Solving the Nonlinear Economic Equations by Modified Newton Method

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Abstract: In this paper two ways how to modify the Newton method for solving general equation are presented. The modifications accelerate convergence of the existing method. The modified methods are useful for application to many economic functions and problems.

Keywords: nonlinear equation, iterative method, Newton method, modified Newton method

JEL classification: C02, C63

Introduction and Motivation

Very often, in economic problems, we have to solve a nonlinear equation or, equivalently, we have to find a zero point of a given function. It can be done exactly only in simple situations (see Soper, 2004), otherwise one has to use an iterative method. We can use the bisection method, the wrong position method (the secant method), the tangent method (the Newton method), the simple iterations method etc. Among them, we shall consider here the Newton method which has generally very fast convergence (see Atkinson, 1989 and Drmač et al., 2003), and has nice applications to monotonous economic functions.

The motivation for our work arises from the following situation. Let us consider an increasing, convex function f, and let us find its zero point z (see Fig. 1).

If x_0 is an approximation of z, then we construct the tangent at the point $(x_0, f(x_0))$ and we take its intersection with x-axis to be the next approximation x_1, etc. This briefly describes the Newton method which will be considered later. From the slopes of the tangents it is obvious that $x_0 > x_1 > x_2 > ... > z$. Thus, at each step we

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are closer to z, and we can approach arbitrarily close to it, but we never reach it. Now, the natural question is: can we take somewhat larger steps?

For example, can we make the transition $x_0 \to x_*$, where $x_* < x_1$ or maybe even $x_* < x_2$, in a single step? Here, we shall answer these questions.

Figure 1: Newton method
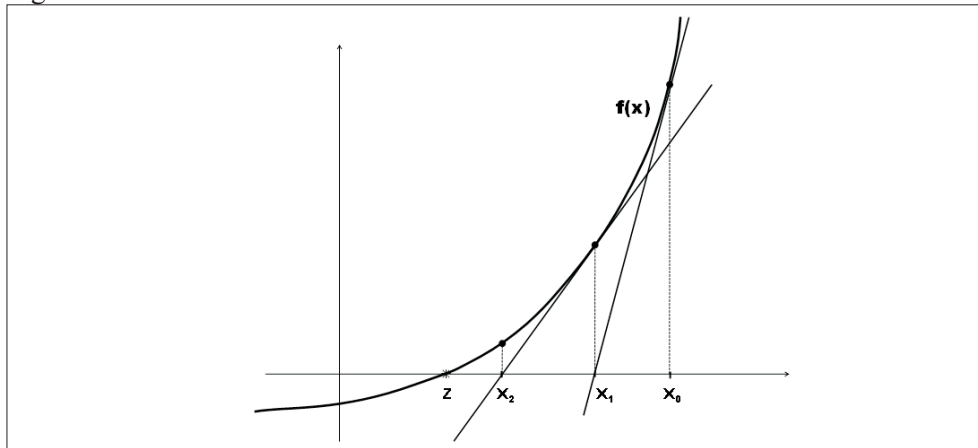
The Newton Iterative Method

The algorithm for the method can be obtained in different ways. We present one of them which uses the Taylor formula. Let $f \in C^{n+1}[a,b]$ and let $x_0 \in [a,b]$. Then for any $x \in [a,b]$ the following formula (Taylor formula) holds.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n +$$
$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^n, \quad \xi = \xi(x) \in \langle x_0, x \rangle \; or \; \xi = \xi(x) \in \langle x, x_0 \rangle \tag{1}$$

On the right side in the first row is the nth Taylor polynomial and in the second row is the residual. Now, we consider the formula (1) with $n = 1$. If $\left| f^{(2)}(t) \right|$ for $t \in [a,b]$ is bounded and if x is close enough to x_0, then the residual can be neglected. Thus, we obtain

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

If x is a zero point of f, i.e. $f(x) = 0$, then we have $0 \approx f(x_0) + f'(x_0)(x - x_0)$. If we put the equality sign in this relation, then the obtained x is an approximation of the root, and we have

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)} \quad or \quad x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2}$$

This x defines the next approximation x_1. In this way the Newton method generates the following sequence of iterations for solving the equation $f(x) = 0$.

Let x_0 be an initial approximation for the solution of $f(x) = 0$. Then

$$x_{n+1} = x_n - \alpha_n \quad where \quad \alpha_n = \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,... \tag{3}$$

The method does not always generate the sequence of iterations which converges to the solution. The sufficient conditions for the convergence are given below.

Let $f \in C^2[a,b]$ and $f(a) \cdot f(b) < 0$. If f' and f'' have fixed signs on $[a,b]$, then f has the unique and simple zero point on $[a,b]$ and x_n converges to it for any $x_0 \in [a,b]$ such that $f(x_0) \cdot f''(x_0) > 0$ holds. $\tag{4}$

If f satisfies the conditions (4), then the method converges fast. The rate of convergence is quadratic which means that the number of correct digits, in the next iteration, is doubled. The method is particularly useful for economic functions which generally satisfy such conditions. They are either decreasing ($f' < 0$) or increasing ($f' > 0$) and either convex ($f'' < 0$) or concave ($f'' > 0$). According to such properties we shall modify the existing Newton method.

First Modification of the Method

As we have seen, the standard Newton method is obtained from the relation (1) using $n = 1$ and by neglecting the residual term. Now, we consider (1) with $n = 2$. We have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(\xi)}{3!}(x - x_0)^3, \tag{5}$$

where $\xi \in \langle x_0, x \rangle$ or $\xi \in \langle x, x_0 \rangle$. Similarly as above, we first neglect the last (residual) term and set $f(x) = 0$. Then, in the quadratic term in (5), according to (2), we replace $x - x_0$ by $-f(x_0)/f'(x_0)$. Thus, we obtain

$$0 \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} \cdot \left(\frac{f(x_0)}{f'(x_0)} \right)^2 .$$

If we replace the sign $\approx$ by $=$, we obtain an approximation for the root,

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0)}{2f'(x_0)} \cdot \left(\frac{f(x_0)}{f'(x_0)} \right)^2 ,$$

that is x_1. Thus, we have obtained the following modified Newton method.

Let x_0 be an initial approximation for the solution of $f(x) = 0$. Then

$$x_{n+1} = x_n - \alpha_n(1 + \alpha_n \beta_n) \; where \; \alpha_n = \frac{f(x_n)}{f'(x_n)} \; and \; \beta_n = \frac{f''(x_n)}{2f'(x_n)}, n = 0,1,2,3,...(6)$$

Second Modification of the Method

As earlier, we consider again the relation (5). We neglect the residual term and set $f(x) = 0$. Then, in the quadratic term in (5), according to (2), we replace only one factor $x - x_0$ by $-f(x_0)/f'(x_0)$. We obtain

$$0 \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} \cdot (x - x_0) \cdot \left(-\frac{f(x_0)}{f'(x_0)} \right).$$

By putting the equality sign in above relation, we have

$$-f(x_0) = (x - x_0) \cdot \left(f'(x_0) - \frac{f(x_0)f''(x_0)}{2f'(x_0)} \right) \quad or \quad x = x_0 - \frac{f(x_0)}{f'(x_0) - \frac{f(x_0)f''(x_0)}{2f'(x_0)}} ,$$

which defines x_1. Simplifying the notation as above, we obtain the following modified Newton method.

Let x_0 be an initial approximation for the solution of $f(x) = 0$. Then

$$x_{n+1} = x_n - \frac{\alpha_n}{1 - \alpha_n \beta_n} \quad where \quad \alpha_n = \frac{f(x_n)}{f'(x_n)} \; and \; \beta_n = \frac{f''(x_n)}{2f'(x_n)}, \quad n = 0,1,2,3,... \quad (7)$$

Comparison and Convergence

Now, let us compare the methods defined by the relations (3), (6) and (7) in some general situations. We also give some notes on the convergence. The full and detailed convergence analysis is beyond the scope of this paper. And it is not so important for the practice. It is obvious that, if any of these three methods converges, it converges to the root. So, in practical computations, we do not have to care about strict convergence conditions. We only have to find an interval $[a,b]$ which contains the desired root and apply the method. If the method fails to converge, then we can reduce the interval and try again (maybe with another method).

To compare the methods we consider single step with the same starting point x_0. Let x_1^*, and x_1^{**} be the next approximations which are generated by the methods (3), (6) and (7), respectively. We have

$$x_1 = x_0 - \alpha_0,$$

$$x_1^* = x_0 - \gamma_0, \qquad \gamma_0 = \alpha_0 \left(1 + \alpha_0 \beta_0\right), \qquad \alpha_0 = \frac{f(x_0)}{f'(x_0)}, \qquad \beta_0 = \frac{f''(x_0)}{2f'(x_0)},$$

$$x_1^{**} = x_0 - \delta_0, \qquad \delta_0 = \frac{\alpha_0}{1 - \alpha_0 \beta_0}, \qquad \alpha_0, \beta_0 \quad \text{as above.}$$

We shall distinguish several cases:

1. f is increasing and convex with $f(x_0) > 0,$
2. f is increasing and convex with $f(x_0) < 0$
3. f is increasing and concave with $f(x_0) > 0$
4. f is increasing and concave with $f(x_0) < 0.$

Similarly, for decreasing f we also have four cases.

Now, we shall compare the methods for increasing function f. The analysis for decreasing f is analogous since we can use the transformation $f \to -f$.

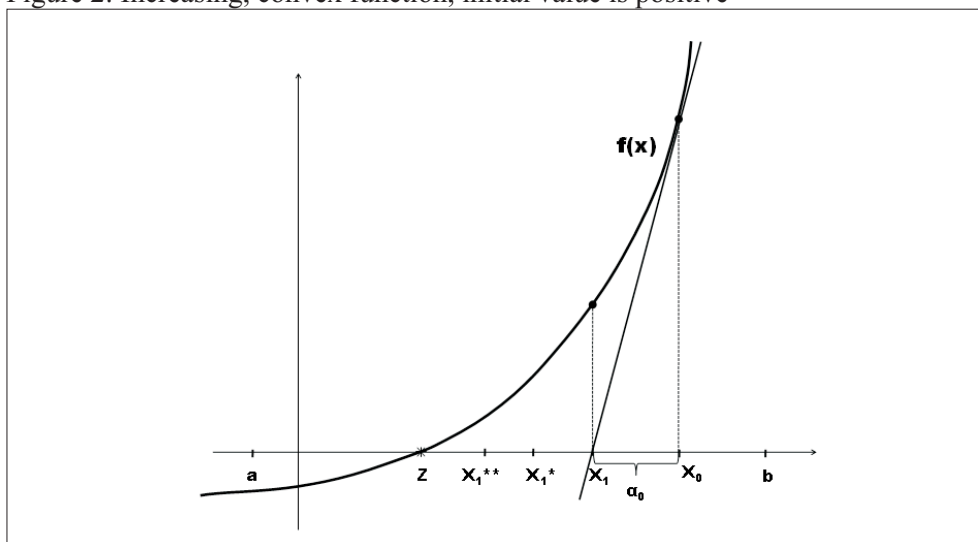1. f is increasing, convex and $f(x_0) > 0$ (see Fig. 2).

Since $f'(x_0) > 0$ and $f''(x_0) > 0,$ we have $\alpha_0 > 0$ and $\beta_0 > 0.$ Thus, we have $\gamma_0 > \alpha_0$ and consequently $x_1^* < x_1.$

If, in addition, $\alpha_0 \beta_0 < 1$ or generally $|\alpha_0 \beta_0| < 1,$ that is

$$|f(x_0) \cdot f''(x_0)| < 2\left[f'(x_0)\right]^2, \tag{8}$$

then $\delta_0 > \alpha_0$ which yields $x_1^{**} < x_1$.

Figure 2: Increasing, convex function, initial value is positive

Now, since $(1+t)(1-t) = 1 - t^2 < 1$, we have for $|t| < 1$, $1 + t < 1/(1-t)$ and $1 - t < 1/(1+t)$, and thus $\gamma_0 < \delta_0$ or $x_1^{**} < x_1^{*}$ (see again Fig. 2). Note that the condition (8) holds provided that (4) holds and $|f(x_0)|$ is small enough, i.e. that x_0 is close enough to the root. Thus, as the process advances, the condition (8) will hold.

We can see that, in the both modified methods, a single step is larger and we are closer to the zero point z than in the standard Newton method. To be sure that we have convergence, it is sufficient to show that the iterates are bounded below, i.e. $z < x_1^{*}$ and $z < x_1^{**}$. This will hold if $f'''(\xi) \leq 0$. Namely, using the relation (5), we have the following exact expression for the root,

$$z = x = \underbrace{x_0 - \frac{f(x_0)}{f'(x_0)}}_{x_1} - \frac{f''(x_0)}{2f'(x_0)}(x - x_0)^2 - \frac{f'''(\xi)}{6f'(x_0)}(x - x_0)^3, \qquad \xi \in \langle x, x_0 \rangle.$$

In the standard Newton method x_1 is obtained by neglecting the last two terms in the above equation. The iterations x_1^{**} and x_1^{*} are obtained by neglecting only the last term and by replacing $x - x_0$ with $x_1 - x_0$ in the quadratic term. Since $|x_1 - x_0| < |x - x_0|$, the quadratic term becomes smaller which gives the larger value of the right side. Thus, if the last term is not positive i.e. f''' is not positive, then $z < x_1^{*}$ and $z < x_1^{**}$, which ensures convergence. If it is not so, then $x_1^{*} < z$ or $x_1^{**} < z$ may

hold. To be sure that we are still in the considered domain (interval $[a,b]$), $|x - x_0|$ must be small, that is x_0 must be close enough to z or $[a,b]$ must not be too narrow.

2. f is increasing, convex and $f(x_0) < 0$ (see Fig. 3).

Figure 3: Increasing, convex function, initial value is negative
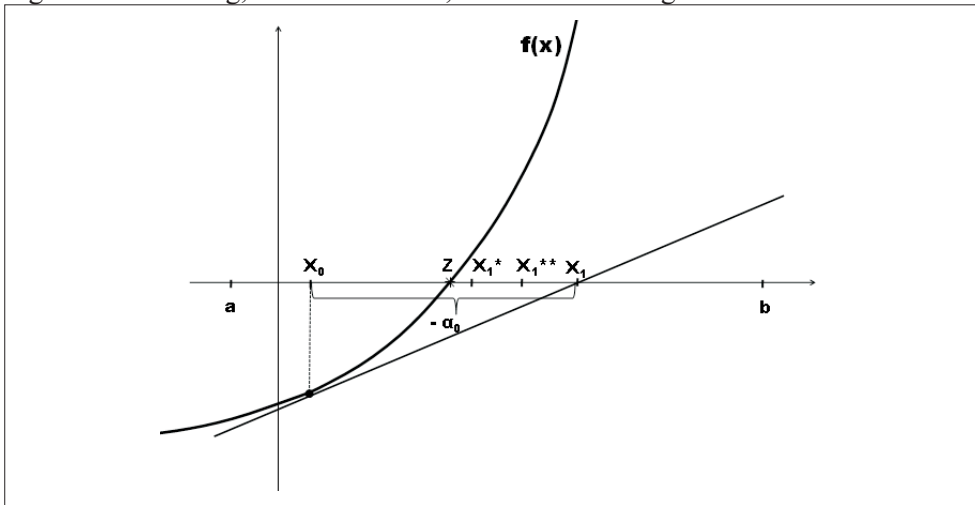
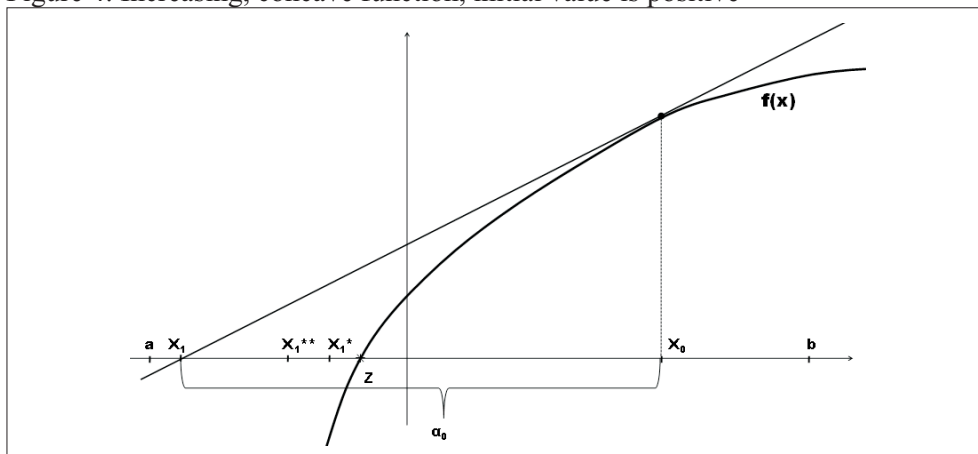We have $\alpha_0 < 0$ and $\beta_0 > 0$. Under the condition (8) we have $0 > \gamma_0 > \alpha_0$ and $x_1^* < x_1$. Similarly, $0 > \delta_0 > \alpha_0$ and $x_1^{**} < x_1$. By the same arguments as above, we also have $0 > \gamma_0 > \delta_0$ which yields $x_1^* < x_1^{**}$. In this case the assumptions (4) are not completely satisfied. We have $f(x_0) \cdot f''(x_0) < 0$. The next approximation moves from the left to the right side of the zero point. Thus, it can fly out of the domain $[a,b]$ and thus ruin the convergence. Sometimes, since $x_1^* < x_1^{**} < x_1$, this may happen for the standard Newton method but not for the modified ones (see again Fig. 3). If the approximation remains in the interval $[a,b]$ (if x_0 is close to z), then the next step will be as in the case 1.

3. f is increasing, concave and $f(x_0) > 0$ (see Fig. 4).

Since $f'(x_0) > 0$ and $f''(x_0) < 0$, we have $\alpha_0 > 0$ and $\beta_0 < 0$. If (8) holds, then we obtain $0 < \gamma_0 < \delta_0 < \alpha_0$ and thus $x_1 < x_1^{**} < x_1^*$. Here the approximation moves from the left to the right side of the zero point which can ruin the convergence (as in the case 2).

Figure 4: Increasing, concave function, initial value is positive

4. f is increasing, concave and $f(x_0) < 0$ (see Fig. 5).

Figure 5: Increasing, concave function, initial value is negative
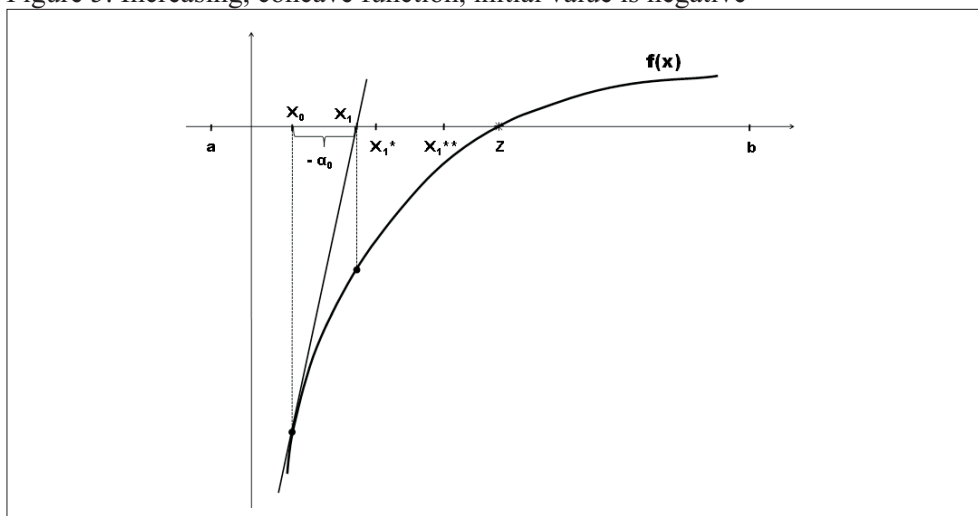
We have $\alpha_0 < 0$ and $\beta_0 < 0$. Under the condition (8) we have $0 > \alpha_0 > \gamma_0 > \delta_0$ or $x_1 < x_1^* < x_1^{**}$. The convergence is ensured if f''' is not positive by the same arguments as in the case 1.

Applications

We provide here some examples which arise from different economic problems where the application of iteration methods is necessary. We also use these examples to compare the accuracy of the methods (3), (6) and (7). All calculations are performed with 12 significant digits (10 digits are displayed).

Example 1. Demand and supply function for oil (in millions bbls) are given by

$$q_1 = 1 - \ln x, \qquad q_2 = x - 0.437,$$

where x is oil price (in hundreds \$ per bbl). What is the equilibrium oil price (in \$) ?

Solution. From the equilibrium condition $q_2 = q_1$, we obtain the equation

$$f(x) = 0, \qquad f(x) = x + \ln x - 1.437 .$$

The solution of this equation is the required price. We have

x	0.5	1	1.5
$f(x)$	- 1.630	- 0.437	0.468

So, the initial interval which contains the solution is $[a,b] = [1,15]$. Since

$$f'(x) = 1 + \frac{1}{x} > 0, \qquad f''(x) = -\frac{1}{x^2} < 0, \qquad x \in [1, 1.5],$$

the function is increasing and concave on $[1,15]$. Let us take the left initial approximation, $x_0 = 1$. This is the case 4 in the previous section. We shall solve the problem by using each of the three methods which we have considered. We have

$$\alpha_n = \frac{f(x_n)}{f'(x_n)} = \frac{x_n f(x_n)}{1 + x_n}, \qquad \beta_n = \frac{f''(x_n)}{2f'(x_n)} = \frac{-1}{2x_n(1 + x_n)}, \qquad \alpha_n \beta_n = \frac{-f(x_n)}{2(1 + x_n)^2}, \qquad n = 0,1,2,...$$

We obtain the following results.

method	x_0	x_1	x_2	x_3
(3)	1	1.2185	1.229967909	1.229992185
(6)	1	1.230435563	1.229992185	1.229992185
(7)	1	1.231125215	1.229992185	1.229992185

The exact value (10 digits) is $x = 1.229992185$. We can see that the standard Newton method, in the first iteration, gives two correct digits while the modified methods give three. In the second iteration we have five correct digits in the standard

method and ten in the modified ones. The standard Newton method reaches accuracy of ten digits in the third iteration. We also see that the first iteration in the methods (6) and (7) jumps over the zero point (from the left side to the right) and then, in the second iteration, it comes back.

Thus, the required oil price is \$123 $(x = 1.23)$, which is obtained by the modified methods (6) and (7) in the first iteration.

The methods (3), (6) and (7) can be also used to find extremes of the given function. We present it in the following example.

Example 2. A firm's total cost of production is given by

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

where Q is its output. What is the minimum of the average cost per unit of output?

Solution. We have the following formulation of the problem

$$AC = \frac{TC}{Q} = \frac{120}{Q} + 45 - Q + 0.4Q^2 \quad \rightarrow \quad \min.$$

The necessary condition for the minimum is

$$\frac{d(AC)}{dQ} = -\frac{120}{Q^2} - 1 + 0.8Q = 0.$$

Coming back to our standard notation, we have

$$f(x) = 0, \qquad f(x) = -\frac{120}{x^2} - 1 + 0.8x \quad.$$

Since

x	2	4	6	8
$f(x)$	- 29.4	- 5.3	0.467	3.525

we have the interval $[a,b] = [4,6]$. Since $f'(x) = 240 / x^3 + 0.8 > 0$ and $f''(x) = -720 / x^4 < 0$ for $x \in [4,6]$, we have again the increasing concave function. Let us take here the right initial approximation, $x_0 = 6$ (the case 3 from the previous section). We again solve the problem by using the methods (3), (6) and (7). We have

$$\alpha_n = \frac{x_n g(x_n)}{240 + 0.8x_n^3}, \quad \beta_n = \frac{-360}{x_n(240 + 0.8x_n^3)}, \quad \alpha_n \beta_n = \frac{-360 g(x_n)}{(240 + 0.8x_n^3)^2}, \quad n = 0, 1, 2, \dots \quad ,$$

where $g(x_n) = x_n^2 f(x_n) = -120 - x_n^2 + 0.8x_n^3$. We obtain the following results.

method	x_0	x_1	x_2	x_3
(3)	6	5.755813953	5.764319618	5.764331153
(6)	6	5.764480643	5.764331153	5.764331153
(7)	6	5.764183587	5.764331153	5.764331153

The accuracy of the methods is given in the following table.

The number of correct digits			
method	x_1	x_2	x_3
(3)	2	5	10
(6)	4	10	10
(7)	4	10	10

Note that the real accuracy in the third iteration is better than 10 digits but it is the computational limit.

Thus, we obtained that for $Q = 5.764331153$ units of output the average cost per unit is minimal and it is $AC_{min} (5.764331153) = 73.344$ currency units.

The next example shows that the considered methods are suitable in the situations where we need to solve several equations with the same expression (function).

Example 3. A factory used to produce 5500 products weekly and sell them by 150 HRK each. After reconstruction, the factory has three possible production capacity levels: basic (8000 products weekly), extended (10000 products) and high (12000 products). If the market demand is given by

$$Q = 740\, x e^{-0.02x}$$

where Q is the number of products and x is its price, what is the price and the total revenue at each production level?

Solution. The required prices are solutions of the equations

$$740\, x e^{-0.02x} = C_i, \quad i = 1, 2, 3,$$

where $C_1 = 8000$ (basic level), $C_2 = 10000$ (extended level), $C_3 = 12000$ (high level). Thus, for each $i = 1, 2, 3$ we define the function $f_i(x) = 740x e^{-0.02x} - C_i$ and the problem $f_i(x) = 0$. Since the first and the second derivation,

$$f_i'(x) = 740 \left(1 - 0.02x \right) e^{-0.02x}, \qquad f_i''(x) = 14.8 \left(0.02x - 2 \right) e^{-0.02x}, \qquad i = 1, 2, 3$$

are independent of i, we have

$$\alpha_n = \alpha_n(i) = \frac{740\, x_n\, e^{-0.02\, x_n} - C_i}{740\left(1-0.02 x_n\right) e^{-0.02\, x_n}}, \qquad \beta_n = \frac{0.01 x_n - 1}{50 - x_n} \qquad i = 1, 2, 3; \quad n = 0, 1, 2, \dots$$

We consider now each equation separately.

The case $i = 1$. Since $f_1(150) < 0$ and $f_1(150) < 0$, we know that the solution belongs to an interval $[a, 150]$, $a < 150$. But the actual value of a is not important. We just take the pre-reconstruction price 150 to be the initial approximation, $x_0 = 150$. We obtain

method	x_0	x_1	x_2	x_3
(3)	150	116.4295301	120.5490389	120.5942415
(6)	150	122.0644124	120.5943902	120.5942475
(7)	150	121.2545256	120.5942578	120.5942475

We can also compare the accuracy,

The number of correct digits			
method	x_1	x_2	x_3
(3)	1	4	8
(6)	2	6	10
(7)	2	7	10

The case $i = 2$. According to the obtained result for $i = 1$, we can take the starting approximation here $x_0 = 120$. We obtain

method	x_0	x_1	x_2	x_3
(3)	120	99.3129693	100.1479483	100.1478908
(6)	120	100.5356928	100.1478946	100.1478908
(7)	120	100.4674560	100.1478929	100.1478908

and thus, we have the following accuracy

The number of correct digits			
method	x_1	x_2	x_3
(3)	0	6	10

(6)	3	8	10
(7)	3	8	10

The case $i = 3$. Similarly as above we take $x_0 = 100$. We obtain

method	x_0	x_1	x_2	x_3
(3)	100	80.17746867	79.46904894	79.46566495
(6)	100	80.17746867	79.46575241	79.46566487
(7)	100	80.17746867	79.46573699	79.46566487

and

The number of correct digits			
method	x_1	x_2	x_3
(3)	0	4	8
(6)	0	5	10
(7)	0	5	10

Note that, in the first iteration, all methods yield the same result. The reason for that is $\beta_0 = 1$. Such step reduces the modified methods (6) and (7) to the standard one (3).

Note also that, in this example, we have not taken care about conditions, signs or even the interval $[a,b]$. We have just applied the methods and solved the problem.

We resume the obtained results. Since the total revenue is obtained as $TR = Q \cdot x$, we have

level	capacity	price	TR
basic	8000	120.59	964720
extended	10000	100.15	1001500
high	12000	79.47	953640

In the following example we consider a general yield (interest rate) problem for any security or any other financial instrument.

Example 4. Let the initial price of some financial instrument be c_0. Let c_i be the money flow at the end of the ith period (see Fig. 6). What is the yield for this instrument?

Figure 6: Yield problem
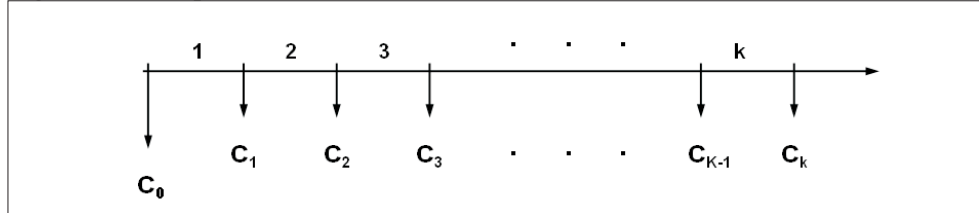
Solution. The yield equation has the form

$$C_0 = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + \cdots + \frac{C_{k-1}}{x^{k-1}} + \frac{C_k}{x^k} , \qquad x = 1 + \frac{p}{100} .$$

where p is the interest rate and $p/100$ is the yield. Multiplying this equation by x^k, we obtain

$$C_0 x^k - C_1 x^{k-1} - C_2 x^{k-2} - \cdots - C_{k-1} x^1 - C_k = 0, \qquad x = 1 + \frac{p}{100} . \tag{9}$$

Thus, x is a zero point of the polynomial (9). When x is once computed, $p/100$ and p are then easily obtained from x. According to the relation (9) we shall apply the iterative methods (3), (6) and (7) to a general polynomial,

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 .$$

We have

$$f'(x) = k a_k x^{k-1} + (k-1) a_{k-1} x^{k-2} + (k-2) a_{k-2} x^{k-3} + K + 3 a_3 x^2 + 2 a_2 x + a_1 ,$$
$$f''(x) = k(k-1) a_k x^{k-2} + (k-1)(k-2) a_{k-1} x^{k-3} + (k-2)(k-3) a_{k-2} x^{k-4} + K + 3 \cdot 2 a_3 x + 2 \cdot 1 a_2 .$$

If we introduce the vectors

$$X = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{k-2} \\ x^{k-1} \\ x^k \end{bmatrix}, \quad A_0 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-2} \\ a_{k-1} \\ a_k \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ (k-1)a_{k-1} \\ ka_k \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 \cdot 1 a_2 \\ 3 \cdot 2 a_3 \\ 4 \cdot 3 a_4 \\ \vdots \\ k(k-1)a_k \\ 0 \\ 0 \end{bmatrix} \in \mathbf{R}^{k+1},$$

we have the following simple expressions: $f(x) = A_0^T X, f'(x) = A_1^T X, f''(x) = A_2^T X$
Thus, we obtained

$$\alpha_n = \frac{A_0^T X_n}{A_1^T X_n}, \qquad \beta_n = \frac{A_2^T X_n}{2 A_1^T X_n}, \qquad n = 1,2,3,... \tag{10}$$

This approach is suitable in applications with software which has the scalar product as built in function (WOLFRAM MATHEMATICA, MATLAB, even HP calculators etc.).

Suppose now the following situation. Five years ago we bought the shares for 25000 Euro. The dividends after each year was as follows.

year	1.	2.	3.	4.	5.
dividend	2000	0	3000	1000	4000

If the recent price of these shares is 27000 EUR, what is the yield ?

Since $C_0 = 25000$ and the money flows are: $C_1 = 2000, C_2 = 0, C_3 = 3000,$
$C_4 = 1000$ and $C_5 = 4000 + 27000 = 31000,$ we have

$$25000 = \frac{2000}{x} + \frac{3000}{x^3} + \frac{1000}{x^4} + \frac{31000}{x^5}, \qquad x = 1 + \frac{p}{100} .$$

Now, the relation (9) reads $25000x^5 - 2000x^4 - 3000x^2 - 1000x - 31000 = 0.$
Thus, we need to find the root of the polynomial

$$f(x) = 25x^5 - 2x^4 - 3x^2 - x - 31 .$$

We have

$$X = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -31 \\ -1 \\ -3 \\ 0 \\ -2 \\ 25 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 \\ -6 \\ 0 \\ -8 \\ 125 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -6 \\ 0 \\ -24 \\ 500 \\ 0 \\ 0 \end{bmatrix} \in \mathbf{R}^6 .$$

We apply the methods using the relation (10). Since $x = 1 + p/100$, we take $x_0 = 1.$ We obtain

method	x_0	x_1	x_2	x_3
(3)	1	1.109090909	1.090748306	1.090075696
(6)	1	1.083666416	1.090073289	1.090074821
(7)	1	1.088471850	1.090074813	1.090074821

We have the following accuracy.

The number of correct digits			
method	x_1	x_2	x_3
(3)	1	4	6
(6)	2	6	10
(7)	2	8	10

Thus, the interest rate is $p = 9.0074821\% \approx 9\%$ and the yield is 0.09.

We conclude the applications with the following example.

Example 5. The number of products which can be sold on the market is given by

$$Q = 9\ln(10 - x) + x ,$$

where x is the product price. What is the highest price which can be achieved on the market?

Solution.   Obviously x may be increased as far as $Q > 0$. Thus, the boundary condition for the problem is $Q = 0$, or

$$f(x) = 0, \qquad f(x) = 9\ln(10 - x) + x .$$

It is clear that $x > 0$ because x denotes the price. Since the function must be well defined, we have $10 - x > 0$ or $x < 10$. Thus, the natural domain is $D(f) = \langle 0, 10 \rangle$. Now, it is obvious that $\ln(10 - x) > 0$ implies $f(x) > 0$. But, $\ln(10 - x) > 0$ holds if $10 - x > 1$ holds, that is $x < 9$. Thus, if x is the root, then $x \geq 9$. We have

x	9	9.9
$f(x)$	9	- 10.823

Note that we could not check the function value at $x = 10$ because it is out of the domain. Let us take $x_0 = 9$. Since

$$f'(x) = \frac{-9}{10 - x} + 1 = -\frac{x - 1}{10 - x}, \qquad f''(x) = \frac{-9}{(10 - x)^2},$$

we have

$$\alpha_n = -\frac{(10-x_n)f(x_n)}{x_n-1}, \qquad \beta_n = \frac{9}{2(x_n-1)(10-x_n)}, \qquad \alpha_n\beta_n = \frac{-9f(x_n)}{2(x_n-1)^2}, \qquad n=0,1,2,\text{K}$$

Now, we apply the methods. We obtain

method	x_0	x_1	x_2	x_3
(3)	9	10.125	does not exist	does not exist
(6)	9	9.413085938	9.640637394	9.658050710
(7)	9	9.688995215	9.658036764	9.658059212

and

The number of correct digits			
method	x_1	x_2	x_3
(3)	0	-	-
(6)	1	2	6
(7)	2	5	10

We can see that the first approximation in the standard Newton method (3) flew out of the domain (10.125 > 10) and thus the method broke up (which is not the case for the modified methods). If we chose the starting approximation close enough to the zero point ($x_0 \geq 9.2$ is good), then the method (3) would also converge.

Thus, we have obtained the highest price . If the price was higher than this one then we could not sell any product on that market.

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