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Perspective Collineation and Osculating Circle of Conic in PE-plane and I-plane

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ABSTRACT

All perspective collineations in a real affine plane are classified according to a constant cross-ratio and the position of the center and axis. A special attention will be given to the conditions which basic elements of perspective collineation have to fulfill in order to obtain the touch or osculation or hyperosculation of two conics. On the affine models of an isotropic and pseudo - Euclidean plane the osculating circle of a conic is constructed by using perspective collineations.

Key words: perspective collineation, homology, elation, constant cross - ratio, conic, osculating circle

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Perspektivna kolineacija i oskulacijska kružnica konike u PE-ravnini i I-ravnini

SAŽETAK

Sve perspektivne kolineacije u realnoj afinoj ravnini klasificiraju se s obzirom na karakterističnu konstantu te položaj središta i osi. Pokazuje se, kako odabrati temeljne elemente perspektivne kolineacije kako bi se neka konika i njezina slika dodirivale u jednoj ili dvije točke, oskulirale se ili hiperoskulirale. Na afinim se modelima izotropne i pseudoeuklidske ravnine pomoću perspektivne kolineacije konstruiraju oskulacijske kružnice konika.

Ključne riječi: perspektivna kolineacija, homologija, elacija, karakteristična konstanta, konika, oskulacijska kružnica

1 Introduction

The transformation of a real projective plane known as a **collineation** maps points to points, lines to lines and preserves the incidence relation. Any collineation that has one range of invariant points (o) and a pencil of invariant lines (S) is called a **perspective collineation**. The fixed line o is called the **axis** and the fixed point S is called the **center** of the perspective collineation. All lines joining the pairs of corresponding points are called **rays** and pass through the center S. The intersection point of two corresponding lines lies on the axis o. Every perspective collineation is a projective transformation since the cross rations of four distinct points of a range of points and four distinct lines of a pencil of lines are invariant. Here are some basic properties of a perspective collineation ([1], [2]).

Teorem 1. *A perspective collineation is uniquely determined by its center S, axis o and one pair of corresponding points A, A_1. (Instead of corresponding points a pair of corresponding lines can be given.)*

Teorem 2. *If A, A_1 denote a pair of corresponding points, S the center of perspective collineation and K the intersection point of the ray SA and the axis o ($K = SA \cap o$), then the cross ratio (AA_1, KS) is constant. This constant cross ratio is marked by k and generally, is a real nonzero number.*

Teorem 3. *All perspective collineations form a group under the operation of composition.*

2 Classification of plane perspective collineations

According to the mutual position of the center and axis all perspective collineations are divided into two subsets: **homologies and elations**. A perspective collineation is called an elation if its axis o and center S are incident, otherwise it is called a homology. In each of these subsets the affine elations and affine homologies known as a point reflection, line reflection and translation are extracted. (Table 1)

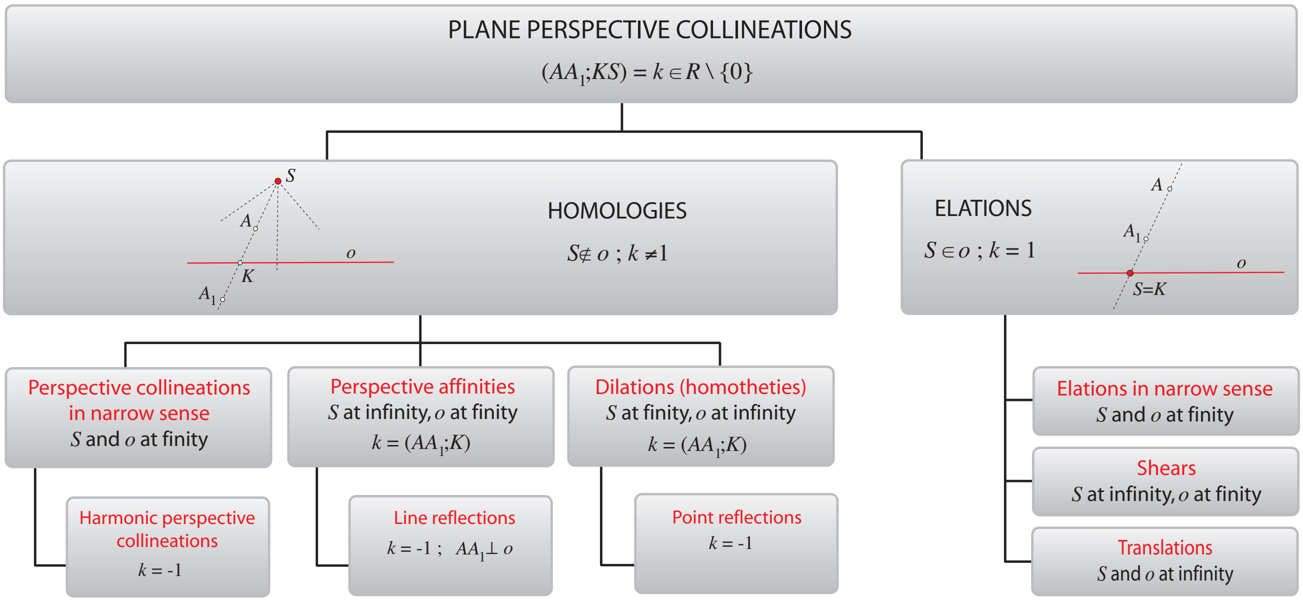

Table 1

All the cases may be summarized as follows:

Homologies may be classified into:

2.1 **Perspective collineations in narrow sense** are homologies with the finite center S and finite axis o, the constant cross ratio of a perspective collineation is a real nonzero number. An involutive perspective collineation in narrow sense is called a *harmonic perspective collineation*, its constant cross ratio equals -1.

2.2 **Perspective affinities** are homologies with the center S at infinity and the finite axis o. The constant cross ratio of a perspective affinity is division ratio $k = (AA_1;K)$ where A, A_1 is a pair of corresponding points and K is the intersection point of the ray AA_1 and the axis o ($K = AA_1 \cap o$). A division ratio of three collinear points is an invariant of a perspective affinity. An involutive perspective affinity is called *line reflection*. Consequently, its constant cross ratio equals -1.

2.3 **Perspective similarities or dilations or homotheties** are homologies with the finite center S and axis o at infinity. The constant cross ratio of a perspective similarity is the division ratio $k = (AA_1;S)$ where A, A_1 is a pair of corresponding points. An involutive perspective similarity is called a *point reflection*. Consequently, its constant cross ratio equals -1.

Elations are perspective collineations for which the center and axis are incident, that is $S = K$. The constant cross ratio of all elations is equal to 1.

The *elations* may be classified as follows:

2.4 **Elations in narrow sense**- with the finite center and finite axis

2.5 **Shears**- with the center at infinity

2.6 **Translations** - with the center and axis at infinity

Shears and translations map the line at infinity to itself. Thus, they are affine transformations. An affine transformation preserves division ratio of three collinear points.

3 Construction of the osculating circle of a conic at an arbitrary point

The order of a conic is an invariant of perspective collineation, i.e., a perspective collineation maps conics into conics. Affine transformations preserve the line at infnity, hence they map a conic into a conic of the same type (i.e. ellipse is mapped into ellipse, hyperbola is mapped into hyperbola and parabola is mapped into parabola). Two conics intersect in four points, some of which may coincide or be real or imaginary. If two real intersection points coincide, the conics c and c_1 *touch* at this so-called touching point. If three real intersection points coincide, c and c_1 are *osculating conics* at this point. If four real intersection points coincide, c and c_1 are *hyperosculating conics* at this point. A special attention will be given to the conditions which basic elements of perspective collineation have to fulfill in order to obtain the touch or osculation or hyperosculation of two conics.

If a conic c touches the axis o at a point A or passes through the center S ($S \notin o$) of a perspective collineation, then the conic c will be mapped into a conic c_1 which touches the conic c at the point A or at the point S, respectively.

If a conic c touches the axis o at a point A and passes through the center S ($S \notin o$) of a perspective collineation, the points S and A are the touching points of the conics c and c_1. A conics with two common touching points can also be obtained if the point S and the line o are a pole and a polar with respect to a conic c. In this case the intersections of the axis and the conic c are common points of tangency for c and c_1. If the center of a perspective collineation is within the conic c than the intersection points will be a pair of conjugate imaginary points.

If the conics c and c_1 are osculating conics, they determine an elation with the common tangent at the point of tangency as its axis (the point of tangency is different from the center of the elation). Also if a conic c passes through the center of an elation and doesn't touch the axis, then the conics c and c_1 are osculating conics.

If a conic c touches axis o at the center S of elation then conic c will be mapped into hyperosculating conic c_1.

All these aforementioned facts provide that by using the appropriate perspective collineation for given conic c it is possible to construct an osculating or hyperosculating conic or a conic c_1 which touches the conic c at one or two points.

By applying a perspective collineation an osculating circle of a conic in the Euclidean plane and on the projective models of some projective - metric planes is constructed in [4] and [5]. By using an elation the same constructions can be made on the affine models of the pseudo - Euclidean and isotropic plane.

3.1 Pseudo - Euclidean plane

The ordered triple $\{f, I, J\}$ is called the absolute figure of the pseudo - Euclidean plane where I and J are two real absolute points on the absolute line f. According to their position with respect to the absolute figure, the proper conics of the pseudo - Euclidean plane may be divided into nine types [4]. A circle is a conic incidental with both absolute points.

Let the absolute figure of PE - plane be $\{f, I, J\}$ where the line f is a line at infinity and the points I and J are the points at infinity on perpendicular lines i and j. Let the pseudo - Euclidean ellipse c be presented by a circle in the Euclidean sense. It needs to construct the osculating circle of c at its arbitrarily chosen point A. By using the appropriate elation the given ellipse c can be mapped into its osculating circle c_1 at the given point A. The construction is carried out in steps:

The point A is selected for the center of an elation. The intersection points of the rays AI and AJ with the ellipse c are marked by I' and J'. I', I and J', J are the pairs of corresponding points of that elation. The line $f' = I'J'$ corresponds to the absolute line f. The lines f and f' intersect at the point at infinity, thus the axis o is parallel to the line f' and passes through the center S. The elation (A, o, I, I') maps the given conic c into an osculating circle c_1 (Figure 1).

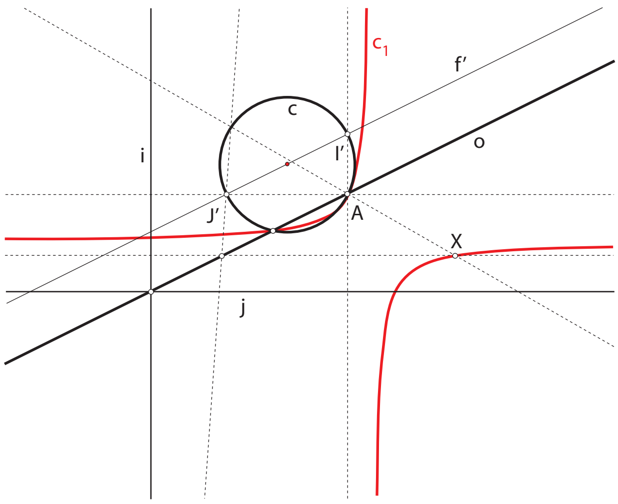

Figure 1

3.2 Isotropic plane

The ordered pair $\{f, F\}$ is called the absolute figure of the isotropic plane where the point F is called the absolute point on the absolute line f [3]. Let the affine model of an isotropic plane with the absolute figure at infinity be given. Let the absolute point F be on the line f_F. A circle of the isotropic plane is a conic touching f at F. Let a conic c and the tangent at a point A on the conic are given.There are two ways to find an elation that will map the given conic c into the osculating circle c_1 at the point A:

The first way is to take the tangent a of the conic c in the point A for the axis of an elation, and then find the center S of the elation on the axis. If the tangent a is taken for the axis of an elation then the tangent f' to the conic c corresponds to the absolute tangent f of the osculating conic c_1 passes through the intersection point of the lines a and f. The point of tangency F' of f' and c and the absolute point F are a pair of corresponding points, and the ray FF' intersects the axis a at the center S of an elation (Figure 2).

The second way is to take the point A for the center of elation and then find the axis of the elation. If the point A is taken for the center of an elation, then AF is the ray the elation. The point F' is the intersection point of the ray AF and the conic c. The line pair f, f' is a pair of corresponding lines. The axis o passes through the point A and intersection point of the lines f' and f, thus it is parallel to f'. (Figure 3).

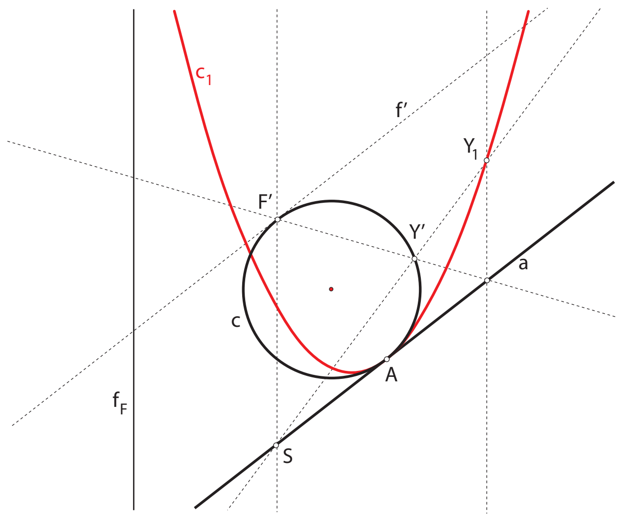

Figure 2

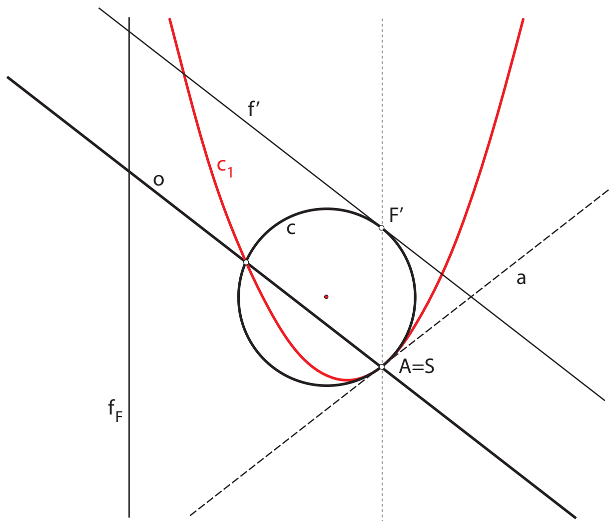

Figure 3

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