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Endogenous Business Cycles in the Ramsey Growth Model

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Abstract: The Ramsey model is an analytical structure aimed at explaining intertemporal optimal growth. As a consequence, business cycles cannot be generated resorting to this structure, unless one introduces some source of inefficiency. Our central argument is that firms forecast future demand using a simple rule and thus they fail to perceive the full extent in which demand is capable of growing. Hence, firms will not invest as much as it is economically feasible in each moment of time, and this mechanism leads eventually to business cycles. The paper contributes to the endogenous business cycles literature with an important new feature: we do not have to consider the labour market in order to generate fluctuations – the framework just assumes consumption and investment decisions.

Key words: endogenous business cycles, Ramsey growth model, nonlinear dynamics, chaos, logistic equation

JEL Classification: C61, E32, O41

Introduction

In its original form, the Ramsey model of intertemporal utility maximisation rests upon a competitive market environment, where externalities, asymmetric information and other imperfections are absent. As a corollary of this Walrasian market structure, the Ramsey model is essentially an optimisation framework from which one can extract time paths concerning efficient levels of consumption, output and accumulated capital.

In other words, the important and powerful analytical instrument that this model represents is aimed at explaining optimal growth and the time series it furnishes must be viewed as reflecting potential consumption, potential output and potential capital stock. In the long run, according to mainstream growth theory, these trajectories will reflect constant growth (zero or positive).

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However, there is a significant difference between optimal benchmarks and what is effectively undertaken in terms of economic activity. Business cycles, the various macroeconomic theories would agree, are essentially triggered by the inability to fulfil in all time moments the complete available potential. Production functions reflect how much the economy produces when it makes complete use of its capital stock, labour force and technological capabilities, but it is unreasonable to think that inputs are always used under maximum capacity utilisation. This simple claim will be taken in this paper to modify the optimal growth Ramsey setup in order to explain a possible mechanism underlying economic fluctuations.

Broadly speaking, macroeconomics identifies two main strands of literature concerning business cycles theory. The Keynesian view, originally developed by Phelps (1970), Lucas (1972), Fischer (1977) and Phelps and Taylor (1977), among others, relies on nominal rigidities; because nominal prices and wages do not adjust instantaneously, firms are unable to understand with precision how relative prices evolve and therefore coordination failures may arise. Firms do not decide with complete knowledge of real conditions, rather they support their decisions on strategic complementarities, i.e., there is a kind of herd behaviour, triggered by the absence of full knowledge, that produces periods of high confidence, where firms invest heavily, and periods of pessimism that place the economy at low levels of activity, generating recession phases. Essentially, the Keynesian theories of fluctuations state that monetary disturbances have real effects.

The other well known interpretation of business cycles is the Real Business Cycle (RBC) theory. The RBC model, that goes back to Kydland and Prescott (1982), Long and Plosser (1983), Prescott (1986), Christiano and Eichenbaum (1992) and Baxter and King (1993), just to cite the most influential, takes, as a starting point, the optimal growth framework. Macroeconomic fluctuations arise because two features are added to the Ramsey model; first, some source of real disturbances (most commonly, technology shocks or changes in government spending); second, employment is not considered constant over time – employment varies in RBC models because the representative agent utility depends not only on consumption but also on leisure time.

The RBC model has three major implications:

(i) nominal variables and monetary transmission mechanisms are absent. Business cycles arise as a purely real phenomenon;

(ii) stochastic variables have to be considered in order to generate cycles, i.e., RBC theory is useless under a purely deterministic framework. The way through which the disturbances in real variables propagate in order to produce cycles relates to the role of the labour market. A positive technology shock, e.g., implies that firms will be willing to hire more work; a higher demand for labour means higher wages that will decrease leisure in favour of labour; the increased generated income that this process implies leads to higher consumption. Thus, innovation will give place to a change in the representative agent priorities: given the incentive of higher wages, the supply of labour rises and more output is produced; every time a technological development takes place, an expansion period begins;

(iii) the strongest statement of RBC theory is that cycles are generated under market efficiency. There is no departure from the Walrasian general equilibrium setup. Nominal frictions, externalities or other departure from the optimal growth benchmark are not needed, just a process of innovation alongside with a response of the labour force to changes in the conditions governing the productive sector.

In this paper we present an alternative view of business cycles. We follow the RBC theory in the sense that nominal issues are overlooked. The Ramsey growth model will be our guiding structure. However, unlike RBC models we aim at explaining cycles under a purely deterministic framework. This is important because a central critique that we can point relatively to the referred class of models is that fluctuations are not endogenously explained, i.e., as in the Solow growth model, RBC models explain a crucial economic mechanism by saying that, in its essence, what determines it is outside the economic system – uncontrollable innovation processes or political decisions regarding government purchases are the key causes of the observed macroeconomic aggregates evolution over time. In this way, the interaction of demand and supply in the several markets would not be truly important; efficiency is from the beginning assured, and exogenous factors are the ones able to justify how the economy evolves.

If one wants to use an optimality framework to explain fluctuations in the absence of external random shocks, some kind of market imperfection must be added. Recent literature has chosen to assume an external effect over production to generate endogenous cycles. Work by Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002) and Weder (2004), among others, takes a deterministic version of the RBC model (that is, a Ramsey model with optimal decisions concerning leisure and labour choices) and allows for increasing returns to scale in the production of goods and services; these increasing returns are justified on the basis of an externality effect (firms benefit from higher levels of inputs utilisation by other firms).

Other work, developed by Aloi, Dixon and Lloyd-Braga (2000) and Aloi, Jacobsen and Lloyd-Braga (2003), emphasises as well the relevance of increasing returns in order to generate endogenous cycles. Nevertheless, these authors prefer to assume an overlapping generations model, following the pioneer analytical structure proposed by Grandmont (1985), rather than the intertemporal optimal growth framework. An important remark is made in Aloi, Dixon and Lloyd-Braga (2000, p. 99): ‘it is the degree of increasing returns (internal or external) that is the most fundamental requirement: if there are constant or decreasing returns to scale, then endogenous fluctuations are not possible’.

Solving the optimal program that the above described growth model with externalities involves, one finds, for some specific parameter values, the possibility of endogenous cycles. Mathematically speaking, endogenous fluctuations correspond to the outcome of difference equations systems when to the long term solution does not correspond a stable fixed point neither an unstable result (trajectories diverging to infinity). Instead, there is a periodic solution (of period larger than 1) or a periodic/chaotic outcome. This last one involves the notion of sensitive dependence on initial conditions (SDIC). A system displays SDIC when, for some variable, starting from distinct initial points no matter how close to each other the corresponding trajectories will rapidly evolve towards clearly distinct states. Hence, SDIC is synonymous of unpredictability in deterministic setups, a phenomenon that is generally associated with chaotic motion.

There is an important strand of literature that debates, since the early 1980s, nonlinear dynamics in economics, and this literature focus heavily, among other economic issues, on business cycles. The idea that purely deterministic problems regarding models of intertemporal utility maximisation can produce cyclical or chaotic motion is indeed appealing because it provides a reasonable explanation for real business cycles without resorting to exogenous disturbances. Further insights on the economic theory of nonlinear dynamics may be found on, e.g., Chiarella (1992), Bullard and Butler (1983), Barnett, Medio and Serletis (1997) and Medio and Lines (2001).

The main caveat of the production externalities setup is that, according to Coury and Wen (2005), the level of externalities needed to generate endogenous fluctuations is too high to be considered empirically realistic, and therefore we end up by getting smooth long run growth for plausible parameter values and one obtains cycles and chaos under non plausible circumstances. These authors try to give a more realistic flavour to the model's structure by introducing capacity utilisation constraints and through the consideration of durable consumption goods, i.e., an aggregate consumption variable that is not exhausted in the first time period it is used.

Although we aim at contributing to the endogenous cycles literature, our approach is somehow different. First, we ignore the leisure – labour trade-off; the original Ramsey model is considered. Second, as discussed previously, endogenous fluctuations can only arise under some market imperfection; instead of assuming production externalities, we take the intuitive view that firms do not make an optimal use of the available resources, that is, investment decisions are such that the economy will not be located in all time moments over the efficient production possibilities frontier. The reason for inefficiency is that the representative firm uses a kind of rule of thumb when forecasting future demand (for consumption and investment), since it is unable to perceive accurately which is the level of future demand corresponding to the one giving place to optimal decisions.

This argument follows Nelson and Winter (1982), Dosi (1988) and Cyert and March (1989), who recognise that firms tend to employ routinised behavioural investment rules rather than a profit maximising behaviour reflecting full rationality. We will focus on the idea that Dosi, Fagiolo and Roventini (2005) support about firm behaviour: firms are seen as prudent, risk-averse entities that are not able to fully anticipate their future level of demand. The previous statement has major implications on the interpretation of economic growth, since it contains a plausible candidate explanation of inefficiencies capable of generating endogenous cycles.

In what follows firms update demand expectations adopting a simple linear rule where the predicted growth rate of demand is dependent on the level of expected demand; under this rule, we will be able to observe that endogenous fluctuations might arise for reasonable parameter values. The bounded rationality rule chosen to characterise investors' behaviour will lead to a dynamic equation that resembles the discrete time logistic map, which is known to lead to chaotic motion for specific values of the underlying parameter.

Synthesising, we propose a model of deterministic real business cycles, where the market imperfection leading to fluctuations is associated to how investors evaluate expected changes in demand. There are, in fact, many other causes of deviation of the output level from its trend path, but one cannot ignore the importance of expectations about demand over the utilisation of resources. Bounded rational firms do not have the sufficient knowledge and skills to make always the investment decisions that are optimal under the maximisation of utility setup; they just adopt a simple rule that approximates to the optimal behaviour but that is not the optimal behaviour. This is sufficient to conduct to a situation where firms will change systematically over time the amount of investment relatively to its potential level. Movements in output, consumption and accumulated capital follow the fluctuations in investment.

The proposed framework introduces a factor of inefficiency, and therefore output and capital will, in the long run, correspond to amounts below the potential levels (the levels of the variables that would correspond to a precise evaluation about future demand and investment decisions). Nevertheless, one of the most relevant results of the analysis is that, with inefficient firms, there are long term time periods where consumption is higher than the optimal level (when investment is low) and periods where investment is above its potential level (this occurs simultaneously with low levels of consumption). Hence, among other features, the model is capable of explaining how deviations from optimal growth through the introduction of production inefficiencies can be compatible with periods where some component of demand attains higher levels than the ones ever accomplishable for the optimal case.

The remainder of the paper is organised as follows. Section 2 explains the mechanism through which firms form expectations about future demand. Section 3 integrates the previous mechanism in the Ramsey growth model assuming that investment decisions are conditioned by demand expectations. Section 4 studies the underlying dynamics; we analyze local bifurcations and make a numerical analysis of global dynamics. Finally, section 5 concludes.

Firms Expectations about Future Demand

We assume a market economy with a continuum of identical firms; the number of firms can be normalised to one in order to simplify the analysis. The resulting representative firm has the potential to generate output according to a Cobb-Douglas production function with constant returns to scale. Let the per capita output be represented by y_t and the per capita stock of capital be given by k_t; the production function is

$$y_t = Ak_t^{\alpha} \tag{1}$$

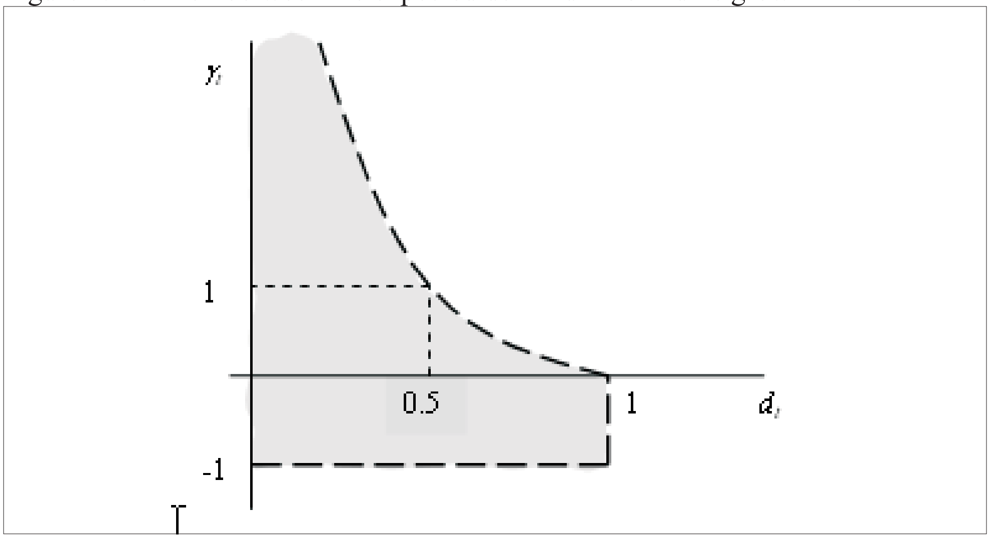
In (1), *A>0* represents a technology index and *0<a<1* is the output - capital elasticity. Note that this is the level of per capita output in the absence of any productive inefficiency what, in our framework, will correspond to say that it is the level of output respecting to the case where firms' demand expectations meet the optimal demand expectations.

We assume that the representative firm forms and updates demand expectations in the following way. We begin by considering a risk averse behaviour, in the sense that firms prefer to under invest than to over invest (generally, economic agents tend to regret more the losses originated by decisions they effectively make than the losses provoked by the absence of decisions). Therefore, we take a rational representative firm with a reasonable perception about future demand but that essentially asks 'if the worst possible expected economic scenario does in fact occur which will be, in this case, the demand level we should expect?'. Accordingly, let $0 \leq d_t \leq 1$ represent the ratio between the expected demand considered by the representative firm and the expected demand level that would imply an optimal process of investment and, consequently, an optimal growth process. Under this formulation, firms will never consider a level of demand above the one that corresponds to the optimal benchmark case.

Let $d_0 < 1$ be the initial expected demand relative level. Starting with this initial expectation, the representative firm has then to choose how it will update expectations for the subsequent time periods. This updating procedure has bounds. For every d_t, this share cannot grow less than minus 100% (otherwise, the share would decline to negative values, what has no meaning). The above bound, in turn, will be such that it avoids relation $d_{t+1} > 1$ to be verified; that is, condition

$d_t(1+\gamma_t) \le 1$ must hold, with γ_t representing the growth rate of d_t. Putting the two bounds together, we get the inequality $-1 < \gamma_t < 1/d_t - 1$. Figure 1 represents the area inside which the growth rate of the demand expectation may vary.

Figure 1: Relation between the expected demand ratio and its growth rate



According to figure 1, there is an area in which firms make expectations about the evolution of demand. If firms were fully rational and had complete information they would choose γ_t in order to remain forever on the upper bound, because optimal consumption, investment and output levels would be attained. For this to happen the representative firm would have to compute a nonlinear relation between d_t and γ_t, that is, there is not, in this case, any constant marginal rate of substitution between the level and the growth of expected demand. It will be easier for firms to consider that when d_t varies, γ_t should vary always in the same amount; this reduces the costs and the uncertainty associated with using complex tools when forecasting the way demand will evolve.

Note that the previous reasoning does not imply that the representative firm is irrational or that managers tend to be unable to make relatively accurate predictions. It just means that it is impossible to fully understand how the economic environment will evolve, and therefore there is always an error associated with forecasting demand (besides this, as stated before, firms tend to 'play it safe', i.e., to be prudent).

According to our previous arguments, we take the following rule of thumb: the representative firm linearises the upper bound relation in figure 1 around a given chosen point. A straight line obtained in this way is contained on the referred bounds only for $d_t \geq 0.5$. Therefore, one considers a first-order linear approximation of $\gamma_t = 1/d_t - 1$ around a point $0.5 \leq \tilde{d} \leq 1$, leading to equation (2),

$$\gamma_t = \frac{2}{\tilde{d}} - 1 - \frac{1}{\tilde{d}^2} d_t \tag{2}$$

Noticing that $\gamma_t = \frac{d_{t+1} - d_t}{d_t}$, then the adopted rule corresponds to

$$d_{t+1} = \frac{2}{\tilde{d}} d_t \left(1 - \frac{1}{2\tilde{d}} d_t \right) \tag{3}$$

Note as well that a line with a high slope is more likely to reflect expectations than a less sloped line, because this makes it more pronounced the reaction to expected demand: if d_t is high, it is expected that this will decline; if d_t is near zero, firms should expect demand to rise more rapidly. Thus, $\tilde{d}$ is probably located near 0.5 (the lower possible value). Regard that $\tilde{d} = 0.5$ corresponds exactly to the logistic equation $d_{t+1} = 4d_t(1 - d_t)$, relatively to which chaotic motion is known to be generated.

Looking at figure 1, we understand why the linearisation process makes sense: when firms expect today's demand to be low (d_t near zero), they will not expect it to grow at infinitely large rates for the next period; hence, instead of being rational, the representative firm acts 'reasonably', adopting a simple rule of behaviour. Nevertheless, we attribute to the representative firm a relatively high degree of information, in the sense that the straight line it chooses in figure 1 touches the upper bound in a given point, and probably the selected line will correspond to a line close to the steepest one inside the feasible set, because this reflects the highest possible constant marginal rate of substitution between expected demand and expected demand growth.

One now needs to incorporate the demand expectations formation rule (3) in the Ramsey model. A central assumption in our analysis is that firms do not invest according to the full potential of the economy, but considering their expectations about future demand. Consequently, we assume two investment variables: potential investment, j_t, and effective investment, i_t (they are both per capita variables). Only if $d_t = 1$ we will have $i_t = j_t$; for any other value of d_t we assume that $i_t = f(d_t) \cdot j_t$, with $f(d^t) = d_t^\theta$, $\theta > 0$. This function can be convex, linear or concave, but the most intuitive form is the concave one ($\theta < 1$), i.e., for values of d_t near 1, the investment level corresponds almost to the potential benchmark level.

Considering a closed economy without government, the demand equation is $y_t = c_t + j_t$, with c_t per capita consumption. We consider as well a standard rule of capital accumulation, $k_{t+1} = i_t + (1-\delta)k_t$, k_0 given, with $\delta > 0$ a depreciation rate.

Demand and capital accumulation equations, together with the inefficiency function f and the production function (1), are the necessary ingredients to develop the business cycles version of the Ramsey model. This is done along the next section.

The Ramsey Model with a Non Optimal Investment Level

We can write the expression of demand as $y_t = c_t + i_t / d_t^{\theta}$, or, rearranging, $i_t = d_t^{\theta}(y_t - c_t)$. Note that although the economy produces y_t and consumes c_t, it just invests i_t, and not the optimal level j_t, meaning that there is an inefficiency problem – the economic system does not work over its production possibilities frontier, that is, a part $j_t(1-d_t^{\theta})$ of resources is simply wasted, because firms do not fully understand that they would benefit from investing under complete resource utilisation capacity.

Replacing the investment expression above in the capital accumulation constraint one obtains

$$k_{t+1} = d_t^{\theta}(y_t - c_t) + (1-\delta)k_t \tag{4}$$

Our modified Ramsey model is translated in the optimal control problem $Max \sum_{t=0}^{+\infty} U(c_t)\beta^t$ subject to (4). The utility function, *U,* is required to possess the standard constant intertemporal elasticity of substitution property; tractability of the model requires an explicit functional form, which we assume to be $U(c_t) = \ln c_t$. Parameter $0.5 < \beta < 1$ is a discount factor.[1]

To solve the model we define a Hamiltonian function, with p_t a co-state variable,

$$H(k_t, p_t, c_t) = \ln c_t + \beta p_{t+1}\left[d_t^{\theta}\left(Ak_t^{\alpha} - c_t\right) - \delta k_t\right] \tag{5}$$

First-order conditions are,

$$H_c = 0 \Rightarrow \beta p_{t+1} = \frac{1}{d_t^{\theta} c_t} \tag{6}$$

$$\beta p_{t+1} - p_t = \left[\delta - \alpha d_t^{\theta} A k_t^{-(1-\alpha)}\right]\beta p_{t+1} \tag{7}$$

$$\lim_{t\to+\infty} k_t \beta^t p_t = 0 \text{ (transversality condition)} \tag{8}$$

Given (6), expression (7) can be rewritten as

$$c_{t+1} = \beta(d_t / d_{t+1})^{\theta} c_t \left\{1 - \delta + \alpha d_{t+1}^{\theta} A\left[d_t^{\theta}(Ak_t^{\alpha} - c_t) + (1-\delta)k_t\right]^{-(1-\alpha)}\right\} \tag{9}$$

Equations (4) and (9) form the Ramsey system with endogenous variables k_t and c_t. Variable d_t can be studied outside this system. The Ramsey model is known to have a saddle-path equilibrium, meaning that convergence to a long run steady state is possible only if consumption is chosen by the representative agent in order to follow the stable trajectory. Otherwise, divergence will prevail.

To rule out instability, we assume that consumption follows the stable path. To find an analytical expression for this path, we begin by defining the steady state; this is the state in which the values of variables are defined in the following way: $\left\{\bar{k} \equiv k_{t+1} = k_t, \bar{c} \equiv c_{t+1} = c_t, \bar{d} \equiv d_{t+1} = d_t\right\}$.

In the steady state, equations (4) and (9) imply the following relations: $\alpha \bar{d}^{\theta} A \bar{k}^{-1(1-\alpha)} = \frac{1-\beta}{\beta} + \delta$ and $\alpha \bar{d}^{\theta} A \frac{\bar{c}}{\bar{k}} = \frac{1-\beta}{\beta} + \delta(1-\alpha)$. The logistic equation that we have defined for the expected demand variable has two steady states: $\bar{d} = 0$ and $\bar{d} = 2\tilde{d} - d^2$; we will see below that the second is the only with meaning, in what concerns the expression of the saddle-path.

The linearisation of the Ramsey system in the vicinity of the steady state allows to obtain a new system, in matricial form,

$$\begin{bmatrix} k_{t+1} & -\bar{k} \\ c_{t+1} & -\bar{c} \end{bmatrix} = \begin{bmatrix} 1/\beta & -\bar{d}^{\theta} \\ -\sigma / \bar{d}^{\theta} & 1+\sigma \end{bmatrix} \tag{10}$$

with $\sigma \equiv \frac{1-\alpha}{\alpha} \beta \left(\frac{1-\beta}{\beta} + \delta \right) \left(\frac{1-\beta}{\beta} + \delta(1-\alpha) \right) > 0$. Saddle-path stability can be confirmed through the observation of the following conditions [with J the Jacobian matrix in (10)],

$1 - Tr(J) + Det(J) = \frac{\sigma(1-2\beta)}{\beta} < 0$ (the negative sign is ensured by condition $\beta > 0.5$)

$1 + Tr(J) + Det(J) = 2\frac{2+\sigma}{\beta} > 0$

The previous conditions guarantee the existence of two eigenvalues, $\lambda_1 \in (-1,1)$ and $\lambda_2 > 1$. The existence of one and only one eigenvalue inside the unit circle, independently of parameter values, is the necessary condition for the type of stability we have claimed to be observable.

For eigenvalue λ_1, a corresponding eigenvector is $P = \begin{bmatrix} 1 & \frac{1-\beta\lambda_1}{\beta \bar{d}^{\theta}} \end{bmatrix}$. The second element in the vector represents the slope of the stable arm, and this is

$$c_t - \bar{c} = \frac{1-\beta\lambda_1}{\beta \bar{d}^{\theta}}(k_t - \bar{k}) \tag{11}$$

Trajectory (11) is positively sloped (as in the original Ramsey model) if one considers the value of $\bar{d}$ different from zero (for $\bar{d} = 0$, the saddle-path is infinitely sloped implying $k_t = \bar{k}$).

If c_t obeys to the relation in (11), this variable can be replaced in (4) to obtain a capital accumulation difference equation with k_t and d_t the only endogenous variables,

$$k_{t+1} = d_t^{\theta} A k_t^{\alpha} + \left[1 - \frac{1-\beta\lambda_1}{\beta \bar{d}^{\theta}} d_t^{\theta} - \delta\right] k_t + \left[\frac{1-\beta\lambda_1}{\beta \bar{d}^{\theta}} \bar{k} - \bar{c}\right] d_t^{\theta} \tag{12}$$

Section 4 analyzes system (3)-(12). From the study of this model's global dynamics emerges an interpretation of endogenous business cycles.

Dynamic Analysis

Local Bifurcation Analysis

Because one of the equations of the dynamic system is known to produce strange dynamics (high periodicity and chaos) we realise from the start that a purely local analysis of the model's behaviour generates misleading results. Nevertheless, the study of local bifurcations is an important first step for the characterisation of the Ramsey-cycles analytical structure.

The central result at this level is the one stated in proposition 1.

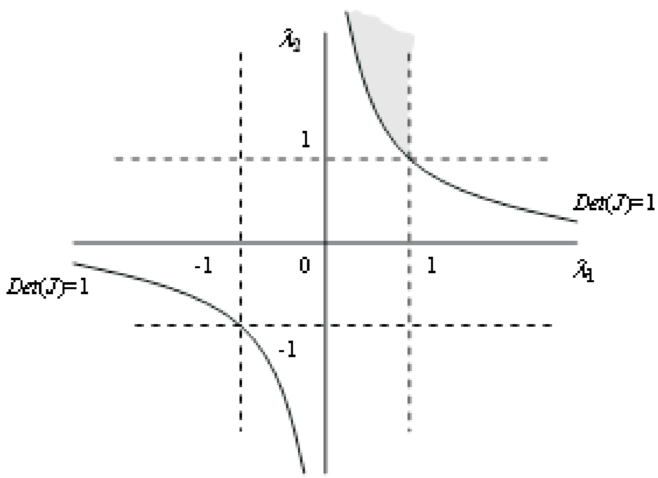
Proposition 1. For the Ramsey model with inefficient levels of investment, a flip bifurcation occurs at $\tilde{d} = 2/3$. Under the defined parameter constraints no other bifurcation (fold, transcritical, pitchfork or Neimark-Sacker) is observable.

Proof: Through the linearisation of system (3)-(12) in the steady state vicinity, we compute Jacobian matrix.

$$G = \begin{bmatrix} 2(1-1/\tilde{d} & 0 \\ \dfrac{\theta\delta\bar{k}}{\tilde{d}(2-\tilde{d})} & \lambda_1 \end{bmatrix}$$

A flip bifurcation occurs when conditions $1 + Tr(G) + Det(G) = 0, Tr(G) \in (-2,0)$ and $Det(G) \in (-1,1)$ are satisfied. The first of these conditions is satisfied for $\tilde{d} = 2/3$, and in this case we observe that $Tr(G) = \lambda_1 - 1$ and $Det(G) = -\lambda_1$. Remind that λ_1 is the eigenvalue of the Ramsey system inside the unit circle; because $Det(J) = 1/\beta + \sigma(1-\beta)/\beta > 1$ (see figure 2 for a graphical representation of the location of the Ramsey model eigenvalues), we confirm that $Tr(G) \in (-1{,}0)$ and $Det(G) \in (-1{,}0)$. Therefore, a flip bifurcation occurs for $\tilde{d} = 2/3$.

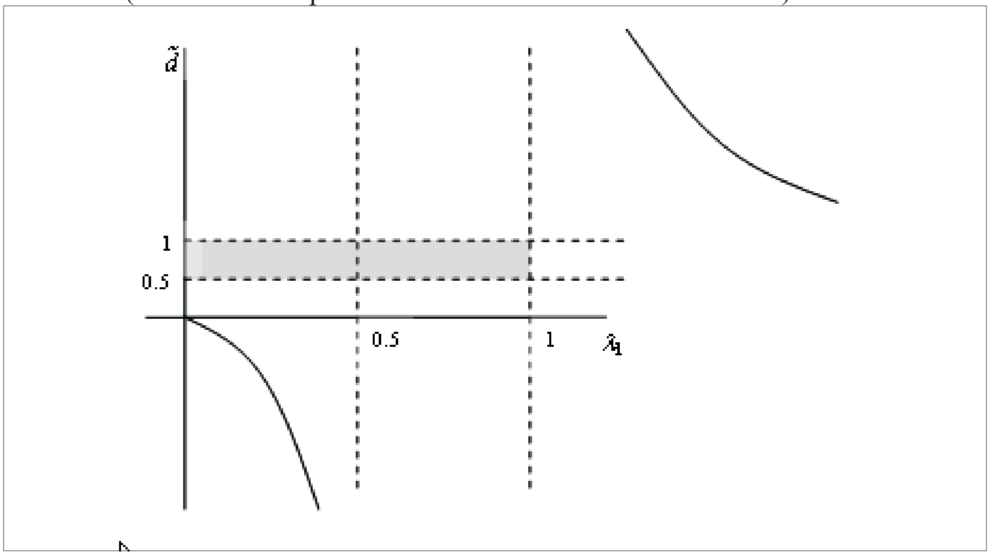
Figure 2: Location of the eigenvalues of the Ramsey model (grey area)



Other types of bifurcations are outside parameter bounds. Fold, transcritical or pitchfork bifurcations would require condition $1 - Tr(G) + Det(G) = 0$ to be satisfied for some parameter values. This condition is satisfied only if $\tilde{d} = 2$; since we have imposed the condition $0.5 \leq \tilde{d} \leq 1$, these bifurcations are outside the scope of our system.

Finally, a Neimark-Sacker bifurcation requires the following condition: $Det(G) = 1$. This implies the relation $\tilde{d} = \dfrac{2\lambda_1}{2\lambda_1 - 1}$. Because $0 < \lambda_1 < 1$ and $0.5 \leq \tilde{d} \leq 1$, the condition is never met and this type of bifurcation is ruled out of the analysis. Figure 3 allows a visual characterisation of the issue in question, since it plots the found relation and easily we regard that it is outside the required bounds.

Figure 3: Graphical representation of the condition for a Neimark-Sacker bifurcation (this occurs for parameter values that are not admissible)



The flip bifurcation indicates the point in which the system undertakes a qualitative change in its dynamic properties. Note that the eigenvalues of G are $g_1 = \lambda_1$ and $g_2 = 2(1-1/d)$. The first eigenvalue is located inside the unit circle, while the second is negative but can be lower or higher than -1; the point where the change occurs is precisely bifurcation point $d = 2/3$: for $d > 2/3$, eigenvalue g_2 is lower than one in absolute value, and thus both eigenvalues are inside the unit circle, i.e., the system is locally stable; for $d < 2/3$ we have $g_2 < -1$, and we characterise the system as exhibiting local saddle-path dynamics.

Numerical examples concerning a global dynamic analysis in the following sub-section will reveal that the local stability result found for $2/3 < d \leq 1$ is maintained under a global perspective and that local saddle-path stability ($0.5 \leq d < 2/3$) is a misleading result, because after the first bifurcation, for , a series of other bifurcations take place, leading to a region of chaos for the lower admissible values of d

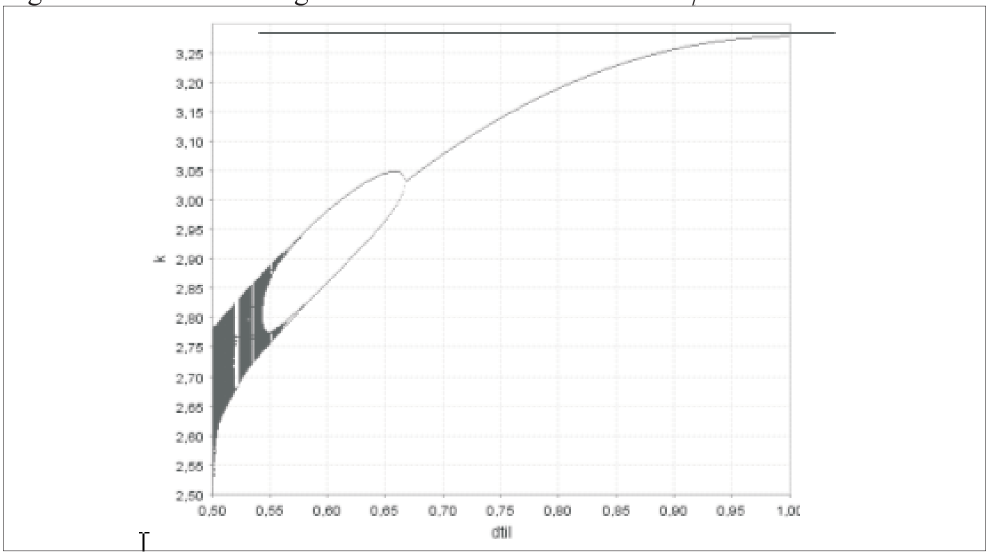
Numerical Analysis of Global Dynamics

The global dynamic analysis of the model involves the need to work with concrete parameter values. Recall that our parameters are $A > 0$, $0 < \alpha < 1$, $\theta > 0$, $\frac{1}{2} < \beta < 1$, $\delta > 0$ and $0.5 \leq d \leq 1$. In reality, only the last parameter has an important effect over the qualitative behaviour of the model. For the other parameters we can consider reasonable constant values as a benchmark for the following discussion (no significant changes would occur in terms of qualitative behaviour if these values were switched by other meaningful values): $A=1$, $\alpha=0.25$, $\theta=0.5$, $\beta=0.95$, $\delta=0.05$.

The dynamic analysis will take in consideration not only the endogenous variables d_t and k_t, but also the other economic aggregates that were previously discussed in our Ramsey model with inefficiency, namely, c_t, y_t, i_t and j_t.

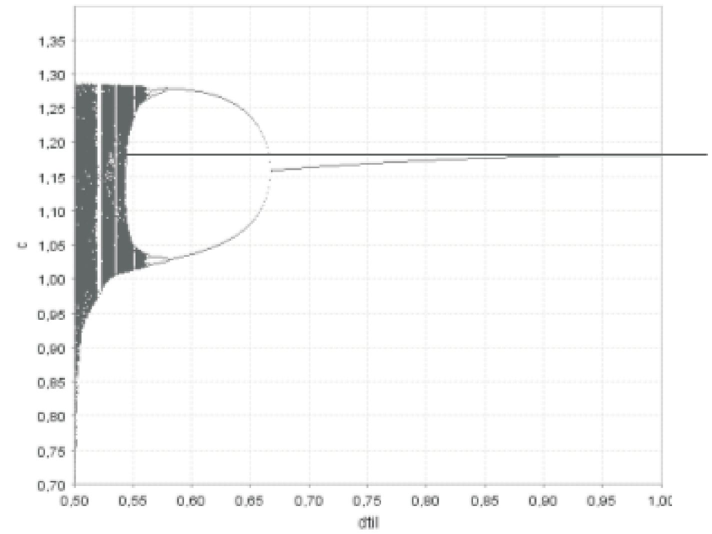
We begin by presenting the bifurcation diagrams for $0.5 \le d \le 1$, concerning variables k_t and c_t (figures 4 and 5).[2] For the other variables of the model we find a similar qualitative behaviour. In this example, $d_0=0.2$ and $k_0=1$; we also consider 1,000 observations after the first 1,000 transients.

Figure 4: Bifurcation diagram for $0.5 \le \tilde{d} \le 1$ and variable k_t



Figures 4 and 5 allow to confirm previous subsection generic results: a flip bifurcation occurs at $\tilde{d}=2/3$; above this value, stability holds, and, according to figure 4, the higher the value of this parameter (the lower the sensitivity of expected demand growth relatively to expected demand), the larger will be the stock of accumulated capital. To the left of the bifurcation point, a series of bifurcations emerge as a route to chaos.

Figure 5: Bifurcation diagram for $0.5 \leq \tilde{d} \leq 1$ and variable c_t



Our main concern is to compare the results of the endogenous cycles model with the benchmark optimal case where $d_t = 1$. The optimal case implies, for our set of parameter values, that the steady state is quantified in the following manner: $\bar{k} = 3.277$, $\bar{c} = 1.182$, $\bar{y} = 1.345$, $\bar{i} = \bar{j} = 0.164$. The horizontal lines in figures 4 and 5 highlight the optimal state for variables consumption and capital stock, allowing for a comparison between the two cases. This takes us to our first numerical result:

Numerical result 1: for the chosen parameter values, the endogenous business cycles model produces long term results that are equal to the optimal growth outcome only for $\tilde{d} = 1$. Below this value, the accumulated capital is always a lower quantity relatively to the benchmark case, and this tends to be the more pronounced the lower is the value of the parameter (a same result would be found for the output variable). Relatively to consumption, we regard that for low levels of the parameter in the expected demand equation, per capita consumption can be higher or lower than the optimal level; a similar result could be displayed for the investment variable.

Figures 4 and 5 indicate that endogenous irregular cycles occur for values of $\tilde{d}$ near 0.5; we have argued that this is a perfectly reasonable scenario since it implies a relatively sharp trade-off between demand expectations and the way these expectations are likely to evolve. To present long term time trajectories for our main variables, we consider $\tilde{d} = 0.507$. Figures 6 and 7 plot time trajectories for variables capital stock and consumption.

Figure 6: Time trajectory of variable k_t

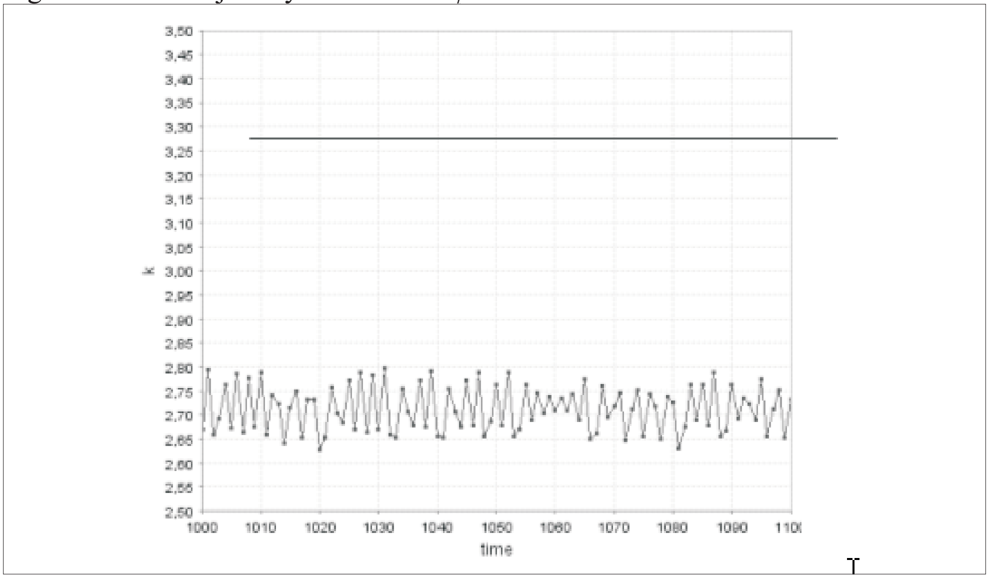
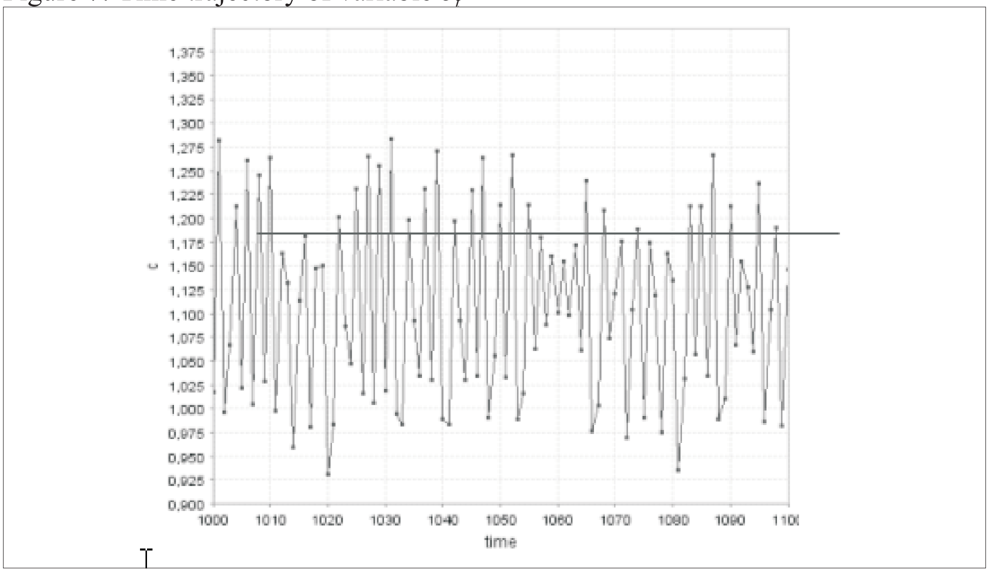


Figure 7: Time trajectory of variable c_t



Irregular cycles are observable and, as stated, these are a direct consequence of the inefficiency produced by inaccurate firms' demand expectations. Once again the horisontal lines compare our fluctuations outcome with the benchmark result of the

Ramsey model. The figures are drawn for 100 iterations after the first 1,000 transients and the initial values of variables are the same as for the bifurcation diagrams.

Note an important result that the bifurcation diagram was not able to show:

Numerical result 2: for the chosen set of parameter values, the time path of consumption is such that there are periods when consumption is higher and periods when it is lower than the optimal value. Nevertheless, and this is our new result, it is straightforward to regard that, in average, consumption is lower than its optimal level.

The aggregates output and investment (both potential and effective) are not drawn, but they display as well irregular cycles in their long term time movement. Once again, we would be able to regard that output is below its optimal level in every time period, while investment is higher than optimal in some periods of time.

The previous results are reinforced with the presentation of the correspondent chaotic attractors. We draw three attractors, the ones that give more meaningful results: the one relating the capital stock with the consumption variable; the one that establishes a relation between consumption and investment; and, finally, the one concerning potential and effective investment. Figures 8, 9 and 10 consider 100,000 iterations after the first 1,000 transients; the values of parameters are the same as before, including the demand expectations parameter. Also, the initial values are the previously taken.

Figure 8: Attractor (k_t, c_t)

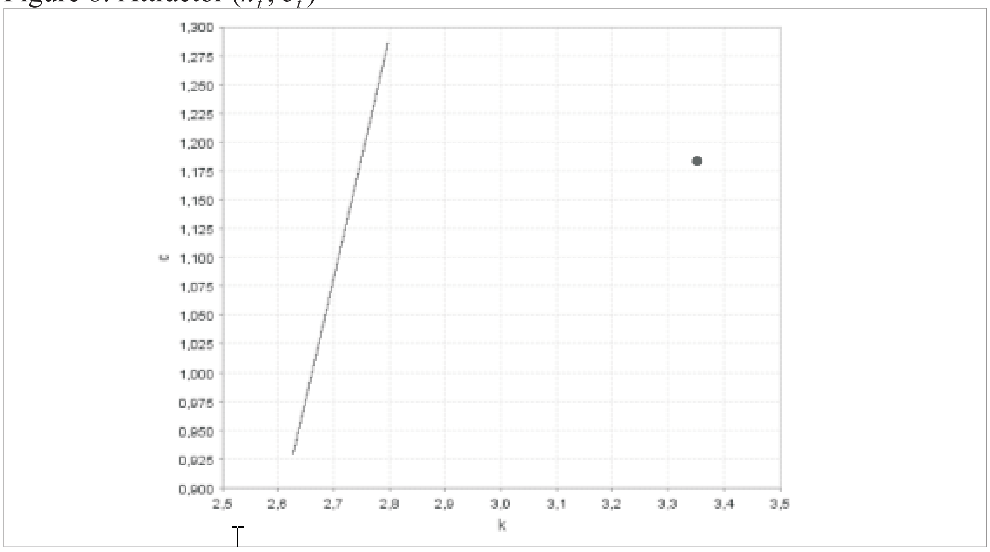


Figure 9: Attractor (c_t, i_t)

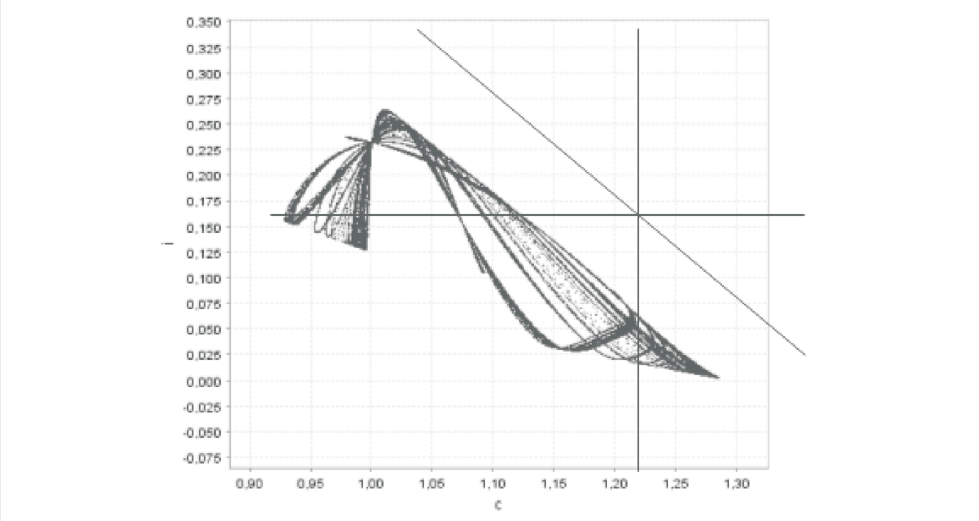


Figure 10: Attractor (i_t, j_t)

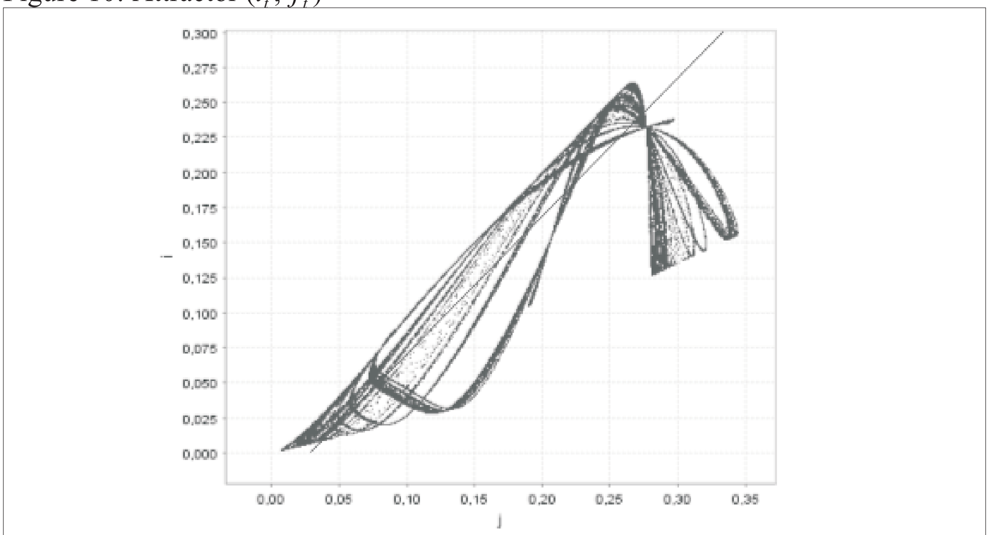


Figure 8 does not bring any surprise into the analysis; we have defined a saddle-path stable trajectory linking capital and consumption variables and the attractor just reflects this linear relation. The comparison with the optimal case implies looking at the optimal point presented to the right of the line.

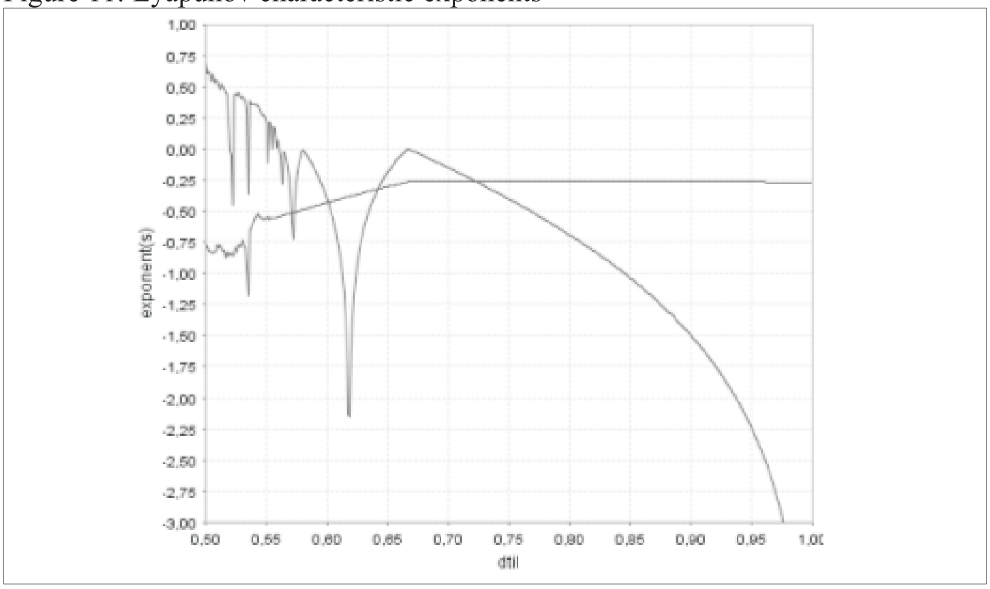
Relatively to figure 9, this is the picture that unveils the mystery of consumption and investment that can be higher with inefficiency than in the optimal case. As one observes, periods of higher than optimal investment are possible only under very low levels of consumption and exceptionally high consumption is observable in periods of low investment. The negatively sloped straight line represents potential output (optimal consumption plus optimal investment); as we see, inefficiency makes consumption plus investment to be always below the optimal income level. The horizontal and vertical lines correspond to the optimal levels of both variables.

Numerical result 3. Under the selected set of parameter values, exceptionally high levels of consumption (investment) exist in periods where investment (consumption) is exceptionally low.

With respect to figure 10, we confirm that inefficiency is translated in an effective investment level that is never above the potential level of investment. The straight line indicates precisely the optimal case where $i_t = j_t$.

To confirm the existence of chaos in system (3)-(12), we compute through figure 11, the two Lyapunov characteristic exponents of the system; we find that the largest Lyapunov exponent is almost always positive for $0.5 \leq \tilde{d} < 0.56$; this is the chaotic region of the model (more rigorously, the region where it is possible to observe an exponential divergence of nearby orbits), that is, the region in which irregular endogenous cycles are generated. The Lyapunov exponents are computed for the same benchmark parameter values we have been considering.

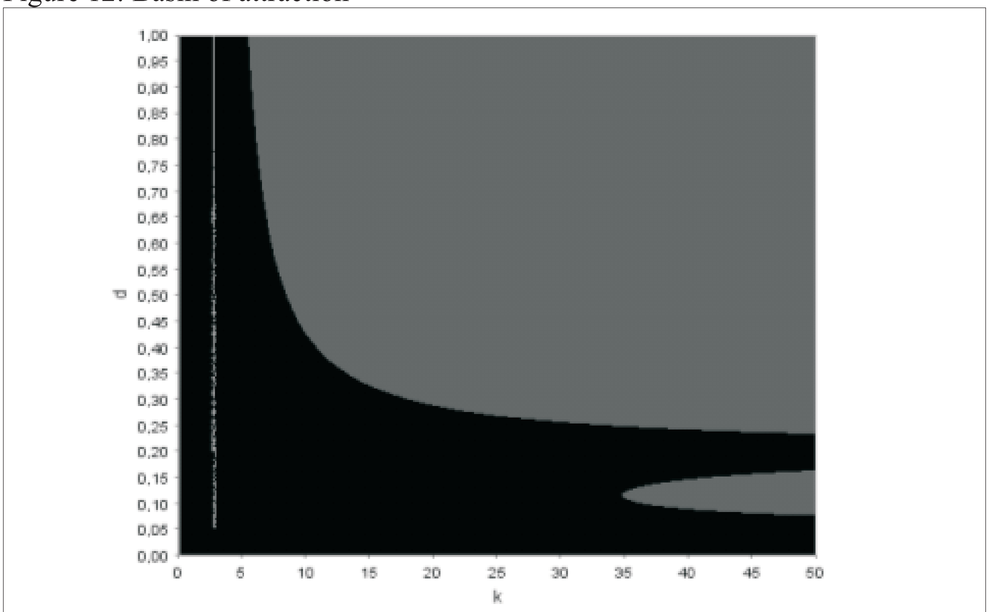
Figure 11: Lyapunov characteristic exponents



The computation of Lyapunov exponents allows for the determination of the dimension of the attractors in figures 9 and 10. The Lyapunov dimension is an approximation of the fractal dimension, and it corresponds to $D_L = 1 - \varepsilon_1 / \varepsilon_2$, where ε_1 is the positive Lyapunov exponent and ε_2 is the negative Lyapunov exponent. For $d = 0.507$, we have $\varepsilon_1 = 0.568$ and $\varepsilon_2 = -0.859$. Hence, the non integer dimension of our attractor is $D_L = 1.661$. The attractor is in fact an object with a dimension higher than one but below the dimension 2 that defines the plan.

Our global dynamic numerical analysis becomes complete with the representation of the basin of attraction. This is the set of initial values that allows for the convergence through time of the endogenous variables of the model to the attracting set. Any initial value outside the basin simply implies that variables diverge (to zero or infinity). Adopting the same parameter values as before, figure 12 indicates that for initial values near the attractor, the system tends to converge to the attractor (the basin is the darker region). Although it is not represented, we know that given the logistic nature of the demand expectations equation, any d_0 higher than 1 or lower than 0 is outside the basin; nevertheless, this is not relevant because the ratio d_t cannot logically assume values outside the specified bounds.

Figure 12: Basin of attraction



Final Remarks

The various theories aiming at explaining business cycles have a point in common: they consider that there is some mechanism in the economic system that deviates aggregate variables from some benchmark value. In the Keynesian view, nominal rigidities are the responsible for such process; in RBC theory there are fluctuations in labour participation as a result of the evaluation that is made in each moment concerning the labour-leisure trade-off; in the new endogenous cycles literature, what produces the departure from the optimum growth setup is a production externality.

In this paper, we have presented an alternative explanation of business cycles. This explanation is endogenous, in the sense that fluctuations are produced by a purely deterministic model, but the source of cycles is related to how firms understand and predict the evolution of demand for investment and consumption. Our benchmark model is the optimal growth process underlying the Ramsey intertemporal utility maximisation problem that in its original inefficiencies free version is unable to produce cycles.

Once we introduce a bounded rationality rule in the Ramsey problem, it is possible to characterise an economy working under irregular cycles. Because firms have not the full knowledge to accurately predict how demand will evolve, they will accordingly invest less than it would be optimal; as a consequence we encounter an economy where some resources are in each time moment simply wasted, that is, besides consumption and investment, demand also incorporates a component of non used resources that cannot have any use in posterior time periods.

Therefore, in this model, cycles are a synonymous of inefficiency; because firms are not willing to take chances, they under invest, and this underinvestment can have irregular features given the simple rule that is considered for the behaviour of the representative firm concerning the expected evolution of demand.

We have proceeded with a dynamic analysis that was undertaken both locally and globally. Although generic, the local analysis is misleading; a bifurcation point seems to divide long term dynamics into a region where the equilibrium corresponds to a stable node and another region with saddle-path stability. The global analysis, although this is undertaken only through a numerical example, allows understanding that saddle-path stability does not in fact characterise the dynamic behaviour of variables; a series of bifurcations lead to a region of chaotic motion, where irregular cycles describe the movement of the various considered economic aggregates.

Since we introduce inefficiency into the Ramsey model, obviously the capital stock and the output will be lower than the correspondent optimal values. Nevertheless, the cyclical nature of the aggregates' time paths produces a curious result: there are time periods where consumption is higher than when no inefficiency is present (this happens for very low levels of investment) and periods with more investment than it would be optimal (for low levels of consumption). Hence, inefficiency means low average consumption and investment, but inefficiency also means pronounced fluctuations that incorporate periods where some agents will be better off than in the optimal case.

One last word for the role of government. As regarded, the model has clear implications on welfare. The risk averse way through which firms predict demand and make investment decisions is translated on less than optimal income and on less than optimal average consumption and investment. The model has also Keynesian features in the sense that the government can act in order to attenuate business cycles, with welfare enhancing effects. This may be done in two ways:

(i) aiding to improve information flows in the economy and furnishing the necessary guarantees for firms to feel safe to invest (and thus helping firms demand expectations to be closer to the respective optimal value); and

(ii) undertaking public investment to offset (or attenuate) private underinvestment in the periods of time when effective investment is away from the optimal level. Fluctuations would be completely eliminated if the government would be able to invest $j_t(1-d_t^{\theta})$ per capita units in every time moment, because in this way demand would exactly correspond to generated income. Under this interpretation, cycles will never be fully eliminated because public resources are not flexible enough to adjust to a process of private investment over time that is not stochastic but that is impossible to predict with accuracy, given a rule underlying demand expectations which is characterised by dependence on initial conditions.

NOTES

[1] In fact, condition β>0.5 does not necessarily hold. Nevertheless, it imposes a reasonable upper bound to the discount rate (the presented condition implies a discount rate below 100%).

[2] The program used to draw figures 4 and 5, and all the following, is iDMC (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

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