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INVOLUTES AND EVOLUTES IN n-DIMENSIONAL SIMPLY ISOTROPIC SPACE $I_n^{(1)}$

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In this paper, the notions of the isotropic involutes (of order k) and the isotropic evolutes in n-dimensional simply isotropic space $I_n^{(1)}$ *are defined. We determine the formula of involutes of a given admissible curve in* $I_n^{(1)}$ *and the curvature and the torsion of involutes and evolutes in* $I_3^{(1)}$. *The system of differential equations which determines the evolute of a given admissible curve in* $I_n^{(1)}$ *is found. The explicit formula of the evolutes of admissible curve in* $I_3^{(1)}$ *is given. The definitions of involutes and evolutes, which are used in this article, are motivated by the analogous definitions for Euclidean case from [2].*

Keywords: admissible curve, involutes, evolutes, n-dimensional simply isotropic space.

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1. CURVES IN $I_n^{(1)}$

Let I be an interval, $I \subseteq \mathbf{R}$ and $f: I \to I_n^{(1)}$ vector function given in affine coordinates as

$$\overrightarrow{OX}(t) = (x_1(t), ..., x_n(t)) : = \mathrm{x}(t), \quad t \in I.$$

The set of points $c \in I_n^{(1)}$ is called a *C^r-curve* if there is an open interval $I \subseteq \mathbf{R}$ and C^r-function ($r \geq 1$) $f: I \to I_n^{(1)}$ with $f(I)=c$.

A C^r-curve is a *regular C^r-curve* provided

$$\dot{\mathbf{x}}(t) = (\dot{x}_1(t), ..., \dot{x}_n(t)) \neq \mathbf{0}, t \in I,$$

and if f is an injective transformation a curve is called a *simple C^r-curve*.

A regular C^r-curve ($r \geq n-1$) is *nondegenerate* if the set of vectors

$$\left\{\dot{\mathbf{x}}(t), ..., \mathbf{x}^{(n-1)}(t)\right\}$$

is linearly independent for all $t \in I$.

A curve $c \subset I_n^{(1)}$ is said to be an *admissible C^r-curve* (r≥n-1) when c is a simple, nondegenerate C^r-curve (r≥n-1) without the isotropic osculating hyperplanes.

Let c, which is defined on a closed interval [a,b], be an admissible curve in $I_n^{(1)}$. Then

$$s := \int_{t=a}^{b} \left(\dot{x}_1^2 + \ldots + \dot{x}_{n-1}^2\right)^{1/2} dt$$

is called the *isotropic arc length* of the curve c from $\mathbf{x}(a)$ to $\mathbf{x}(b)$. (From now on, *s* always denotes a parameter of the arc length.)

For the admissible curve c ($\mathbf{x}=\mathbf{x}(s)$) we can define *n-frame* $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$ in any point, as has been done in [1] or [4] for example. Then, there are functions $\kappa_1(s), \ldots, \kappa_{n-1}(s)$ so that the Frenet formulae

$$(1.1) \qquad \begin{cases} \mathbf{t}_1' = \kappa_1 \mathbf{t}_2 \\ \mathbf{t}_i' = \kappa_i \mathbf{t}_{i+1} - \kappa_{i-1}\mathbf{t}_{i-1} \quad i = 2, \ldots, n-1, \\ \mathbf{t}_n' = 0 \end{cases}$$

hold. The functions $\kappa_1(s), \ldots, \kappa_{n-1}(s)$ are called the *isotropic curvatures* of the curve c.

The definitions of involutes and evolutes, which are used in this article, are motivated by the analogous definitions for the Euclidean case from [2].

2. INVOLUTES

2.1. Involutes in $I_n^{(1)}$

Definition 1. Let c, given by $\mathbf{x} = \mathbf{x}(s)$, $\mathbf{x}: I \to I_n^{(1)}$, $I \subseteq \mathbf{R}$ be an admissible C^r-curve (r≥n) parameterized by the parameter of the arc length. The orthogonal trajectories of the first tangents of the curve c are called the *involutes* of the curve c.

Theorem 1. A one-parameter family of involutes of an admissible curve c is represented by the formula

$$(2.1) \qquad \bar{\mathbf{x}}(s) = \mathbf{x}(s) + \mathbf{t}_1(s)(k - s),$$

where *k* is an arbitrary constant and *s* is the arc length of the curve c.

Proof. The involute of the curve c ($\mathbf{x}=\mathbf{x}(s)$) is characterized by

$$(2.2) \qquad \bar{\mathbf{x}}(s) = \mathbf{x}(s) + u(s)\mathbf{t}_1(s)$$

where *u*(s) is a function of *s* on *I*. Then the differentiation of the relation (2.2) and the Frenet formulae (1.1) give the following equation

(2.3) $\bar{\mathbf{x}}'(s) = (1+u'(s))\mathbf{t}_1(s) + \kappa_1(s)u(s)\mathbf{t}_2(s)$.

In accordance with $\bar{\mathbf{x}}'\mathbf{t}_1 = 0$, we have $1+u'(s)=0$ and furthermore,

(2.4) $u(s)=k-s$, k=const.

Inserting the relation (2.4) into (2.2) we obtain the expression (2.1) as desired.

Corollary 1. Two different involutes of an admissible curve c are equidistant.

In addition, we wish to generalize the notion of an involute.

Definition 2. Let c ($\mathbf{x}=\mathbf{x}(s)$) be an admissible curve. Curves, which are orthogonal to the system of k-dimensional osculating hyperplanes of c, are called the *involutes of order k* of the curve c.

The involutes of order k are given by

(2.5) $\bar{\mathbf{x}} = \mathbf{x}(s) + u_1(s)\mathbf{t}_1(s)+\ldots+u_k(s)\mathbf{t}_k(s),\ k \le n-1.$

In order to determine the functions u_1, ..., u_k from (2.5) we differentiate (2.5) and by using the Frenet formulae (1.1) we have

$$\begin{aligned}(2.6)\qquad \bar{\mathbf{x}}' &= (1+u_1' - u_2\kappa_2)\mathbf{t}_1 + \sum_{l=2}^{k-1}(u_l' + \kappa_{l-1}u_{l-1} - \kappa_l u_{l+1})\mathbf{t}_l + \\ &+(u_k' + \kappa_{k-1}u_{k-1})\mathbf{t}_k + \kappa_k u_k \mathbf{t}_{k+1}.\end{aligned}$$

Since we have $\mathbf{t}_i\mathrm{y}'=0$ for i=1, ...,k; k≤n-1 we obtain

$$(2.7)\qquad \begin{cases} 1+\mathrm{u}_1' - \kappa_2\mathrm{u}_2 = 0 \\ \mathrm{u}_l' + \kappa_{l-1}\mathrm{u}_{l-1} - \kappa_l\mathrm{u}_{l+1} = 0,\ \ l = 2,\ldots,k- \\ \mathrm{u}_k' + \kappa_{k-1}\mathrm{u}_{k-1} = 0. \end{cases}$$

from (2.6) after a scalar multiplication by $\mathbf{t}_1$, $\mathbf{t}_2$, ..., $\mathbf{t}_k$.

The system of differential equations (2.7) is the same as in the Euclidean case and admits a uniquely determined set of solutions u_1,...,u_k, having already prescribed the initial values at the point s=a of the curve c.

According to the above, the involute, which is defined in *Definition 1*, is actually the involute of order 1 and then the relations (2.7) are reduced to (2.4).

If $\mathrm{c} \subset \mathrm{I}_n^{(m)}$ $(m<n)$ is an admissible curve the involutes of c could be defined in the same way as is done above. Obviously, *Theorem 1, Corollary 1* and *Corollary 2* are true in a case when *k≤n-m-1*.

2.2. Involutes in $\mathrm{I}_3^{(1)}$

Corollary 2. Let c, given by $\mathbf{x}=\mathbf{x}(s)$, be an admissible curve in $\mathrm{I}_3^{(1)}$ where s is the parameter of the arc length and $\{\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s)\}$ the 3-frame of the given curve. Then the involute $\bar{c}$ ($\bar{\mathbf{x}} = \bar{\mathbf{x}}(s)$) of curve c has the following form

(2.8) $$\bar{\mathbf{x}}(s)=\mathbf{x}(s)+(k-s)\mathbf{t}(s).$$

The proof is analogues to that of *Theorem 1.*

Corollary 3. If $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of an admissible curve c, then the curvature $\bar{\kappa}$ and the torsion ¯ of the involute $\bar{c}$ of the curve c are given by

(2.9) $$\bar{\kappa}(s)=\frac{\operatorname{sgn}\,\kappa}{|s-k|}\ ,\quad \bar{\tau}(s)=\frac{\left(\frac{\tau}{\kappa}\right)'}{\kappa(k-s)}.$$

Proof. The parameter s is not the parameter of the arc length of $\bar{c}$, so, as is shown in [4], we have

(2.10) $$\bar{\kappa}(s)=\frac{Det\left(\dot{\tilde{\bar{\mathbf{x}}}},\ddot{\tilde{\bar{\mathbf{x}}}}\right)}{\left|\dot{\tilde{\bar{\mathbf{x}}}}\right|^{3}},\quad \bar{\tau}(s)=\frac{Det\left(\dot{\bar{\mathbf{x}}},\ddot{\bar{\mathbf{x}}},\dddot{\bar{\mathbf{x}}}\right)}{Det^{2}\left(\dot{\tilde{\bar{\mathbf{x}}}},\ddot{\tilde{\bar{\mathbf{x}}}}\right)}\ .$$

On the other hand, the differentiation of equation (2.8) implies that

$$\dot{\bar{\mathbf{x}}}(s)=(k-s)\kappa\mathbf{n},$$
$$\ddot{\bar{\mathbf{x}}}(s)=-(k-s)\kappa^{2}\mathbf{t}+\left[(k-s)\kappa'-\kappa\right]\mathbf{n}+(k-s)\kappa\tau\mathbf{b},$$
$$\dddot{\bar{\mathbf{x}}}(s)=\left[2\kappa^{2}-3(k-s)\kappa\kappa'\right]\mathbf{t}+\left[-(k-s)\kappa^{3}-2\kappa'+(k-s)\kappa''\right]\mathbf{n}+\left[2(k-s)\kappa'\tau+(k-s)\kappa\tau'-2\kappa\tau\right]\mathbf{b},$$
$$\dot{\tilde{\bar{\mathbf{x}}}}(s)=(k-s)\kappa\tilde{\mathbf{n}},$$
$$\ddot{\tilde{\bar{\mathbf{x}}}}(s)=-(k-s)\kappa^{2}\tilde{\mathbf{t}}+\left[(k-s)\kappa'-\kappa\right]\tilde{\mathbf{n}}\ .$$

Now, it is easy to see that

$$\left|\dot{\tilde{\bar{\mathbf{x}}}}\right|=\left|(k-s)\kappa\right|,$$
$$Det\left(\dot{\tilde{\bar{\mathbf{x}}}},\ddot{\tilde{\bar{\mathbf{x}}}}\right)=(k-s)^{2}\kappa^{3}\ ,$$
$$Det\left(\dot{\bar{\mathbf{x}}},\ddot{\bar{\mathbf{x}}},\dddot{\bar{\mathbf{x}}}\right)=(k-s)^{3}(\kappa\tau'-\kappa'\tau)\kappa^{3}\ .$$

And now from the above relations and (2.10) we deduce (2.9).

Example 1. The involutes of the helix

(2.11) $$\mathbf{x}(s)=\left(a\,\cos\frac{s}{a},\ a\,\sin\frac{s}{a},\ \frac{p}{a}s\right)$$

are the plane curves

$$\mathbf{x}(s)=\left(a\,\cos\frac{s}{a}+(s-k)\sin\frac{s}{a},\ a\,\sin\frac{s}{a}+(k-s)\cos\frac{s}{a},\ \frac{p}{a}k\right)\quad k=const.$$

We could ask ourselves if there are any other admissible curves in $I_3^{(1)}$ which have plane involutes. Because of (2.9) we may conclude that $\bar{\tau}\equiv 0$ if and only if

$\left(\frac{\tau}{\kappa}\right)' = 0$. Thus, $\frac{\tau}{\kappa} = \text{const}$. So, only those admissible curves in $I_3^{(1)}$ which have plane involutes are the helices.

3. EVOLUTES

3.1. Evolutes in $I_n^{(1)}$

Definition 3. We say that a curve $\mathbf{c}^*(\mathbf{x}^*=\mathbf{x}^*(s))$ is an evolute of an admissible C^n-curve $c(\mathbf{x}=\mathbf{x}(s))$, $c \subseteq I_n^{(1)}$ if $\mathbf{c}$ is the involute of $\mathbf{c}^*$. The parameter s is the parameter of the arc length of c.

The question that must be asked is: when does an evolute of a given curve exist and what does this evolute look like? The following theorem, which has the same form as in the Euclidean case (see [2]), answers the first part of this question.

Theorem 2. Let $c: I \to I_n^{(1)}$ be an admissible curve and s the parameter of the arc length. The evolute of c exists if and only if there is a nonisotropic unit field $\mathbf{a}(s)$ and a real function p(s) such that

(3.1) $$\mathbf{t}_1 + \mathbf{a}'\mathrm{p} = 0.$$

Proof. $\Rightarrow$ Let c* ($\mathbf{x}^*=\mathbf{x}^*(s)$) be the involute of c ($\mathbf{x}=\mathbf{x}$(s)). Then, there is a unit field $\mathbf{a}(s)$ and a function $p(s)$ so that

(3.2) $$\mathbf{x}^*(s)=\mathbf{x}(s)+p(s)\mathbf{a}(s)$$

and

(3.3) $$\mathbf{x}^{*\prime}(s)=\lambda(s)\mathbf{a}(s).$$

By differentiating (3.2) we get

(3.4) $$(\lambda\text{-p}')\mathbf{a}=\mathbf{t}_1+\mathbf{a}'p.$$

Multiplying the relation (3.4) by $\mathbf{a}$ we obtain

$$\lambda\text{-}p'=0$$

and then, the relation (3.4) becomes (3.1).

$\Leftarrow$ Now we suppose that (3.1) holds. Define $\mathbf{x}^*(s)$ by

(3.5) $$\mathbf{x}^*(s)=\mathbf{x}(s)+\mathbf{a}(s)p(s)$$

and by differentiating that by s we get

(3.6) $$\mathbf{x}^{*\prime}(s)=\mathbf{t}_1+\mathbf{a}'p+\mathbf{a}p'.$$

Comparing (3.6) and (3.1) we conclude that

$$\mathbf{x}^{*\prime}=\mathbf{a}p'$$

which means that vectors $\mathbf{x}^*\text{-}\mathbf{x}$ and $\mathbf{x}^{*\prime}$ are linearly dependent. In addition, we have

$$\mathbf{t}_1\mathbf{x}^{*\prime}=\mathbf{t}_1\mathbf{a}p'=-\mathbf{a}'p\mathbf{a}p'=0,$$

since **a** is a unit field. Therefore **x'** is orthogonal to **x*'** which implies c is the involute of c*.

Now, we shall try to find the expression for the evolute c* of a given admissible curve c ($\mathbf{x}=\mathbf{x}(s)$), $c \subseteq I_n^{(1)}$ which is referred to as the parameter of the arc length s.

Obviously,

$$(3.7)\qquad c^* \ldots\ \mathbf{x}^*(s)=\mathbf{x}(s)+p(s)\mathbf{a}(s),\ \ p(s)\neq 0,$$

where $\mathbf{a}(s)$ is a unit field orthogonal to c and therefore, collinear with the first tangent of c* ($\mathbf{x}^{*\prime}=\lambda\mathbf{a}$). So, we have

$$(3.8)\qquad \mathbf{a}=\sum_{i=2}^{n} a_i\mathbf{t}_i$$

and since $|\mathbf{a}|=1$ it follows that

$$(3.9)\qquad \sum_{i=2}^{n-1} a_i^2=1.$$

By differentiating (3.7) by s, we get

$$\lambda\mathbf{a}=\mathbf{t}_1+p'\mathbf{a}+p\sum_{i=2}^{n}[a_i'\mathrm{t}_i+a_i(\kappa_i\mathbf{t}_{i+1}-\kappa_{i-1}\mathbf{t}_{i-1})],\quad (\kappa_n=0)$$

and then,

$$(\lambda-p')\mathrm{a}=(1-\kappa_1a_2p)\mathrm{t}_1+p[(a_2'-\kappa_2a_3)\mathrm{t}_2+\sum_{i=3}^{n-2}(a_i'+a_{i-1}\kappa_{i-1}-a_{i+1}\kappa_i)\mathrm{t}_i+$$
$$(a_{n-1}'+a_{n-2}\kappa_{n-2})\mathrm{t}_{n-1}+(a_n'+\kappa_{n-1}a_{n-1})\mathrm{t}_n].$$

So now we have

$$\lambda-p'=0$$

and

$$(1-\kappa_1a_2p)\mathrm{t}_1+p[(a_2'-\kappa_2a_3)\mathrm{t}_2+\sum_{i=3}^{n-2}(a_i'+a_{i-1}\kappa_{i-1}-a_{i+1}\kappa_i)\mathrm{t}_i+$$
$$(a_{n-1}'+a_{n-2}\kappa_{n-2})\mathrm{t}_{n-1}+(a_n'+\kappa_{n-1}a_{n-1})\mathrm{t}_n]=0.$$

At the end, we will have the following system

$$(3.10)\qquad \begin{cases} 1-\kappa_1a_2p=0 \\ a_2'-\kappa_2a_3=0 \\ a_i'+a_{i-1}\kappa_{i-1}-a_{l+1}\kappa_l=0 \qquad i=3,\ldots,n-2 \\ a_{n-1}'+a_{n-2}\kappa_{n-2}=0 \\ a_n'+a_{n-1}\kappa_{n-1}=0 \\ \sum_{i=2}^{n-1} a_i^2=1 \end{cases}$$

which gives us the evolute of c (up to a constant).

When we consider an admissible curve c from $I_n^{(m)}$ the analogous system to system (3.10) does not determine the evolutes of c completely because the constants $a_{n-m+1},\ldots,a_n$ aren't actually in that system.

3.2. Evolutes in $I_3^{(1)}$

If we put n=3, the system (3.10) becomes

$$(3.11)\begin{cases} a_2 = \pm 1 \\ 1 \mp p\kappa = 0 \\ a_3' \pm \tau = 0 \ , \end{cases}$$

and then, we have

$$(3.12) \qquad p(s) = \pm\frac{1}{\kappa(s)}(\equiv \pm\rho(s)), \ a_3(s) = k - \int_0^s \tau(\sigma)d\sigma.$$

Inserting this into (3.7) we get the following corollary:

Corollary 4. The equation of evolute c* of an admissible curve c (**x**=**x**(s)) in $I_3^{(1)}$, where *s* is the parameter of the arc length on c, has the following form:

$$(3.13) \qquad c^* \ldots \mathbf{x}^*(s) = \mathbf{x}(s) + \rho(s)\left[\mathbf{n}(s) + (k - \int_0^s \tau(\sigma)d\sigma)\mathbf{b}\right].$$

The projection of (3.13) on the basic plane $x_3=0$ is

$$\tilde{\mathbf{x}}^* = \tilde{\mathbf{x}}(s) + \rho(s)\tilde{\mathbf{n}}(s)$$

and this is a formula of an evolute in the Euclidean case.

Corollary 5. The curvature κ^* and the torsion τ^* of the evolute c* of a curve $c \subset I_3^{(1)}$ depend on the curvature κ and torsion τ of c in the following way:

$$(3.14) \qquad \kappa^*(s) = \frac{\kappa^3(s)}{|\kappa'(s)|}, \quad \tau^*(s) = -\frac{\kappa^3}{\kappa'}(k - \int_0^s \tau(\sigma)d\sigma).$$

Proof. If c* is given by (3.13) we have

$$\dot{\mathbf{x}}^* = \rho'\mathbf{n} + \rho'(k - \int_0^s \tau(\sigma)d\sigma)\mathbf{b},$$

$$\ddot{\mathbf{x}}^* = -\rho'\kappa\mathbf{t} + \rho''\mathbf{n} + \rho''(k - \int_0^s \tau(\sigma)d\sigma)\mathbf{b},$$

$$\dddot{\mathbf{x}}^* = -(2\rho''\kappa + \rho'\kappa')\mathbf{t} + (\rho''' - \rho'\kappa^2)\mathbf{n} + \rho'''(k - \int_0^s \tau(\sigma)d\sigma)\mathbf{b},$$

$$|\dot{\tilde{\mathbf{x}}}^*| = |\rho'|,$$

$$Det(\dot{\tilde{\mathbf{x}}}^*, \ddot{\tilde{\mathbf{x}}}^*) = \rho'^2\kappa,$$

$$Det(\dot{\mathbf{x}}^*, \ddot{\mathbf{x}}^*, \dddot{\mathbf{x}}^*) = (\rho')^2\kappa^3(c - \int_0^s \tau(\sigma)d\sigma).$$

Formulae (2.10) complete the proof.

Example 2. The evolute c* of the helix, given by (2.11), is an isotropic straight line

c*...**x**(s)=(0,0,ak).

In the projection on x_3=0 it shows that the evolute of a circle is the point.

Corollary 6. The evolute of a given curve is a plain curve if and only if c is a plane curve.

Proof. Namely, τ*=0 if and only if κ=0 or $\int \tau(\sigma)d\sigma = k$. The condition κ=0 contradicts the fact that c is admissible. The second condition can be written as τ=0 which means that c lies in a nonisotropic plane.

Corollary 7. If a curve c has a constant torsion $\tau_0 \neq 0$, then the torsion of its evolute has the form

$$\tau^* = -\frac{\kappa^3}{\kappa'}(k - \tau_0 s).$$

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EVOLVENTE I EVOLUTE U n-DIMENZIONALNOM JEDNOSTRUKO IZOTROPNOM PROSTORU

Sažetak

Članak se sastoji od tri dijela. Prvi, uvodni dio, definira pojam dopustive krivulje u n-dimenzionalnom jednostruko izotropnom prostoru i navodi Frenetove formule kao specijalni slučaj situacije u n-dimenzionalnom m-struko izotropnom prostoru $I_n^{(m)}$ *opisane u [3]. U drugom dijelu dana je formula evolventi dopustive krivulje, kao i sustav diferencijalnih jednadžbi koji određuje evolvente k-tog reda u* $I_n^{(1)}$. *Nadalje, izvedena je fleksija i torzija* *evolventi dopustive krivulje u trodimenzionalnom jednostruko izotropnom prostoru* $I_3^{(1)}$ *u ovisnosti o fleksiji i torziji dane krivulje, a dan je primjer evolvente cilindrične spirale. Treći dio bavi se evolutama dopustive krivulje u* $I_n^{(1)}$. *Nađen je sustav diferencijalnih jednadžbi koje određuju evolutu dane dopustive krivulje u* $I_n^{(1)}$, *dana je eksplicitna formula evolute dopustive krivulje u* $I_3^{(1)}$, *kao i fleksija i torzija takve evolute u ovisnosti o fleksiji i torziji dane krivulje. Razmotrene su i neke posljedice izvedenih formula, te pitanje evolventi i evoluta dopustivih krivulja u općem slučaju* $I_n^{(m)}$. *Upotrijebljene definicije evolvente i evolute motivirane su analognim definicijama za euklidski slučaj koje su izrečene u [2].*

Ključne riječi: dopustiva krivulja, evolvente, evolute, n-dimenzionalni jednostruko izotropni prostor.