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A NOTE ON APPROXIMATE INVERSE SYSTEMS AND SUBSYSTEMS

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The main purpose of this paper is to study the relationships between the limit of an approximate inverse system and the limits of its approximate subsystems.

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1 INTRODUCTION

All spaces in this paper are Tychonoff spaces. Cov(X) is the set of all normal coverings of a topological space X. For other details see [1]. If a covering $\mathcal{V}$ is a refinement of a covering $\mathcal{U}$, then we write $\mathcal{V} \prec \mathcal{U}$.

In this paper we study the approximate inverse system in the sense of S. Mardešić [10].

DEFINITION 1.1 *An approximate inverse system is a collection* $\mathbf{X} = \{X_a, p_{ab}, A\}$, *where* $(A,\leq)$ *is a directed preordered set,* $X_a, a \in A$, *is a topological space and* $p_{ab}: X_b \to X_a, a \leq b$, *are mappings such that* $p_{aa} = id$ *and the following condition (A2) is satisfied:*

(A2) For each $a \in A$ *and each normal cover* $\mathcal{U} \in Cov(X_a)$ *there is an index* $b \geq a$ *such that* $(p_{ac}p_{cd}, p_{ad}) \prec \mathcal{U}$, *whenever* $a \leq b \leq c \leq d$.

DEFINITION 1.2 *An approximate map* $p = \{p_a : a \in A\} : X \to X_a$ *into an approximate inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is a collection of maps* $p_a : X \to X_a$, $a \in A$, *such that the following condition holds*

(AS) *For any* $a \in A$ *and any* $\mathcal{U} \in Cov(X_a)$ *there is* $b \geq a$ *such that* $(p_{ac}p_c, p_a) \prec \mathcal{U}$ *for each* $c \geq b$. (See [12]).

DEFINITION 1.3 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system and let* $\mathbf{p} = \{p_a : a \in A\} : X \to X_a$ *be an approximate map. We say that* $\mathbf{p}$ *is a limit of* $\mathbf{X}$ *provided it has the following universal property* [12, p. 592]:

(UL) *For any approximate map* $\mathbf{q} = \{q_a : a \in A\} : Y \to X_a$ *of a space* Y *there exists a unique map* $g : Y \to X$ *such that* $p_a g = q_a$ *for any* $a \in A$.

REMARK 1.4 If $\mathbf{p}: X \to \mathbf{X}$ is a limit of $\mathbf{X}$, then the space X is determined up to a unique homeomorphism.Therefore, we often speak of the limit X of $\mathbf{X}$ and we write $X = \lim \mathbf{X}$.

DEFINITION 1.5 [12, p. 592, Definition (1.12)]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system. A point* $x = (x_a) \in \prod\{X_a : a \in A\}$ *is called a thread of* $\mathbf{X}$ *provided it satisfies the following condition:*

(L) $(\forall a \in A)(\forall \mathcal{U} \in Cov(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in st(x_a, \mathcal{U})$.

REMARK 1.6 If X_a is a $T_{3.5}$ space, then the sets $st(x_a, \mathcal{U})$, $\mathcal{U} \in Cov(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of Tychonoff spaces, the condition (L) is equivalent to the following condition [12, Remark (1.13)]:

(L)* $(\forall a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a$.

The following theorem shows that the set of threads is a limit of $\mathbf{X}$.

THEOREM 1.7 [12, Theorem (1.14)]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system. Let* $X \subset \prod X_a$ *be the set of all threads of* $\mathbf{X}$ *and let* $p_a : X \to X_a$ *be the restriction* $p_a = \pi_a | X$ *of the projection* $\pi_a : \prod X_a \to X_a, a \in A$. *Then* $\mathbf{p} = \{p_a : a \in A\} \to \mathbf{X}$ *is a limit of* $\mathbf{X}$.

The *canonical limit* of $\mathbf{X}$ is the set of all threads of $\mathbf{X}$ [12, p. 593] .

An approximate inverse system is said to be *commutative* provided it satisfies the commutativity condition [12, (1.4) Definition]:

(C) $$p_{ab}p_{bc} = p_{ac} \quad \text{for } a<b<c.$$

An inverse system in the sense of [3, p. 135] we call a *usual inverse system.* By virtue of [12, Remark (1.15)] if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a commutative approximate inverse system and all X_a are Tychonoff spaces, then the limit of $\mathbf{X}$ in the usual sense and in the approximate sense coincide.

A basis of (open) normal coverings of a space X is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U}\in\text{Cov}(X)$ admits a refinement $\mathcal{V}\in \mathcal{C}$. We denote by $cw(X)$ (*covering weight*) the minimal cardinal of a basis of normal coverings of X [13, p. 181].

LEMMA 1.8 [13, Example 2.2]. *If X is a compact Hausdorff space, then $cw(X) = w(X)$.*

2 WELL-ORDERED APPROXIMATE INVERSE SYSTEMS

Let τ be a cardinal number. We say that $(A,\leq)$ is *τ-directed* if for each $B\subseteq A$ with $\text{card}(B)\leq \tau$ there exists an $a\in A$ such that $a\geq b$ for each $b\in B$. An approximate inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-directed if A is τ-directed. We say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if it is $\aleph_0$-directed.

LEMMA 2.1 *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a τ-directed approximate inverse system of Tychonoff spaces X_a such that $cw(X_a)\leq \tau$ for each $a\in A$. Then for each $a\in A$ there exists an $a^* \in A$ such that*

$$p_{ac} = p_{ab}p_{bc} \quad \textit{and} \quad p_a = p_{ab}p_b \qquad a^* \leq b \leq c. \tag{1}$$

Proof. Let $\mathcal{U}_a$ be a basis of the normal coverings of X_a. By virtue of (A2), (AS) and the directedness of A for each normal covering $\mathcal{U} \in \mathcal{U}_a$ there exists an $a(\mathcal{U})\in A$ such that

$$(p_{ac}, p_{ab}p_{bc}) \prec \mathcal{U} \quad \textit{and} \quad (p_a, p_{ab}p_b) \prec \mathcal{U} \qquad a(\mathcal{U}) \leq b \leq c. \tag{2}$$

The set $\{a(\mathcal{U}):\mathcal{U}\in\text{Cov}(X_a)\}$ has the cardinality equal to $cw(X_a)\leq \tau$. There exists an $a^* \in A$ such that $a^* \geq a(\mathcal{U})$ for each $\mathcal{U}$ since A is τ-directed. Let us prove (1).

Suppose that $p_{ac}(x) \neq p_{ab}p_{bc}(x)$ for a given point x of X_c. There exists a pair of disjoint open sets U and V such that $p_{ac}(x) \in U$ and $p_{ab}p_{bc}(x) \in V$. By virtue of Remark 1.6, the sets $st(p_{ac}(x),\mathcal{U})$, $\mathcal{U} \in Cov(X_a)$ form a basis of the topology at the point $p_{ac}(x)$. This means that there is $\mathcal{V} \in \mathcal{U}_a$ such that $st(p_{ac}(x),\mathcal{V}) \subseteq U$. We infer that $(p_{ab}p_{bc}, p_{ac}) \not\prec \mathcal{V}$. This contradicts the definition of a*. Similarly, it follows that $p_a = p_{ab}p_b$.■

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system. In the sequel p_a^X denotes the natural projection $p_a^X : \lim\mathbf{X} \to X_a$.

THEOREM 2.2 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of topologically complete spaces such that* $cw(X_a) < \tau$, $a \in A$, *and* $card(cf(A)) \geq \tau$. *Then there exist:*

1. *a set B cofinal in A,*
2. *a usual inverse system* $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ *such that* $Y_b = X_a$ *for some* $a \in A$,
3. *a homeomorphism* $H : \lim\mathbf{X} \to \lim\mathbf{Y}$ *such that* $p_a^Y H = p_a^X$, $a \in A$.

Proof. Let $card(A) = \aleph_\mu$. We may assume that A is the set of all ordinal numbers α of the cardinality $< \aleph_\mu$. Thus

$$A = \{\alpha : \alpha < \omega_\mu\}.$$

If B′ is a cofinal subset of A, then $\{X_b, p_{ab}, B'\}$ has the limit homeomorphic to $\lim\mathbf{X}$ [12, Theorem (1.19)]. Thus, passing to a cofinal subsystem (if it is necessary), we may assume that ω_μ is a regular ordinal number and $\tau < \aleph_\mu$. Let us observe that A is τ-directed. Let a be any member of A. By transfinite induction we define a set

$$B = \{b_\alpha : \alpha < \omega_\mu\}$$

cofinal in A such that

$$b_1 < b_2 < ... < b_\alpha < ..., \quad \alpha < \omega_\mu.$$

By virtue of Lemma 2.1 there exists an a*. Let $b_1 = a^*$. Suppose that b_α is defined for each $\alpha < \beta < \omega_\mu$. Let us define b_β. If β is a non-limit ordinal, then there exists $\gamma = \beta - 1$. Define $b_\beta = (b_\gamma)^*$. If β is a limit ordinal, then $card(\{b_\alpha : \alpha < \beta\}) < \aleph_\mu$. Thus $\{b_\alpha : \alpha < \beta\}$ is not cofinal in A. This means that there exists an a∈A such that a>b_α for each $\alpha < \beta$. We set $b_\beta = a$. The set B is defined. It is clear that card(B) = $\aleph_\mu$. Hence B is cofinal in A. It remains to prove that if $b_\alpha < b_\beta < b_\gamma$ then

$$p_{b_\alpha b_\gamma} = p_{b_\alpha b_\beta} p_{b_\beta b_\gamma}.$$

It is clear that $(b_\alpha)^* = b_{\alpha+1} \leq b_\beta$. By virtue of (1) for $a = b_\alpha$, $b = b_\beta$, $c = b_\gamma$ we have

$$p_{b_\alpha b_\gamma} = p_{b_\alpha b_\beta} p_{b_\beta b_\gamma}.$$

Thus, **Y** is a usual inverse system. By virtue of [12, Theorem (1.19)] there exists a homeomorphism H : lim**X**→lim**Y**. ■

COROLLARY 2.3 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of compact spaces such that* $w(X_a) < \tau$, $a \in A$, *and card(cf(A))*$\geq \tau$. *Then there exist:*

1. *a set B cofinal in A,*
2. *a usual inverse system* $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ *such that* $Y_b = X_a$ *for some* $a \in A$,
3. *a homeomorphism H : lim***X***→lim***Y** *such that* $p_a^Y H = p_a^X$, $a \in A$.

Proof. Each compact space X is topologically complete and cw(X) = w(X) (Lemma 1.8). Apply Theorem 2.2. ■

COROLLARY 2.4 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of compact spaces such that* $w(X_a) < \tau$, $a \in A$, *and card(cf(A))*$\geq \tau$. *Then w(lim***X***)*$\leq \tau$.

Proof. By virtue of Theorem 2.3 there exists a usual inverse system $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ such that $Y_b = X_a$ for some a∈A and there exists a homeomorphism H : lim**X**→lim**Y**. Applying [18, Teorema 2.2.] we complete the proof.■

COROLLARY 2.5 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of compact metric spaces such that card(cf(A))*$\geq \aleph_1$. *Then there exist:*

1. *a set B cofinal in A,*
2. *a usual inverse system* $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ *such that* $Y_b = X_a$ *for some* $a \in A$,
3. *a homeomorphism H : lim***X***→lim***Y** *such that* $p_a^Y H = p_a^X$, $a \in A$.

3 APPROXIMATE SYSTEMS AND APPROXIMATE SUBSYSTEMS

We start with the following definition.

DEFINITION 3.1 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system and let* B *be a directed subset of* A *such that* $\{X_b, p_{bc}, B\}$ *is an approximate inverse system. We say that* $\{X_b, p_{bc}, B\}$ *is an approximate subsystem of* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *if there exists a mapping* $q : lim\mathbf{X} \rightarrow lim\{X_b, p_{bc}, B\}$ *such that*

$$p_b q = P_b, \quad b \in B,$$

where $p_b : lim\{X_b, p_{bc}, B\} \rightarrow X_b$ *and* $P_b : lim\mathbf{X} \rightarrow X_b$, $b \in B$, *are natural projections.*

The next theorem is the main theorem of this Section.

THEOREM 3.2 *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces. If* $w(X_a) < card(A)$ *for each* $a \in A$, *then* $lim\boldsymbol{X}$ *is homeomorphic to a limit of a well-ordered usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < card(A)\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $card(\Phi) < card(A)$.

Proof. The proof consists of several steps.
Step 1. *For each subset* B *of* A *there exists a directed set* $F_\infty(B)$ *such that* $card(F_\infty(B)) = card(B)$.
Proof of Step 1. See [9, pp. 238 - 239, Hilfssatz]. For the sake of completeness we give proof for Step 1. We consider two cases.
Card(A)$\leq \aleph_0$. Let ν be any finite subset of A. There exists a $\delta(\nu) \in$A such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. Since A is infinite, there exists a sequence $\{\nu_n\text{: n} \in I\!N\}$ such that $\nu_1 \subseteq ... \nu_n \subseteq ...$ and A $= \bigcup\{\nu_n\text{: n} \in I\!N\}$. Recursively, we define the sets $A_1, ..., A_n, ...$ by

$$A_1 = \nu_1 \bigcup \{\delta(\nu_1)\},$$

and

$$A_{n+1} = A_n \bigcup \nu_{n+1} \bigcup \{\delta(A_n \bigcup \nu_{n+1})\}.$$

Card(A)$> \aleph_0$. For each B$\subseteq$A there exists a set F_1(B) = B$\bigcup\{\delta(\nu)\text{: } \nu \in$B$\}$, where ν is a finite subset of B and $\delta(\nu)$ is defined as in the first case. Put

$$F_{n+1} = F_1(F_n(B),$$

and

$$F_\infty(B) = \bigcup\{F_n(B) : n \in I\!N\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq ... \subseteq F_n(B) \subseteq ...$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$.

If B is finite, then card($F_\infty(B)$) $= \aleph_0$. If card(B)$\geq \aleph_0$, then we have card($\{\delta(\nu)$: $\nu \in$B$\}$)$\leq$card(B)$\aleph_0$. We infer that card($F_1(B)$)$\leq$card(B)$\aleph_0$. Similarly, card($F_n(B)$) $\leq$card(B)$\aleph_0$. This means that card($F_\infty(B)$)$\leq$card(B)$\aleph_0$. Thus

$$card(F_\infty(B)) \leq card(B)\aleph_0, \qquad if \qquad card(B) < card(A).$$

Step 2. *For each subset B of A with card(B)<card(A), there exists a directed set $G_\infty(B) \supseteq B$ such that the collection $\{X_a, p_{ab}, G_\infty(B)\}$ is an approximate system.*
For each subset B of A we define $G_\infty(B)$ by induction as follows:

a) Let $G_1(B) = F_\infty(B)$,

b) For each n>1 we define $G_n(B)$ as follows:

1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$,

2) If n is even, then $G_n(B) = G_{n-1}(B) \cup\{a^* : a \in G_{n-1}(B)\}$.

Now we define $G_\infty(B) = \cup\{G_n(B) : n \in I\!N\}$. It is obvious that card($G_\infty(B)$)$\leq$ card(A).

The set $G_\infty(B)$ is directed. Let a,b be a pair of the elements of $G_\infty(B)$. There exists a n$\in I\!N$ such that a,b$\in G_n(B)$.We may assume that n is odd. Then a,b$\in F_\infty(G_{n-1}(B))$. Thus there exists a c$\in F_\infty(G_{n-1}(B))$ such that c$\geq$a,b. It is clear that c$\in G_\infty(B)$. The proof of directedness of $G_\infty(B)$ is completed.

The collection $\{X_a, p_{ab}, G_\infty(B)\}$ is an approximate system. It suffices to prove that the condition (A2) is satisfied. Let a be any member of $G_\infty(B)$. There exists an n$\in$N such that a$\in G_n(B)$. We have two cases.

1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$. This means that a$\in F_\infty(G_{n-1}(B))$. By definition of $F_\infty(G_{n-1}(B))$ we infer that a* $\in F_\infty(G_{n-1}(B))$. Thus (A2) is satisfied.

2) If n is even, then $G_n(B) = G_{n-1}(B) \cup\{a^* : a \in G_{n-1}(B)$ such that for each normal cover of X_a, a$\in G_1(B)$, there exists a* with the property (A2) and (AS) $\}$. In this case a$\in G_{n+1}(B) \subseteq G_\infty(B)$. Arguing as in the case 1, we infer that (A2) is satisfied.

Step 3. Let card(A)$>\aleph_0$. There exists an initial ordinal number Ω such that all members of A are indexed by the ordinal numbers $\alpha<\Omega$. Hence, A $=\{a_\alpha: \alpha<\Omega\}$. Put $B_\alpha=\{a_\mu: \mu<\alpha<\Omega\}$. We have a transfinite sequence $\{B_\alpha: \alpha<\Omega\}$ such that

a) card(B_α)$<$ card(A),

b) $\alpha<\beta<\Omega$ implies $B_\alpha \subseteq B_\beta$,

c) $A=\bigcup\{B_\alpha: \alpha<\Omega\}$.

Put $A_\alpha = G_\infty(B_\alpha)$ and let $\Delta=\{A_\alpha : B_\alpha \subseteq A\}$ be ordered by inclusion $\subseteq$. It is obvious that Δ is well-ordered by inclusion.

Step 4. *If Φ and Ψ are in Δ such that $\Phi\subset\Psi$, then there exists a mapping $q_{\Phi\Psi}: lim\{X_\alpha, p_{\alpha\beta}, \Psi\}\to lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$.*

Namely, if $x=(x_\alpha, \alpha\in\Psi)\in\lim\{X_\alpha,p_{\alpha\beta},\Psi\}$, then by Definition 1.5 of the threads of $\{X_\alpha,p_{\alpha\beta},\Psi\}$ the condition (L) is satisfied. If (L) is satisfied for $x=(x_\alpha, \alpha\in\Psi)\in\lim\{X_\alpha,p_{\alpha\beta},\Psi\}$, then it is satisfied for $(x_\gamma,\gamma\in\Phi)$ since the required a' in (L) lies - by definition of the set Φ - in the set Φ. This means that $(x_\gamma,\gamma\in\Phi)\in\lim\{X_\gamma, p_{\alpha\beta},\Phi\}$. Now we define $q_{\Phi\Psi}(x)=(x_\gamma,\gamma\in\Phi)$.

Step 5. *The collection $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ is a usual inverse system.* It suffices to prove the transitivity, i.e., if $\Phi\subseteq\Psi\subseteq\Omega$, then $q_{\Phi\Psi}q_{\Psi\Omega}=q_{\Phi\Omega}$. This easily follows from the definition of $q_{\Phi\Psi}$.

Step 6. *The space limX is homeomorphic to $lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$*, where $X_\Phi=\lim\{X_\gamma, p_{\alpha\beta},\Phi\}$. We shall define a homeomorphism $H:\lim X\to\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Let $x=(x_a : a\in A)$ be any point of limX. Each collection $\{x_a : a\in\Phi\in\Delta\}$ is a point x_Φ of X_Φ since $X_\Phi=\lim\{X_a, p_{ab}, \Phi\}$. Moreover, from the definition of $q_{\Phi\Psi}$ (Step 4) it follows that $q_{\Phi\Psi}(x_\Psi)=x_\Phi$, $\Psi\supseteq\Phi$. Thus, the collection $\{x_\Phi : \Phi\in\Delta\}$ is a point of $\lim\{X_\Phi, q_{\Phi\Psi}, \Delta\}$. Let $H(x)=\{x_\Phi, \Phi\in\in\Delta\}$. Thus, H is a continuous mapping of limX to $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. In order to complete the proof it suffices to prove that H is 1 - 1 and onto. Let us prove that H is 1 - 1. Let $x=(x_a : a\in A)$ and $=(y_a : a\in A)$ be a pair of points of limX. This means that there exists an $a\in A$ such that $y_a\neq x_a$. There exists an $\Phi\in\Delta$ such that $a\in\Phi$. Thus, the collections $\{x_a : a\in\Phi\}$ and $\{x_a : a\in\Phi\}$ are different. From this we conclude that $x_\Phi\neq y_\Phi$, $x_\Phi,y_\Phi\in X_\Phi=\lim\{X_a, p_{ab}, \Phi\}$. Hence H is 1 - 1. Let us prove that H is onto. Let $y=(y_\Phi : \Phi\in\Delta)$ be any point of $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Each y_Φ is a collection $\{x_a : a \in \Phi\}$ and if $\Psi \supseteq \Phi$, then the collection $\{x_a : a \in \Phi\}$ is the restriction of the collection $\{x_a : a \in \Psi\}$ on Φ. Let x be the collection which is the union of all collections $\{x_a : a \in \Phi\}$, $\Phi \in \Delta$. Hence x is a collection $(x_a : a \in A)$ which is a point of lim**X** and H(x) = y. The proof is completed. ■

COROLLARY 3.3 *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact metric spaces. Then lim*$\boldsymbol{X}$ *is homeomorphic to the limit of a well-ordered usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < card(A)\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $card(\Phi) < card(A)$.

COROLLARY 3.4 *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact metric spaces such that* $card(A) = \aleph_1$. *Then lim*$\boldsymbol{X}$ *is homeomorphic to the limit of a well-ordered usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \omega_1\}$, *where each* X_α *is a metric space as a limit of an approximate inverse sequence.*

Considering only the countable subsets B of A and arguing as in the proof of Theorem 3.2, we obtain the following theorem.

THEOREM 3.5 *For each approximate inverse system* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$, $card(A) \geq \aleph_1$, *of metric compact spaces, there exists a usual* σ*-directed inverse system* $\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ *such that each* X_Φ *is the limit of a countable approximate subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ *of the system* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *and lim*$\boldsymbol{X}$ *is homeomorphic to* $lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$.

If $\mathbf{X} = \{X_a, p_{ab}, A\}$is an approximate inverse system such that $card(A) = \aleph_0$, then we have the following theorem.

THEOREM 3.6 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$*be an approximate inverse system of topologically complete spaces such that* $card(A) = \aleph_0$. *Then there exists a countable well-ordered subset B of A such that the collection* $\{X_b, p_{bc}, B\}$ *is an approximate inverse sequence and lim*$\mathbf{X}$ *is homeomorphic to* $lim\{X_b, p_{bc}, B\}$.

Proof. From the Step 1 of the proof of Theorem 3.2 it follows that there exists a sequence

$$A_1 \subseteq A_2 \subseteq ... \subseteq A_n ...$$

of fine sets A_n such that $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Using a $\delta(A_n)$ for each A_n, we obtain a sequence $B = \{b_n : n \in \mathbb{N}\}$ such that B is cofinal in A. Let us prove that $\{X_b, p_{bc}, B\}$ is an approximate inverse system, i.e., that (A2) is satisfied for $\{X_b, p_{bc}, B\}$. For each X_b and each normal cover of X_b there exists an $a' \in A$ such that (A2) is satisfied for $b \leq a' \leq c \leq d$ since (A2) is satisfied for $\mathbf{X} = \{X_a, p_{ab}, A\}$. There exists a b' such that $b' \in B$, $b' \geq a'$, since B is cofinal in A. It is obvious that (A2) is satisfied for each $c,d \in B$ such that $b \leq b' \leq c \leq d$. By virtue of [12, Theorem (1.19)] it follows that $\lim\mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. ■

THEOREM 3.7 *If* $\mathbf{X}=\{X_n, p_{MN}, \mathbb{N}\}$ *is an approximate inverse sequence of complete metric spaces, then there exist:*

a) *a cofinal subset* $M = \{n_i, i \in \mathbb{N}\}$ *of* $\mathbb{N}$,

b) *a usual inverse sequence* $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ *such that* $Y_i = X_{n_i}$ *and* $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$ *for each* $i,j \in \mathbb{N}$,

c) *a homeomorphism* $H : \lim\mathbf{X} \to \lim\mathbf{Y}$.

Proof. See [7, Theorem 2.11] or [2, Proposition 8].■

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an approximate commutative (or usual) inverse system, then the assumption $w(X_a) < card(A)$ and Step 2. in the proof of Theorem 3.2 can be omitted and we have the following theorem.

THEOREM 3.8 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a usual inverse system of compact spaces. Then* $\lim\mathbf{X}$ *is homeomorphic to the limit of a well-ordered usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < card(A)\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $card(\Phi) < card(A)$.

4 APPLICATIONS

A continuum is a *tree* if each pair of points is separated by a third point. A continuum with precisely two nonseparating points is called a *generalized arc* (or an *ordered continuum*). A continuum X is a tree if and only if X is locally connected and hereditarily unicoherent. Each tree is hereditarily locally connected. A tree is a generalized arc if and only if it is atriodic. A *dendrite (arc)* is a metrizable tree (generalized arc).

THEOREM 4.1 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of compact locally connected metric spaces such that* $card(cf(A)) \geq \aleph_1$. *Then* $X = \lim\mathbf{X}$ *is locally connected.*

Proof. By virtue of Theorem 2.5 there exists a usual inverse system $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ such that $Y_b = X_a$ for some $a \in A$ and there exists a homeomorphism $H : \lim\mathbf{X} \rightarrow \lim\mathbf{Y}$. Using [4, Theorem 3] we infer that $\lim\mathbf{Y}$ is locally connected.■

THEOREM 4.2 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of locally connected continua* X_a *and surjective bonding mappings* p_{ab} *such that* $w(X_a) \leq \lambda$. *Then, either* $X = \lim\mathbf{X}$ *is locally connected or* $w(X) \leq \lambda$.

Proof. If $\mathrm{card}(\mathrm{cf}(A)) < \lambda$, then $w(X) \leq \lambda$. If $\mathrm{card}(\mathrm{cf}(A)) \geq \lambda$, then from Theorem 2.5 it follows that there exists a usual inverse system $\mathbf{Y} = \{Y_b, p_{cd}, B\}$ such that $Y_b = X_a$ for some $a \in A$ and there exists a homeomorphism $H : \lim\mathbf{X} \rightarrow \lim\mathbf{Y}$. Using [4, Theorem 4] we infer that $\lim\mathbf{Y}$ is locally connected.■

COROLLARY 4.3 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system with surjective bonding mappings. If* X_a, $a \in A$, *are locally connected metric continua, then, either* $X = \lim\mathbf{X}$ *is metrizable or* X *is locally connected.*

THEOREM 4.4 *Let* X *be the limit of a well-ordered approximate inverse system of trees (generalized arcs) such that* $w(X_a) \leq \lambda$. *Then, either* X *is a tree (generalized arc) or* $w(X) \leq \lambda$.

Proof. This follows from the Theorem above and the fact that X is hereditarily unicoherent (atriodic) [8, Corollary 4.3], [8, The proof of Lemma 5.14].■

COROLLARY 4.5 *Let* X *be the limit of a well-ordered approximate inverse system of dendrite (arcs). Then, either* X *is metrizable or* X *is tree (generalized arc).*

THEOREM 4.6 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate* σ*-directed inverse system of trees (generalized arcs). Then* $X = \lim\mathbf{X}$ *is a tree (generalized arc).*

Proof. It suffices to prove that X is hereditarily locally connected since X is hereditarily unicoherent (atriodic) [8, Corollary 4.3], [8, The proof of Lemma 5.14]. Suppose that X is not hereditarily locally connected. By virtue of [17] there exists in X a non-degenerate continuum of convergence Y such that there exists a net $\{Y_\gamma : \gamma \in \Gamma\}$ of subcontinua of X such that $\mathrm{Lim}Y_\gamma = Y$, $Y \cap Y_\gamma = \emptyset$ for all $\gamma \in \Gamma$, and if $\gamma, \delta \in \Gamma$ then either $Y_\gamma = Y_\delta$ or $Y_\gamma \cap Y_\delta = \emptyset$. Let x, y be a pair of distinct points of Y and let U, V be a pair of open subsets of X such that x∈U, y∈V and $Cl(U)\bigcap Cl(V) = \emptyset$. There exists a $\gamma_1 \in \Gamma$ such that $Y_{\gamma_1} \bigcap U \neq \emptyset \neq Y_{\gamma_1} \bigcap V$. Let $Z_1 = Y_{\gamma_1}$. From the normality of X, it follows that there exists an open set $V_1 \subseteq X$ such that $Cl(V_1)\bigcap X_1 = \emptyset$ and $Y \subseteq V_1$. There exists a $\gamma_2 > \gamma_1$ such that $Y_{\gamma_2} \subseteq V_1$. Let $Z_2 = Y_{\gamma_2}$. We infer that there exists an open set $V_2 \subseteq X$ such that $Cl(V_2)\bigcap Z_2 = \emptyset$ and $Y \subseteq V_2$. Continuing in this way we obtain a sequence V_1, V_2, ... of the open sets and a sequence Z_1, Z_2, ... such that

$$Cl(V_n) \subseteq V_{n-1}, \quad n = 2, 3, ..., \tag{3}$$

and

$$Cl(V_n) \bigcap Z_n = \emptyset, \quad Z_{n+1} \subseteq V_n. \tag{4}$$

If F and G are closed disjoint subsets of X, then by virtue of [6, Lemma 2.17] there is an a(F , G)∈A such that $p_b(F)\bigcap p_b(G) = \emptyset$ for each b≥a(F , G). Let Z_m be any member of the sequence Z_1, Z_2, ... and let G_n be any member of the sequence $Cl(V_1)$, $Cl(V_2)$, ... There exists an a(m,n) such that $p_b(Z_m)\bigcap p_b(Cl(V_n))$ $= \emptyset$ for each b≥a(m,n). By virtue of the σ-directedness of A, there exists an a∈A such that a>a(m,n) for each m and n. We may assume that

$$p_a(Cl(U)) \bigcap p_a(Cl(V)) = \emptyset. \tag{5}$$

Let $K = p_a(Cl(U))$, $L = p_a(Cl(V))$ and $X_n = p_a(Z_n)$, n = 1, 2, ... By virtue of [15, p. 310, Lemma 2.4] or [17, p. 246., Theorem 4]) X_a is not hereditarily locally connected, a contradiction.■

We say that a mapping f:X→Y is *hereditarily monotone* if the restriction f|:K→f(K) is monotone for each subcontinuum K⊆X.

THEOREM 4.7 *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of hereditarily locally connected metric continua and hereditarily monotone bonding mappings. Then* $X = lim\boldsymbol{X}$ *is hereditarily locally connected.*

Proof. The proof is broken into several steps.

Step 1. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a usual inverse system, then for each subcontinuum K of $X = \lim\mathbf{X}$ there exists a usual inverse system $\mathcal{K} = \{p_a(K), p_{ab}|p_b(K), A\}$ with the monotone bonding mappings $p_{ab}|p_b(K)$. Each $p_a(K)$ is locally connected since X_a is hereditarily locally connected. We infer that $K = \lim\mathcal{K}$ is locally connected. Hence, X is hereditarily locally connected.

Step 2. Let $\mathbf{X}=\{X_n, p_{MN}, \mathbb{N}\}$ be an approximate inverse sequence of hereditarily locally connected metric continua and hereditarily monotone mappings. By virtue of Theorem 3.6 there exist

a) a cofinal subset $M =\{n_i, i\in\mathbb{N}\}$ of $\mathbb{N}$,

b) a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$ and $q_{ij} = p_{n_in_{i+1}}p_{n_{i+1}n_{i+2}} \cdots p_{n_{j-1}n_j}$ for each $i,j\in\mathbb{N}$,

c) a homeomorphism $H : \lim\mathbf{X}\to\lim\mathbf{Y}$.

Now each mapping q_{ij} is hereditarily monotone. From Step 1. it follows that $\lim\mathbf{Y}$ is hereditarily locally connected. Hence $\lim\mathbf{X}$ is locally connected.
Step 3. Let us prove the Theorem. Let now $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system as in the Theorem. By virtue of Theorem 3.5 there exists a usual σ-directed inverse system $\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ such that each X_Φ is a limit of a countable approximate subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ of the system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\lim\mathbf{X}$ is homeomorphic to $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. From Step 2. we infer that each X_Φ is hereditarily locally connected. We infer that $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ is hereditarily locally connected since $\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ is σ-directed. Thus, $\lim\mathbf{X}$ is hereditarily locally connected.■

A *graph* is a 1-dimensional polyhedron. Thus graphs are metrizable and locally connected.

We shall say that a non-empty compact space is *perfect* if it has no isolated point.

A continuum is said to be *totally regular* [14, p. 47] if for each $x\neq y$ in X there is a positive integer n and perfect subsets $A_1, \ldots, A_n, \ldots$ of X such that $x_i \in A_i$ for $i = 1, \ldots, n$ implies that $\{x_1, \ldots, x_n\}$ separates x from y in X.
Each graph is totally regular [14, Theorem 7.5, equivalence (1)⇔(8)].

The following theorem is a part of [14, Theorem 7.15, equivalence (1)⇔(6)]. Each totally regular continuum is hereditarily locally connected.

THEOREM 4.8 *If X is a continuum then the following conditions are equivalent:*

1. *X is totally regular,*

2. *X is homeomorphic to* $\lim\{G_a, f_{ab}, \Gamma\}$ *such that each* G_a *is a graph and each* f_{ab} *is a monotone surjection.*

THEOREM 4.9 [14, Theorem 7.7]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a usual inverse system of totally regular continua* X_a *and the monotone surjective mappings* p_{ab}. *Then* $X = lim\mathbf{X}$ *is totally regular.*

LEMMA 4.10 *Let X be a non-metric totally regular continuum. There exists a* σ*-directed inverse system*

$$\mathbf{X} = \{X_n, f_{nm}, A\} \tag{6}$$

such that each X_n *is totally regular and each* f_{nm} *is a monotone surjection.*

Proof. Apply [14, Theorem 9.4], Theorem 4.9 and Lemma 3.5 of [16] . ■

THEOREM 4.11 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular metric continua and monotone bonding mappings. Then* $X = lim\mathbf{X}$ *is totally regular.*

Proof. If card(A) = $\aleph_0$, then there exists a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$, $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$ for each i,j$\in \mathbb{N}$, and a homeomorphism H : lim$\mathbf{X}\to$lim$\mathbf{Y}$ (Theorems 3.6 and 3.7). By virtue of Theorem 4.9 lim$\mathbf{Y}$ is totally regular. Hence X is totally regular. If card(A)$\geq \aleph_1$, then there exists a usual σ-directed inverse system $\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ such that each X_Φ is a limit of a countable approximate subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ of the system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and lim$\mathbf{X}$ is homeomorphic to lim$\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ (Theorem 3.5). Each X_Φ is totally regular since card(Φ) = $\aleph_0$. Applying Theorem 4.9 we conclude that lim$\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ is totally regular. Thus X = lim$\mathbf{X}$ is totally regular.■

We say that a continuum X is a continuous image of an arc if there exists a generalized arc L and a continuous surjection f : L$\to$X.

LEMMA 4.12 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$*be an approximate system such that* X_a, $a\in A$, *are compact locally connected spaces and* p_{ab} *are monotone surjections. If* $\mathbf{Y} = \{X_b, p_{cd}, B\}$ *is an approximate subsystem of* $\mathbf{X}$, *then the mapping* q_{AB} *:* $lim\mathbf{X}\to lim\mathbf{Y}$ *(defined in Step 4 of the proof of Theorem 3.2) is a monotone surjection.*

Proof. Let P_a : lim$\mathbf{X}\to X_a$, a$\in$A, be the natural projection. Similarly, let p_a : lim$\mathbf{Y}\to X_a$, a$\in$B, be the natural projection. From the definition of q_{AB} (Step 4 of the proof of Theorem 3.2) it follows that $p_a q_{AB} = P_a$ for each a$\in$B. By virtue of [12, Corollary 4.5] and [8, Corollary 5.6] it follows that P_a and p_a are monotone surjections. Let us prove that q_{AB} is a surjection. Let $y = (y_a : a\in B)\in \lim Y$. The sets $P_a^{-1}(y_a)$, $a\in B$, are non-empty since P_a is surjective for each $a\in A$. From the compactness of $\lim X$ it follows that a limit superior $Z = Ls\{P_a^{-1}(y_a), a\in B\}$ is a non-empty subset of $\lim X$. We shall prove that for each $z = (z_a: a\in A)\in Z$ $P_a(z) = y_a$. Suppose that $P_a(z)\neq y_a$. There exists a pair U, V of open disjoint subsets of X_a such that $y_a \in U$ and $P_a(z)\in V$. For sufficiently large $b\in B$ $P_a(P_b^{-1}(_b))$ is in U because (AS). This means that $P_a^{-1}(V)\cap P_b^{-1}(y_b) = \emptyset$ for sufficiently large $b\in B$. This contradicts the assumption $z\in Ls\{P_a^{-1}(y_a), a\in B\}$. Hence q_{AB} is a surjection. In order to complete the proof it suffices to prove that q_{AB} is monotone. Take a point $y\in \lim Y$ and suppose that $q_{AB}^{-1}(y)$ is disconnected. There exists a pair U, V of disjoint open sets in $\lim X$ such that $q_{AB}^{-1}(y)\subseteq U\cup V$. From the compactness of $\lim X$ it follows that q_{AB} is closed. This means that there exists an open neighborhood W of y such that $q_{AB}^{-1}(y)\subseteq q_{AB}^{-1}(W)\subseteq U\cup V$. From the definition of the basis in $\lim Y$ it follows that there exists an open set W_a in some X_a, $a\in B$ such that $y\in p_a^{-1}(W_a)\subseteq W$. Moreover, we may assume that W_a is connected since X_a is locally connected. Then $P_a^{-1}(W_a)$ is connected since P_a is monotone [8, Corollary 5.6]. Moreover, $q_{AB}^{-1}(y)\subseteq P_a^{-1}(W_a)$ and $P_a^{-1}(W_a)\subseteq U\cup$ since $P_a = p_a q_{AB}$. This is impossible since U and V are disjoint open sets and $P_a^{-1}(W_a)$ is connected. The proof is completed.■

THEOREM 4.13 *Let* $X = \{X_a, p_{ab}, A\}$ *be an approximate well-ordered inverse system of continuous images of arcs such that* $w(X_a) < \tau$, $a\in A$, *card(cf(A))*$\geq \tau$ *and card(cf(A))*$> \aleph_1$. *If the mappings* p_{ab} *are monotone surjections, then* $X = \lim X$ *is a continuous image of an arc.*

Proof. By virtue of Theorem 2.3 there exist a usual inverse system $Y = \{Y_b, p_{cd}, B\}$ such that $Y_b = X_a$ for some $a\in A$ and a homeomorphism $H : \lim X\to \lim Y$. From [5, Theorem 2.17] it follows that Y is the continuous image of an arc. Hence X is the continuous image of an arc.■

THEOREM 4.14 *Let* $X = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of continuous images of arcs such that cf(card(A))*$> \aleph_1$ *and* $w(X_a)<card(A)$, $a\in A$. *If the bonding mappings are monotone surjections, then* $X = \lim X$ *is a continuous image of an arc if and only if a limit of each approximate subsystem of* X *is a continuous image of an arc.*

Proof. *Sufficiency.* By virtue of Theorem 3.2 there exists a well-ordered usual inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$, where each X_α is a limit of an approximate inverse subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, card($\Phi$)<card(A) such that lim**X** is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$. By the assumption of the Theorem, each X_γ is the continuous image of an arc. By virtue of Theorem 4.13, X is the continuous image of an arc.
Necessity. If X is a continuous image of an arc, then X_α is a continuous image of an arc for each directed set B⊆A since there exists a natural projection $p_\alpha: X \to X_\alpha$.
■

PROBLEM. Let $\mathbf{X}=\{X_n, p_{MN}, \mathbb{N}\}$ be an approximate inverse sequence of continuous images of arcs and monotone surjective bonding mappings. Is it true that lim**X** is the continuous image of an arc?

THEOREM 4.15 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continuous images of arcs. If cf(card(A))*$\neq \omega_1$*, then X = lim***X** *is a continuous image of an arc if and only if a limit of each subsystem of* **X** *is a continuous image of an arc.*

Proof. If cf(card(A)) = $\aleph_0$, then there exists a well-ordered sequence $B = \{a_n: n \in \mathbb{N}\} \subseteq A$ which is cofinal in A. It is clear that X is homeomorphic to the limit of an inverse sequence $\{X_a, p_{ab}, B\}$. Applying Theorem [14, Theorem 5.1] we complete the proof. If cf(card(A))> $\aleph_1$, then the proof is similar to the proof of Theorem 4.14.■

We close this Section with the following theorem and corollary.

THEOREM 4.16 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continuous images of arcs. If cf(card(A))*$\neq \omega_1$*, then X = lim***X** *is a continuous image of an arc if and only if each proper subsystem* $\{X_a, p_{ab}, B\}$ *of* **X** *with cf(card(B))* $= \omega_1$ *has a limit which is a continuous image of an arc.*

Proof. The "only if part". If X is a continuous image of an arc, then for each subsystem $\{X_a, p_{ab}, B\}$ there exists a natural projection $f_a : X \to \lim\{X_a, p_{ab}, B\}$. Hence, $\lim\{X_a, p_{ab}, B\}$ is a continuous image of an arc.
The "if" part. By virtue of Theorem 3.8 there exists a well-ordered inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$ such that X is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$. If cf(card(A))$\leq \omega_0$, then we have an inverse subsequence of $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$ which is a cofinal subsystem of $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$. By virtue of [14, Theorem 5.1] X is a continuous image of an arc. Let

cf(card(A)) $> \omega_1$. By virtue of Theorems 3.8 and 4.15 it suffices to prove that each subsystem of $\{X_a, p_{ab}, B\}$of $\mathbf{X} = \{X_a, p_{ab}, A\}$ has a limit which is a continuous image of an arc. We shall use the transfinite induction on card(B). If card(B)$\leq \omega_0$, then we use Theorem 5.1 of [14]. If card(B) $= \omega_1$, then $\lim\{X_a,p_{ab},B\}$ is a continuous image of an arc by assumption of the Theorem. Now let $\{X_a,p_{ab},B\}$ be a subsystem of $\{X_a,p_{ab},A\}$ such that card(B)$> \omega_1$. Suppose that Theorem is true for each subsystem of the cardinality $<$ card(B). By virtue of Theorem 3.8 there exists a well-ordered inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card(B)}\}$ such that $\lim\{X_a,p_{ab},B\}$ is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card(B)}\}$. Since each X_α is a limit of a subsystem of the cardinality <card(B), we have the inverse system $\{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card(B)}\}$ which satisfies the conditions of Theorem 2.17 of [5]. Thus, $\lim\{X_a,p_{ab},B\}$ is a continuous image of an arc. By the transfinite induction, the proof is complete. ■

COROLLARY 4.17 *Let X be a locally connected continuum. The following conditions are equivalent:*

a) *X is a continuous image of an arc,*

b) *If $f : X \to Y$ is a continuous mapping and cf(card(w(Y)) $= \omega_1$, then Y is a continuous image of an arc.*

Proof. a) ⇒ b). Obvious.
b) ⇒ a). By virtue of Theorem [11] there exists an inverse system $\mathbf{X} = \{X_a,p_{ab},A\}$ such that X_a are metric locally connected continua, p_{ab} are monotone mappings and X is homeomorphic to lim**X**. If $\mathbf{Y} = \{X_a,p_{ab},B\}$ is any subsystem of $\{X_a,p_{ab},A\}$ with cf(card(w(Y)) $= \omega_1$, then there exists a natural projection P: X→lim**Y**. By virtue of b) it follows that lim**Y** is a continuous image of an arc. Applying Theorem 4.16 we complete the proof. ■

COROLLARY 4.18 *Let X be a locally connected continuum such that $w(X) < \aleph_{\omega_1}$. The following conditions are equivalent:*

a) *X is a continuous image of an arc,*

b) *If $f : X \to Y$ is a continuous mapping and $w(Y) = \aleph_1$, then Y is a continuous image of an arc.*

REFERENCES

[1] A.R. Aló and H.L. Shapiro.*Normal Topological Spaces.* Cambridge Univ. Press, Cambridge,1974.

[2] M.G. Charalambous. Approximate inverse systems of uniform spaces and an application of inverse systems. *Comment. Math. Univ. carolinae,* 32, 33(1991), 551-565.

[3] R. Engelking. *General Topology.* PWN, Warszawa,1977.

[4] G.R. Gordh, Jr. and S. Mardešić. Characterizing local connectedness in inverse limits. *Pacific Journal of Mathematics* 58 (1975), 411-417.

[5] I. Lončar. Inverse limit of continuous images of arcs. *Zbornik radova Fakulteta organizacije i informatike Varaždin* 2(23) (1997), 47 - 60.

[6] I. Lončar. Approximate limits of m-compact spaces. *Glasnik Matematički* 30(50) (1995), 73 - 84.

[7] I. Lončar. A note on approximate limits. *Zbornik radova Fakulteta organizacije i informatike Varaždin* 19 (1995), 1 - 21.

[8] I. Lončar. Set convergence and local connectedness of approximate limits. *Acta Mathematica Hungarica* 77 (3) (1997), 193 - 213.

[9] F. Maeda. *Kontinuierliche Geometrien.* Springer - Verlag, Berlin, 1958.

[10] S. Mardešić. On approximate inverse systems and resolutions. *Fund. Math.* 142(1993), 241-255.

[11] S. Mardesić. Locally connected,ordered and chainable continua. *Rad JAZU Zagreb* 33(4)(1960), 147-166.

[12] S. Mardešić and T. Watanabe. Approximate resolutions of spaces and mappings, *Glasnik Matematički,* 24 (3) (1989), 587-637.

[13] S. Mardešić and N. Uglešić. Approximate inverse systems which admit meshes, *Topology and its Applications* 59 (1994), 179 - 188.

[14] J. Nikiel, H.M. Tuncali and E.D. Tymchatin. Continuous images of arcs and inverse limit methods. *Mem. Amer. Math. Soc.* 1993, 104, 496, 1 - 80.

[15] J. Nikiel. The Hahn - Mazurkiewicz theorem for hereditarily locally connected continua. *Topology and Applications* 32 (1989), 307 - 323.

[16] J. Nikiel. A general theorem on inverse systems and their limits. *Bulletin of the Polish Academy of Sciences, Mathematics,* 32(1989), 127 - 136.

[17] J.N. Simone. Concerning hereditarily locally connected continua. *Colloq. Math.* 39 (1978), 243 - 251.

[18] M.G. Tkačenko. Cepi i kardinaly. *Dokl. Akad. SSSR* 279 (3) (1978), 546-549.

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BILJEŠKA O APROKSIMATIVNIM INVERZNIM SISTEMIMA I PODSISTEMIMA

Sažetak

U radu je dokazano da aproksimativni inverzni sistemi uz neke dodatne uvjete posjeduju kofinalne podsisteme koji su komutativni ili obični inverzni sistemi. Drugi odjeljak sadrži takve teoreme za dobro uređene aproksimativne inverzne sisteme, dok treći odjeljak sadrži teoreme za opći slučaj. U posljednjem, četvrtom, odjelku dane su neke primjene teorema prethodnih odjeljaka.

Ključne riječi : aproksimativni inverzni sistem, aproksimativni inverzni podsistem.