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NEW APPROACH TO INFORMATION AGGREGATION

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In this paper new types of aggregation operators, namely absorbing-norms and parametric type of operator families called distance-based or evolutionary operators are introduced. Absorbing- norms are commutative, associative binary operators having an absorbing element from the unit interval. A detailed discussion of properties and structure of these operators is given in the paper.

Two types of distance-based operators are defined. The maximum and minimum distance operators with respect to e have the value of the element, which is farther, or nearer to e, respectively, where e is an arbitrary element of the unit interval [0,1]. The special cases e = 0 and e = 1 lead to the max and min operators. The new operators are evolutionary types in the sense that if e is increasing starting from zero till e = 1 the min operator is developing into the max operator, while on the other side the max is transformed into the min operator. It is shown that the evolutionary operators can be constructed by means of min and max operators, which are also special cases of the operators. The maximum distance operators are special operators called uninorms and the minimum distance ones are absorbing-norms.

Keywords: information aggregation, t-norms, fuzzy systems.

1. INTRODUCTION

Information aggregation is one of the key issues in development of systems using computational intelligence techniques. Computational Intelligence umbrellas and unifies neural networks, fuzzy systems and evolutionary computing. In literature a great number of publications can be found dealing with the theory and application of these building blocks. For example, some interesting integration of fuzzy logic and ARTMAP neural networks is done in [10, 11, 12] where ARTMAP – like neural networks was used to calculate values of any input membership functions to the fuzzy cluster in the feature space. This approach is called MF-ARTMAP neural fuzzy system.

Although the fuzzy set theory provides a host of attractive aggregation operators for integrating membership values representing uncertain information ([3], [4], [7], [8], [9]) the research in this area is still running. Recently, for the generalization of t-operators, the concept of uninorms was introduced by Yager and Rybalov [13], and their structure was described by Fador et.al. [1],[5] and De Baets [2]. They also studied the functional equations of Frank and Alsina [5] for two classes of commutative, associative and increasing binary operators. The first one is the class of uninorms, while the second one is the class of nullnorms.

In this paper a new approach to construction of such and similar operators is introduced. The new parametric type operator families are constructed on a distance-based approach, i.e. a distance-based operator assigns the variable to the operator which has a minimum (or maximum) distance from a given constant e lying anywhere in the unit interval. It is shown that these operators can be constructed from the max and min operators and the cases $e = 0$ and $e = 1$ lead back to the max and min. Thus the constant e serves as a parameter for the operator family, i.e. the distance-based operators form a *parametric operator family* with parameter e. They are also *evolutionary types* in the sense that if, for example, in case of minimum distance minimum operator while e is increasing starting from zero till 1 the min operator is transforming into the max operator.

The maximum distance operators are uninorms and the minimum distance operators are generalized nullnorms called absorbing-norms. The absorbing-norms introduced in this paper are commutative, associative binary operators having an absorbing element from the unit interval. The properties of these operators are also discussed in the paper.

2. T-OPERATORS, NEGATION AND SOME BASIC PROPERTIES

The original fuzzy set theory was formulated in terms of Zadeh's standard operations of minimum, maximum and complement [4]. Since 1965 for each of these operations several classes of operators, satisfying appropriate axioms, have been introduced. By accepting some basic conditions, a broad class of set of operations for union and intersection is formed by t-operators.

Definition 1 A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a *t-norm* if it is commutative, associative, non-decreasing and $T(x,1)=x$, for all $x\in[0,1]$.

Definition 2 A mapping $S:[0,1]\times[0,1]\to[0,1]$ is a *t-conorm* it is commutative, associative, nondecreasing and $S(x,0)=x$, for all $x\in[0,1]$.

Definition 3 A mapping $N:[0,1]\to[0,1]$ N is a *negation,* if non-increasing and $N(0) = 1$ and $N(1) = 0$.

N is a *strict negation if* if N is strictly decreasing and N is a *continuous function.* N is a *strong negation* if N is strict and $N(N(a)) = a$, that is, N is *involutive.*

Further it is assumed that T is a t-norm, S is a t-conorm and N is a strict negation.

3. UNINORMS

Uninorms are such kind of generations of t-norms and t-conorms where the neutral element can be any number from the unit interval. The class of uninorms seems to play an important role both in theory and application [2, 5, 13].

Definition 4 [13] A uninorm U is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e. $U(x,e) = x, \forall x \in [0,1]$.

The neutral element e is clearly unique. The case $e = 1$ leads to t-conorm and the case $e = 0$ leads to t-norm.

The first uninorms were given by Yager and Rybalov [13]

$$U_c(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{elsewhere} \end{cases} \tag{1}$$

and

$$U_d(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,e]^2 \\ \max(x,y), & \text{elsewhere} \end{cases} \tag{2}$$

U_c is a conjunctive right-continuous uninorm and U_d is a disjunctive left-continuous uninorm.

Regarding the duality of uninorms Yager and Rybalov have proved the following theorem [13].

Theorem 1 *Assume* U *is a uninorm with identity element e, then* $\overline{U}(x,y) = 1 - U(1-x, 1-y)$ *is also a uninorm with neutral element* $1-e$.

4. NULLNORMS

Definition 5 [5] A mapping $V : [0,1] \times [0,1] \to [0,1]$.

nullnorm, if there exists *an absorbing element* $a \in [0,1]$, i.e., $V(x,a) = a, \forall x \in [0,1]$, V is *commutative*, V is *associative, non-decreasing* and satisfies

$$V(x,0) = x \text{ for all } x \in [0,a] \tag{3}$$
$$V(x,1) = x \text{ for all } x \in [a,1] \tag{4}$$

The Frank equation was studied by Calvo, De Baets, and Fodor in case of uninorms and nullnorms, and they found the following [1].

Theorem 1 [1] *1. Consider a uninorm U with neutral element* $e \in [0,1]$, *then there exists no nullnorm V with absorbing element e such that the pair* (U,V) *is a solution of the Frank equation*

$$U(x,y) + V(x,y) = x + y \text{ for all } (x,y) \in [0,1] \times [0,1]. \tag{5}$$

2. Consider a nullnorm V with absorbing element $e \in [0,1]$, *then there exists no uninorm U with neutral element e such that the pair* (U,V) *is a solution of the Frank equation.*

5. ABSORBING-NORMS

We will see that a group of distance-based operators lead to a non-monotonic generalization of nullnorms.

Definition 6 A mapping $A:[0,1]\times[0,1]\to[0,1]$ is an *absorbing-norm,* if there exists an absorbing element $a\in[0,1]$, i.e., $A(x,a)=a, \forall x\in[0,1]$, and if A is commutative and associative.

It is clear that a is an idempotent element $A(a,a)=a$, hence the absorbing element is unique. Otherwise there would exist at least two absorbing elements a_1, a_2, $a_1, \neq a_2$ for which $A(a_1,a_2)=a_1$, and $A(a_1,a_2)=a_2$, so thus $a_1, = a_2$.

T-operators are special absorbing-operators, namely for any t-norm T, $T(0,x)=0, \forall x\in[0,1]$ and for any t-conorm S, $S(1,x)=1, \forall x\in[0,1]$. Nullnorms are also special absorbing-norms.

As a direct consequence of the definition we have

- if $x\le a$ then $A(x,a)=a=\max(x,a)$,
- if $x\ge a$ then $A(x,a)=a=\min(x,a)$.

These properties provide the background to define some simple absorbing-norms.

Proposition 1 *The trivial absorbing-norm* $A_T:[0,1]\times[0,1]\to[0,1]$ *with absorbing element* a *is*

$$A_T:(x,y)\to a, \forall(x,y)\in[0,1]\times[0,1]. \tag{6}$$

Proof. The statement is obvious from the definition. ■

Theorem 2 *The mapping* $A_{\min}:[0,1]\times[0,1]\to[0,1]$ *defined as*

$$A_{\min}(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[0,a]\times[0,a]\\ \min(x,y) & \text{elsewhere}\end{cases} \tag{7}$$

and the mapping $A_{\max}:[0,1]\times[0,1]\to[0,1]$ *defined as*

$$A_{\max}(x,y)=\begin{cases}\min(x,y), & \text{if } (x,y)\in[a,1]\times[a,1]\\ \max(x,y), & \text{elsewhere}\end{cases} \tag{8}$$

are absorbing-norms with absorbing element a.

Proof.
1) *Commutativity.* It follows from the definitions of the operators.
2) *Associativity.*

On the domains $[0,a]\times[0,a]$ and $[a,1]\times[a,1]$ A_{min} and A_{max} inherit the properties of max and min, so associativity is fulfilled.
Consider the rest of the unit square. Without loss of generality we can assume that $x \geq y \geq z$.

a) We have to prove that $A_{min}(A_{min}(x,y),z) = A_{min}(x, A_{min}(y,z))$.

Suppose first $x \geq a \geq y \geq z$.

$A_{min}(A_{min}(x,y),z) = A_{min}(y,z) = z$,

$A_{min}(x, A_{min}(y,z)) = A_{min}(x,z) = z$.

Suppose $x \geq y \geq a \geq z$.

$A_{min}(A_{min}(x,y),z) = A_{min}(y,z) = z$

$A_{min}(x, A_{min}(y,z)) = A_{min}(x,z) = z$.

b) The proof of $A_{max}(A_{max}(x,y),z) = A_{max}(x, A_{max}(y,z))$ can be carried out in a similar way.

3) *Absorbing element.*

a) If $x \leq a$ then $A_{min}(x,a) = \max(x,a) = a$,.

If $x \geq a$ then $A_{min}(x,a) = \min(x,a) = a$,

b) and the same is true for A_{max} . ■

The structures of these absorbing-operators are shown in Figure 1.

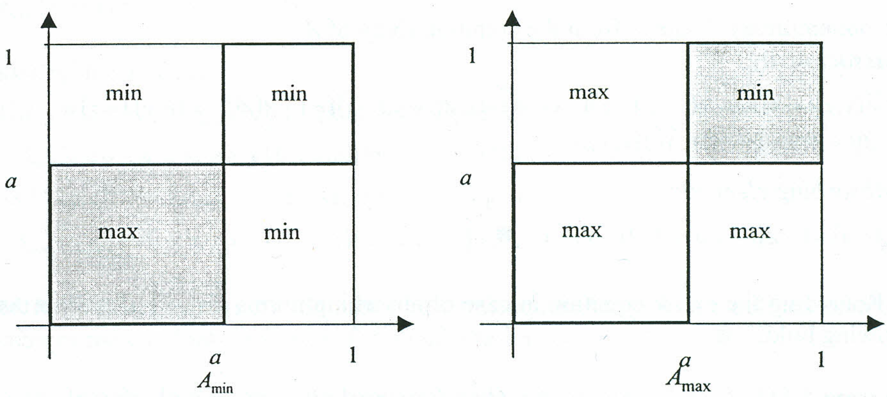

Figure 1. The structure of A_{min} and A_m .

Corollary 1 From the structure of A_{min} and A_m the following properties can be concluded:

- $A_{min}(0,0) = A_{min}(0,1) = A_{min}(1,0) = 0$, i.e. A_{min} is a conjunctive operator,
- $A_{max}(1,1) = A_{max}(0,1) = A_{max}(1,0) = 1$, i.e. A_{max} is a disjunctive operator.

With the combination of A_m , A_{max} .and A_T further absorbing-norms can be defined.

Theorem 3 *The mapping* $A^a_{min} : [0,1]\times[0,1] \to [0,1]$ *defined as*

$$A_{\min}^{a}(x,y)=\begin{cases}a, & \text{if } (x,y)\in[0,a]\times[0,a]\\ \min(x,y), & \text{elsewhere}\end{cases} \quad (9)$$

and the mapping $A_{\max}^{a}:[0,1]\times[0,1]\to[0,1]$ *defined as*

$$A_{\max}^{a}(x,y)=\begin{cases}a, & \text{if } (x,y)\in[a,1]\times[a,1]\\ \max(x,y), & \text{elsewhere}\end{cases} \quad (10)$$

are absorbing-norms with absorbing element a.

Proof. The proof can be carried out analogously to the proof of the previous theorem. ■

Theorem 4 *Assume that A is an absorbing-norm with absorbing element a. The dual operator of A denoted by* $\overline{A}$

$$\overline{A}(x,y)=1-A(1-x,1-y). \quad (11)$$

is an absorbing-norm with absorbing element 1-*a.*

Proof.

1. *Commutativity*: follows from the commutativity of *A*.
2. *Associativity*:

$$\overline{A}(x,\overline{A}(y,z))=\overline{A}(x,1-A(1-y,1-z))=1-A(1-x,A(1-y,1-z))=1-A(A(1-x,1-y),1-z)=$$
$$=1-A(1-\overline{A}(x,y),1-z)=\overline{A}(\overline{A}(x,y),z).$$

3. *Absorbing element*:

$$\overline{A}(x,1-a)=1-A(1-x,a)=1-(1-x)=x\,. \ ■$$

Regarding the Frank equation in case of absorbing norms instead of nullnorms the following hold.

Theorem 5 [1] *Consider a uninorm U with neutral element* $e\in[0,1]$, *then there exists an absorbing-norm A with absorbing element e such that the pair* (U,A) *is a solution of the Frank equation.*

$$U(x,y)+V(x,y)=x+y \text{ for all } (x,y)\in[0,1]\times[0,1]. \quad (12)$$

Proof. From the Frank equation we have $A(x,y)=x+y-U(x,y)$ so *A* is commutative and associative due to the same properties of *U.* If $y=e$ then

$A(x,e)=x+e-U(x,e)=e$ for all $x\in[0,1]$ so all the axioms of absorbing-norms are fulfilled. ■

The reversal of the theorem is also valid in case of some absorbing-norms.

Theorem 6 *Consider the absorbing-norms* $A_{\min}$ *and* $A_{\max}$ *with absorbing element* $a \in [0,1]$, *then the pairs* $(U_d, A_{\min})$ *and* $(U_c, A_{\max})$ *are solutions of the Frank equation, where each of the uninorms* U_d *and* U *has neutral element a.*

Proof. It is easy to see that the min and max operators satisfy the Frank equation, that is $\max(x,y) + \min(x,y) = x + y$ for all $(x,y) \in [0,1]$. By means of this equation the uninorms U_d and U are obtained simply by replacing the min operator with the max and the max operator with the min. ■

The construction of the operators results in that for the pairs $(A_{\min}, U_d)$ and $(A_{\max}, U\ )$ the laws of absorption and distributivity are fulfilled.

Proposition 2. *For the pairs* $(A_{\min}, U_d)$ *and* $(A_{\max}, U_c)$ *the following hold*

1. Absorption laws

$$A_{\min}(U_d(x,z),x) = x \text{ for all } x \in [0,1]\ , \tag{13}$$

$$U_d(A_{\min}(x,z),x) = x \text{ for all } x \in [0,1]\ , \tag{14}$$

$$A_{\max}(U_c(x,z),x) = x \text{ for all } x \in [0,1]\ , \tag{15}$$

$$U_c(A_{\max}(x,z),x) = x \text{ for all } x \in [0,1]\ . \tag{16}$$

2. Laws of distributivity

$$A_{\min}(x, U_d(y,z)) = U_d(A_{\min}(x,y), A_{\min}(x,z)) \text{ for all } x \in [0,1]\ , \tag{17}$$

$$U_d(x, A_{\min}(y,z)) = A_{\min}(U_d(x,y), U_d(x,z)) \text{ for all } x \in [0,1]\ , \tag{18}$$

$$A_m\ (x, U\ (y,z)) = U\ (A_m\ (x,y), A_m\ (x,z)) \text{ for all } x \in [0,1]\ , \tag{19}$$

$$U_c(x, A_{\max}(y,z)) = A_{\max}(U_c(x,y), U_c(x,z)) \text{ for all } x \in [0,1]\ . \tag{20}$$

Proof. In each disjunctive sub-domain of the unit square the pairs are defined as min and max or max and min operators for which these properties hold. ■

6. THE STRUCTURE OF ABSORBING-NORMS

Like uninorms the structure of absorbing-norms is closely related to t-norms and t-conorms on the domains $[0,a] \times [0,a]$ and $[a,1] \times [a,1]$.

Theorem 7. *Let be S and T a t-conorm and a t-norm, respectively, with the properties* $S(x,a) = a$ and $T(x,a) = a$. *The mapping* $A_{\min}^{ST} : [0,1] \times [0,1] \to [0,1]$

$$A_{\min}^{ST}(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [0,a] \times [0,a] \\ T(x,y), & \text{if } (x,y) \in [a,1] \times [a,1] \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{21}$$

and the mapping $A_{\max}:[0,1]\times[0,1]\to[0,1]$

$$A_{\max}^{ST}(x,y)=\begin{cases} S(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ T(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases} \qquad (22)$$

are absorbing-norms with absorbing element a.

Proof.

1) *Commutativity.* It follows from the definitions of the operators.
2) *Associativity.*

On the domains $[0,a]\times[0,a]$ and $[a,1]\times[a,1]$ A_{m}^{ST} and $A_{\max}^{ST}$ inherit the properties of S and T, so associativity is fulfilled.

Consider the rest of the unit square. The inequalities used in the proof are direct consequences of the monotony of S and T.

$$S(x,y)\le S(x,a)=a \text{ if } (\mathrm{x},\mathrm{y})\in[0,a]\times[0,a] \qquad (23)$$

$$a=T(x,a)\le T(x,y) \text{ if } (\mathrm{x},\mathrm{y})\in[a,1]\times[a,1] \qquad (24)$$

a) First we have to prove that $A_{\mathrm{m}}^{ST}\left(A_{\mathrm{m}}^{ST}(x,y),z\right)=A_{\mathrm{m}}^{ST}\left(x,A_{\mathrm{m}}^{ST}(y,z)\right)$.

Suppose first that $x\ge a\ge y\ge z$.

$A_{\min}^{ST}\left(A_{\min}^{ST}(x,y),z\right)=A_{\min}^{ST}(y,z)=S(y,z)$,

$A_{\min}^{ST}\left(x,A_{\min}^{ST}(y,z)\right)=A_{\min}^{ST}(x,S(y,z))=S(y,z)$ since $S(y,z)\le a$.

Suppose $x\ge y\ge a\ge z$.

$A_{\min}^{ST}\left(A_{\min}^{ST}(x,y),z\right)=A_{\min}^{ST}(T(x,z),z)=z$ since $T(x,y)\ge a$

$A_{\min}^{ST}\left(x,A_{\min}^{ST}(y,z)\right)=A_{\min}^{ST}(x,z)=z$.

b) $A_{\max}^{a}\left(A_{\max}^{a}(x,y),z\right)=A_{\max}^{a}\left(x,A_{\max}^{a}(y,z)\right)$ should be proved.

Assume $x\ge a\ge y\ge z$.

$A_{\max}^{ST}\left(A_{\max}^{ST}(x,y),z\right)=A_{\max}^{ST}(x,z)=x$,

$A_{\max}^{ST}\left(x,A_{\max}^{ST}(y,z)\right)=A_{\max}^{ST}(x,S(y,z))=x$ since $S(y,z)\le a$.

Suppose now $x\ge y\ge a\ge z$.

$A_{\max}^{ST}\left(A_{\max}^{ST}(x,y),z\right)=A_{\max}^{ST}(T(x,y),z)=T(x,y)$,

$A_{\max}^{ST}\left(x,A_{\max}^{ST}(y,z)\right)=A_{\max}^{ST}(x,y)=T(x,y.)$.

3) *Absorbing element.* It is satisfied by the assumptions of the theorem. ■

T and *S* are called the underlying t-norm and t-conorm of the absorbing-norms, respectively [5].

7. DEFINITIONS OF DISTANCE-BASED EVOLUTIONARY OPERATORS

Let e be an arbitrary element of the closed unit interval $[0,1]$ and denote by $d(x,y)$ the distance of two elements x and y of $[0,1]$. The idea of definitions of distance-based operators is generated from the reformulation of the definition of the min and max operators as follows.

$$\min(x,y)=\begin{cases} x, & \text{if } d(x,0)\le d(y,0) \\ y, & \text{if } d(x,0)>d(y,0) \end{cases}$$

$$\max(x,y)=\begin{cases} x, & \text{if } d(x,0)\ge d(y,0) \\ y, & \text{if } d(x,0)<d(y,0) \end{cases}$$

Definition 7 The *maximum distance minimum operator with respect to* $e\in[0,1]$ is defined as

$$\max_e^{\min}(x,y)=\begin{cases} x, & \text{if } d(x,e)>d(y,e) \\ y, & \text{if } d(x,e)<d(y,e). \\ \min(x,y), & \text{if } d(x,e)=d(y,e) \end{cases} \tag{25}$$

Definition 8 The *maximum distance maximum operator with respect to* $e\in[0,1]$ is defined as

$$\max_e^{\max}(x,y)=\begin{cases} x, & \text{if } d(x,e)>d(y,e) \\ y, & \text{if } d(x,e)<d(y,e). \\ \max(x,y), & \text{if } d(x,e)=d(y,e) \end{cases} \tag{26}$$

Definition 9 The *minimum distance minimum operator with respect to* $e\in[0,1]$ is defined as

$$\min_e^{\min}(x,y)=\begin{cases} x, & \text{if } d(x,e)<d(y,e) \\ y, & \text{if } d(x,e)>d(y,e). \\ \min(x,y), & \text{if } d(x,e)=d(y,e) \end{cases} \tag{27}$$

Definition 10 The *minimum distance maximum operator with respect to* $e\in[0,1]$ is defined as

$$\min_e^{\max}(x,y)=\begin{cases} x, & \text{if } d(x,e)<d(y,e) \\ y, & \text{if } d(x,e)>d(y,e). \\ \max(x,y), & \text{if } d(x,e)=d(y,e) \end{cases} \tag{28}$$

8. THE STRUCTURE OF EVOLUTIONARY OPERATORS

It can be proved by simple computation that the distance-based evolutionary operators can be expressed by means of the min and max operators as follows.

$$\max_e^{\min}=\begin{cases} \max(x,y), & \text{if } y>2e-x \\ \min(x,y), & \text{if } y<2e-x \\ \min(x,y), & \text{if } y=2e\text{-}x \end{cases} \tag{29}$$

$$\min_e^{\min} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{30}$$

$$\max_e^{\max} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{31}$$

$$\min_e^{\max} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{32}$$

The structures of the evolutionary operators are illustrated in Fig. 2.

9. PROPERTIES OF DISTANCE-BASED OPERATORS

Theorem 8 *The distance-based operators have the following properties.*

$\max_e^{\min}$

- $\max_e^{\min}(x,x) = x, \forall x \in [0,1]$, *that is* $\max_e^{\min}$ *is idempotent,*
- $\max_e^{\min}(e,x) = x$ *that is, e is the neutral element,*
- $\max_e^{\min}$ *is commutative and associative,*
- $\max_e^{\min}$ *is left continuous,*
- $\max_e^{\min}$ *is increasing on each place of* $[0,1] \times [0,1]$.

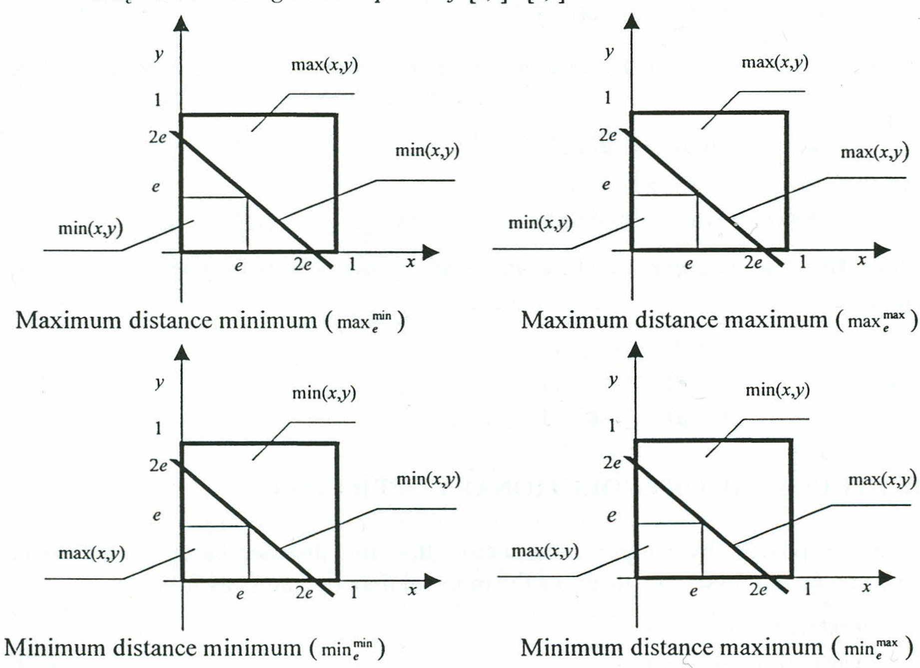

Figure 2. Distance-based operators

$\max_e^{\max}$

- $\max_e^{\max}(x,x)=x, \forall x\in[0,1]$, *that is* $\max_e^{\max}$ *is idempotent,*
- $\max_e^{\max}(e,x)=x$ *that is, e is the neutral element,*
- $\max_e^{\max}$ *is commutative and associative,*
- $\max_e^{\max}$ *is right continuous,*
- $\max_e^{\max}$ *is increasing on each place of* $[0,1]\times[0,1]$.

$\min_e^{\min}$

- $\min_e^{\min}(x,x)=x, \forall x\in[0,1]$, *that is* $\min_e^{\min}$ *is idempotent,*
- $\min_e^{\min}(e,x)=e$ *that is, e is an absorbing element,*
- $\min_e^{\min}$ *is right continuous,*
- $\min_e^{\min}$ *is commutative and associative.*

$\min_e^{\max}$

- $\min_e^{\max}(x,x)=x, \forall x\in[0,1]$, *that is* $\min_e^{\max}$ *is idempotent,*
- $\min_e^{\max}(e,x)=e$ *that is, e is the absorbing element,*
- $\min_e^{\max}$ *is left continuous,*
- $\min_e^{\max}$ *is commutative and associative.*

Corollary 2

- $\max_e^{\min}$ *and* $\max_e^{\max}$ *are uninorms,*
- $\min_e^{m}$ *and* $\min_e^{\max}$ *are absorbing norms,*
- *both of the operators are compensatory ones.*

For duality we have the following corollary.

Corollary 3

a) The dual operators of the uninorm $\max_e^{m}$ *is* $\max_{1-e}^{\max}$, *and*
b) the dual operators of the uninorm $\max_e^{\max}$ *is* $\max_{1-e}^{m}$.
c) The dual operators of the absorbing-norm $\min_e^{m}$ *is* $\min_{1-e}^{\max}$, *and*
d) the dual operators of the absorbing-norm $\min_e^{\max}$ *is* $\min_{1-e}^{m}$.

Proof.

a) $$\overline{\max_e^{\min}}(x,y)=1-\max_e^{\min}(1-x,1-y)=\begin{cases}1-\max(1-x,1-y), & \text{if } y<2(1-e)-x\\ 1-\min(1-x,1-y), & \text{if } y>2(1-e)-x\\ 1-\min(1-x,1-y), & \text{if } y=2(1-e)-x\end{cases}=$$

$$=\begin{cases}\min(x,y), & \text{if } y<2(1-e)-x\\ \max(x,y), & \text{if } y>2(1-e)-x\\ \max(x,y), & \text{if } y=2(1-e)-x\end{cases}=\max_{1-e}^{\max}$$

b) $$\overline{\max_e^{\max}}(x,y)=1-\max_e^{\max}(1-x,1-y)=\begin{cases}1-\max(1-x,1-y), & \text{if } y<2(1-e)-x\\ 1-\min(1-x,1-y), & \text{if } y>2(1-e)-x\\ 1-\max(1-x,1-y), & \text{if } y=2(1-e)-x\end{cases}=$$

$$= \begin{cases} \min(x,y), & \text{if } y < 2(1-e)-x \\ \max(x,y), & \text{if } y > 2(1-e)-x \\ \min(x,y), & \text{if } y = 2(1-e)-x \end{cases} = \max_{1-e}^{\min}.$$

c) $$\overline{\min_e^{\min}}(x,y) = 1 - \min_e^{\min}(1-x, 1-y) = \begin{cases} 1-\min(1-x,1-y), & \text{if } y < 2(1-e)-x \\ 1-\max(1-x,1-y), & \text{if } y > 2(1-e)-x \\ 1-\min(1-x,1-y), & \text{if } y = 2(1-e)-x \end{cases} =$$

$$= \begin{cases} \min(x,y), & \text{if } y > 2(1-e)-x \\ \max(x,y), & \text{if } y < 2(1-e)-x \\ \max(x,y), & \text{if } y = 2(1-e)-x \end{cases} = \min_{1-e}^{\max}$$

d) $$\overline{\min_e^{\max}}(x,y) = 1 - \min_e^{\max}(1-x, 1-y) = \begin{cases} 1-\min(1-x,1-y), & \text{if } y < 2(1-e)-x \\ 1-\max(1-x,1-y), & \text{if } y > 2(1-e)-x \\ 1-\max(1-x,1-y), & \text{if } y = 2(1-e)-x \end{cases} =$$

$$= \begin{cases} \max(x,y), & \text{if } y < 2(1-e)-x \\ \min(x,y), & \text{if } y > 2(1-e)-x \\ \min(x,y), & \text{if } y = 2(1-e)-x \end{cases} = \min_{1-e}^{\min}. \ \blacksquare$$

10. DISTANCE-BASED OPERATORS AS PARAMETRIC EVOLUTIONARY OPERATORS

The min and max operators as special cases of distance-based operators can be obtained depending on e as follows:

a) if $e = 0$ then

$$\max_0^{\min}(x,y) = \max(x,y), \tag{33}$$

$$\max_0^{\max}(x,y) = \max(x,y), \tag{34}$$

$$\min_0^{\min}(x,y) = \min(x,y), \tag{35}$$

$$\min_0^{\max}(x,y) = \min(x,y), \tag{36}$$

b) if $e = 1$ then

$$\max_1^{\min}(x,y) = \min(x,y), \tag{37}$$

$$\max_1^{\max}(x,y) = \min(x,y), \tag{38}$$

$$\min_1^{m}(x,y) = \max(x,y), \tag{39}$$

$$\min_1^{\max}(x,y) = \max(x,y). \tag{40}$$

This means that the distance-based operators form a *parametric family* with parameter e. They are also *evolutionary types* in the sense that if for example in case of $\min_e^{\min}$ while e is increasing starting from zero till 1 the min operator is developing into the max operator.

11. CONCLUSIONS

A distance-based generalization of the conventional min and max operators is introduced.

The maximum and minimum distance operators with respect to $e \in [0,1]$ have the value of the element, which has a longer or shorter distance from e, respectively. The special cases $e = 0$ and $e = 1$ lead to the max and min operators. The new operators are evolutionary types in the sense that if e is increasing starting from zero till $e = 1$ the min operator is developing into the max operator, while on the other side the max is transformed into the min operator. It is shown that the evolutionary operators can be constructed by means of the min and max operators.

The maximum distance operators are uninorms and the minimum distance operators are generalized nullnorms called absorbing-norms. The absorbing norms, introduced in this paper are commutative, associative binary operators having an absorbing element from the unit interval. The properties of these operators are also discussed in the paper.

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NOV PRISTUP AGREGACIJI INFORMACIJA

Sažetak

U ovom radu predstavljeni su novi tipovi agregacijskih operatora, to jest apsorbirajuće norme i parametarski tip porodice operatora pod nazivom operatori bazirani na udaljenosti ili evolucionarni operatori. Apsorbirajuće norme su komutativni, asocijativni binarni operatori koji imaju apsorbirajući element s jediničnog intervala. Podrobna diskusija o svojstvima kao i struktura ovih operatora prikazane su u ovom radu.

Definirana su dva tipa operatora baziranih na udaljenosti. Operatori za maksimalnu i minimalnu udaljenost s obzirom na e imaju vrijednost elementa koji je dalje odnosno bliže e, pri čemu je e proizvoljni element jediničnog intervala [0,1]. Posebni slučajevi e = 0 i e = 1 dovode do maksimalnih i minimalnih operatora. Novi operatori su evolucionarni tipovi u smislu da, ako se ne povećava, počevši od nule do e = 1, minimalni operator se pretvara u maksimalni operator, dok se s druge strane maksimalni operator transformira u minimalni operator. Prikazano je da se evolucionarni operatori mogu konstruirati pomoću minimalnih i maksimalnih operatora koji su također i posebni slučajevi operatora. Operatori za maksimalnu udaljenost su posebni operatori zvani uninorme, a operatori za minimalnu udaljenost su apsorbirajuće norme.

Ključne riječi: agregacija informacija, t-norme, neizraziti sustavi.