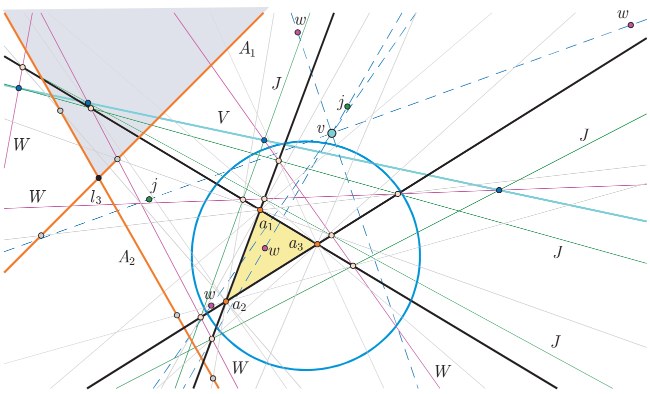

Figure 31: *JayWren points and lines, JayWren center v and axis V*

6 Bridging between the Incenter and Circumcenter hierarchies

Although we have so far emphasized the complete duality between the Incenter and Circumcenter hierarchies, it is also the case that there are numerous remarkable connections between the two. We give a brief indication of this with three examples, leaving proofs to another occasion.

We assume we have a (generic) triangle $\overline{a_1a_2a_3}$ with *both Bilines and Midpoints* (so both hierarchies exist). This is, at least approximately, the situation with our example Triangle.

Theorem 42 *The JayWren center v, the Gergonne-Nagel center u and the Orthocenter h are collinear.*

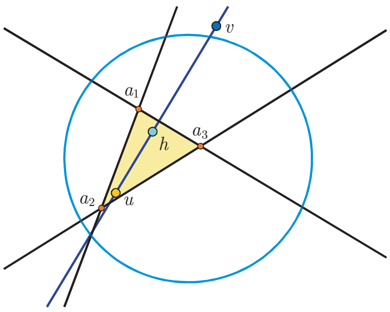

Figure 32: *Jay-Wren-Gergonne-Nagel axis*

Theorem 43 (InCirc joins/meets) *The* 16 ***InCirc joins*** *joining the four Incenters i and the four Circumcenters c meet also four at a time at four* ***InCirc centers*** *r.*

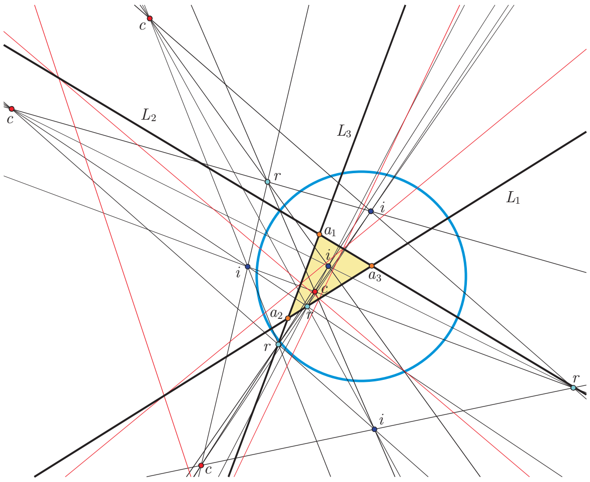

Figure 33: *InCirc joins and InCirc centers*

The four Incenters i, four Circumcenters c and four InCirc centers r form a pleasant symmetrical configuration of 12 points.

Theorem 44 (InCentroid joins/meets) *The* 16 ***InCentroid joins*** *joining the four Incenters i and the four Centroid points g meet two at a time at* 24 ***InCentroid meets*** *which lie,* 8 *each, on the Lines.*

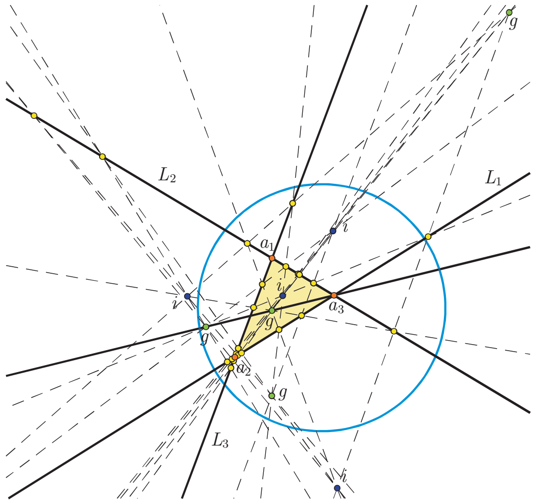

Figure 34: *InCentroid lines and meets*

An interesting direction is to ponder the implications of this work for classical Euclidean triangle geometry. There are also many further phenomenon in the projective setting with no obvious affine/Euclidean parallel, which will be studied in future papers. The author will shortly post Triangle Geometry GSP worksheets on his UNSW website. Hopefully these, together with the formulas and pictures in this paper, will empower and encourage the reader to make his/her own discoveries!

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