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TOWARDS THE GENERALIZATION OF T-OPERATORS: A DISTANCE BASED APPROACH

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Aggregation is one of the key issues in the development of intelligent systems, just like with neural networks, fuzzy knowledge based systems, vision systems, and decision-making systems. From the point of view of a particular application the choice of the most appropriate operator is an important part of system design. This paper gives a brief summary of the best known operators, such as t-norms, t-conorms, uninorms, averaging and compensative operators, and outlines their most important properties. Two new pairs of distances, based on binary operations and their generalizations, are introduced, based on the fuzzy entropy approach, and their properties are outlined.

Keywords: t-operators, uninorms, compensatory operators, aggregation operators, entropy based operators.

1. INTRODUCTION

In the original fuzzy set theory, connectives were formulated in terms of Zadeh's standard operations of minimum, maximum and complement. Since 1965, for each of these operations several classes of operators, satisfying appropriate axioms, have been introduced. By accepting some basic conditions, a broad class of operations for union and intersection is formed by t-operators. The concepts of using *t-norm* and *t-conorm* were originally developed by Menger [12] within the framework of the theory of probabilistic metric spaces. Since then a great number (of various types) of t-operators have been developed [4, 7]. From an algebraic point of view, t-norms and t-conorms are commutative semigroup operations on [0,1] with the neutral element 1 and 0, respectively.

In many applications these conventional operators do not work well, and some additional properties, like compensation behavior are required. Recently, in order to get rid of these disadvantages, several generalizations have been introduced.

For one type of generalization of t-norms and t-conorms the concept of *uninorm* was introduced by Yager and Rybalov [15]. The neutral element of uninorms can be any number in the unit interval. The structure of uninorms was discussed by Fodor et al [5], and an overview of the classes of uninorms is given in [2].

In this paper a new approach to construct uninorms is introduced. The definitions are based on an entropy approach. The concept of an elementary entropy function, derived from fuzzy entropy, forms the basis of our investigation. In fuzzy set theory the entropy was introduced by De Luca and Termini [3]. They gave the axioms of entropy and an example of an entropy of a fuzzy set in the case of a finite universal set. Kaufmann [8] showed that an entropy could be obtained as the distance between the fuzzy set and its nearest crisp set. Knopfmacher [10], and Loo [11] introduced a larger class of entropy that contained the entropy proposed by De Luca and Termini and Kaufmann as special cases. Yager [14] defined the entropy of a fuzzy set by the distance between the fuzzy set and its complement.

In this paper new methods for constructing novel operators are outlined. Based on the entropy of a fuzzy subset, the concept of an elementary entropy function is introduced. This function assigns a value to each element of a fuzzy subset and this characterizes its degree of fuzziness. The new generalized minimum and generalized maximum are defined as minimum and maximum entropy operations.

The definitions of the entropy-based fuzzy operators are generalized to the binary operators defined on $[0,1]^2$. The basic idea of the extension is the reformulation of the entropy of an element as a distance from the fuzziest element 0.5. The properties of the new operators and their further generalization are also discussed.

Throughout this paper the following notations will be used: X is the universal set, X^F is the class of all fuzzy subsets of X, $\Re^+$ is the set of non negative real numbers, $\overline{A}$ is the fuzzy complement of $A \in X^F$ and $|A|$ is the cardinality of A, and the distance of the two elements x and y is denoted by $d(x,y)$.

2. T-NORMS

Definition 1. Let T be a mapping:

$$T:[0,1]\times[0,1]\rightarrow[0,1] \tag{1}$$

T is a *t-norm,* if for all $a,b,c\in[0,1]$ T satisfies the following axioms:

- **A1.a** $T(a,1)=a$,; that, is 1 is the neutral element of T,
- **A1.b** $T(a,b)=T(b,a)$;that is, T is *commutative,*
- **A1.c** $T(T(a,b),c)=T(a,T(b,c))$; that is, T is *associative,*
- **A1.d** $T(a,b)\le T(a,c)$, if $b<c$; that is, T is *nondecrareasing.*

Remarks

1. $T(0,x)\le T(0,1)=0$ implies $T(0,x)=0$, that is, 0 is an *absorbing element* of T.
2. From an algebraic point of view T is a commutative semigroup operation on $[0,1]$ with the neutral element 1.
3. Due to associativity, the extension of a t-norm for more than two arguments is unique.

Definition 2. Given a t-norm T, the *intersection of two fuzzy subsets A and B* of the universe X is defined as

$$A \cap_T B = \{(x, \mu_{A\cap_T B}(x)) | x \in X, \mu_{A\cap_T B}(x) : X \to [0,1]\} \qquad (2)$$

where

$$\mu_{\mu_{A\cap_T B}} : x \mapsto T(\mu_A(x), \mu_B(x)) \qquad (3)$$

Basic t-norms		
1. *Minimum (Zadeh)*	$T_M(x, y) = \min(x, y)$	(4)
2. *Product*	$T_P(x, y) = xy$	(5)
3. *Łukasiewicz t-norm*	$T_L(x, y) = \max(x + y - 1.0)$	(6)
4. *Weakest t-norm*	$T_W(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$	(7)

Definition 3. Let T_1 and T_2 be two t-norms. T_1 is said to be weaker than T_2 (and equivalently, T_2 is stronger than T_1) if $T_1(x, y) \le T_2(x, y)$ for all $(x, y) \in [0,1]^2$.
It is easy to see that for any t-norm

$$T_W \le T \le T_M, \qquad (8)$$

and for the four basic t-norms

$$T_W < T_L < T_P < T_M. \qquad (9)$$

This means that T_W is the weakest and T_M is the strongest t-norm.

3. T-CONORMS

Definition 4. Let S be a mapping

$$S : [0,1] \times [0,1] \to [0,1] \qquad (10)$$

S is a *t-conorm,* if for all $a, b, c \in [0,1]$ S in satisfies the following axioms:

- **A2.a** $S(a,0) = a$; that is, 0 is the neutral element of S,
- **A2.b** $S(a,b) = S(b,a)$; that is, S is *commutative,*
- **A2.c** $S(S(a,b),c) = S(a,S(b,c))$; that is, S is *associative,*
- **A2.d** $S(a,b) \le S(a,c)$, if $b < c$; that is, S is *nondecreasing.*

Definition 5. Given t-conorm S the *union of two fuzzy subsets A and B* of the universe X is defined as

$$A \cup_S B = \{(x, \mu_{A\cup_{ST} B}(x)) | x \in X, \mu_{A\cup_S B}(x) : X \to [0,1]\} \qquad (11)$$

where

$$\mu_{\mu_{A\cup_S B}} : x \mapsto S(\mu_A(x), \mu_B(x)) \qquad (12)$$

Remarks

1. $1 = S(0,1) \le S(x,1)$ implies $S(0,1) = 1$, that is, 1 is an *absorbing element* of S.
2. From an algebraic point of view, S is a commutative semigroup operation on $[0,1]$ with the neutral element 0.
3. Due to associativity, the extension of a t-conorm for more than two arguments is unique.

Basic t-conorms		
1. *Maximum (Zadeh)*	$S_M(x,y) = \max(x,y)$	(13)
2. *Probabilistic Sum*	$S_P(x,y) = x + y - xy$	(14)
3. *Bounded Sum*	$S_L(x,y) = \min(x+y,1)$	(15)
4. *Strongest t-conorm*	$S_W(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 0 & \text{otherwise} \end{cases}$	(16)

4. NEGATIONS

Definition 6. Let N be a mapping

$$N : [0,1] \to [0,1] \tag{17}$$

N is a *negation* if for all $a, b \in [0,1]$ N in satisfies the following axioms:

A3.a $N(0) = 1$ and $N(1) = 0$,
A3.b $N(a) \ge N(b)$, if $a < b$;that is, N is *monotonically nonincreasing*.

The negation is called *strict* if it satisfies the following axioms:

A3.c N is a *continuous function,*
A3.d N is *strictly decreasing,*
A3.e $N(N(a)) = a$, that is, N is *involutive*.

Basic negations		
1. *Standard negation (Zadeh)*	$N(x) = 1 - x$	(18)
2. *Sugeno type*	$N_\lambda(x) = \dfrac{1-x}{1+\lambda x},\ \lambda > 1$	(19)
3. *Intuitionistic negation*	$N_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$	(20)
4. *Dual intuitionistic negation*	$N_i(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$	(21)

5. DUALITY

It is easy to prove that S is a t-conorm if and only if there exists a t-norm T such that for all $(x,y)\in[0,1]^2$

$$S(x,y)=NT(Nx,Ny) \tag{22}$$

Dual basic t-norms and t-conorms in the case of standard negations are:

t-norms	**t-conorms**
$T_M(x,y)=\min(x,y)$	$S_M(x,y)=\max(x,y)$
$T_P(x,y)=xy$	$S_P(x,y)=x+y-xy$
$T_L(x,y)=\max(x+y-1.0)$	$S_L(x,y)=\min(x+y,1)$
$T_W(x,y)=\begin{cases}\min(x,y) & \text{if } \max(x,y)=1\\ 0 & \text{otherwise}\end{cases}$	$S_W(x,y)=\begin{cases}\max(x,y) & \text{if } \min(x,y)=0\\ 0 & \text{otherwise}\end{cases}$

This duality allows us to translate some properties of the t-norms into the corresponding properties of the t-conorms. One such important property is the ordering of the norms.

The duality changes the order, i.e., if T_1, T_2 are t-norms such that $T_1\le T_2$ and S_1, S_2 are the dual t-conorms of T_1 and T_2, respectively, then $S_1\ge S_2$. Consequently, for any t-conorm S

$$S_M\le S\le S_W, \tag{23}$$

and for the four basic t-conorms

$$S_M<S_P<S_L<S_W. \tag{24}$$

This means that S_M is the weakest and S_W is the strongest t-conorm. In summary, for an arbitrary T t-norm and an arbitrary S t-conorm (not necessarily the dual of T) the following holds:

$$T\le T_M\le S_M\le S_W. \tag{25}$$

6. SOME BASIC LAWS

Throughout this section it is assumed that T is a t-norm, S is a t-conorm and N is a strict negation.

Definition 7. T and S are *idempotent* if

$$T(x,x)=x,\ \forall x\in[0,1] \tag{26}$$

$$S(x,x) = x, \forall x \in [0,1] \tag{27}$$

Proposition 1. [4]

1. *T is idempotent if and only if T=min.*
2. *S is idempotent if and only if S=max.*

Definition 8. The absorption laws are

$$T(S(x,y),x) = x, \forall x \in [0,1] \tag{28}$$

$$S(T(x,y),x) = x, \forall x \in [0,1] \tag{29}$$

Proposition 2. [4]

1. $T(S(x,y),x) = x, \forall x \in [0,1]$ *holds if and only if T=min.*
2. $S(T(x,y),x) = x, \forall x \in [0,1]$ *holds if and only if S=max.*

7. UNINORMS

Uninorms are a type of generalization of t-norms and t-conorms where the neutral element can be any number from the unit interval. The class of uninorms seems to play an important role both in theory and applications.

Definition 9. [15]. A uninorm U is a commutative, associative and an increasing binary operator with a neutral element $e \in [0,1]$, i.e.,

$$U(x,e) = x, \forall x \in [0,1]. \tag{30}$$

The neutral element e is clearly unique. The case $e = 1$ leads to t-conorm and the case $e = 0$ leads to t-norm.

Proposition 3. [5]. *Consider a uninorm U with a neutral element e.*

(i) *If* $e \in]0,1]$, *then the binary operator* T_U *defined by*

$$T_U = \frac{U(ex,ey)}{e} \tag{31}$$

is a t-norm.

(ii) *If* $e \in [0,1[$, *then the binary operator* S_U *defined by*

$$S_U = \frac{U(e+(1-e)x, e+(1-e)y) - e}{1-e} \tag{32}$$

is a t-conorm.

Consider the following two linear transformations:

$$\varphi_e = \frac{x}{e} \tag{33}$$

$$\psi_e = \frac{x-e}{1-e} \tag{34}$$

According to Proposition 3, any uninorm U with a neutral element $e \in]0,1[$ has a corresponding t-norm T and a t-conorm S such that

(i) $U(x,y) = \varphi_e^{-1}(T(\varphi_e(x), \varphi_e(y)))$ for all $(x,y) \in [0,e]^2$,

(ii) $U(x,y) = \psi_e^{-1}(T(\psi_e(x), \psi_e(y)))$ for all $(x,y) \in [e,1]^2$.

The following proposition gives information concerning the other parts of the unit square.

Proposition 4. [5]. *Consider a uninorm U with neutral element e, then*

$$\min(x,y) \le U(x,y) \le \max(x,y) \text{ for all } (x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e].$$

This means that U acts as a mean on this domain (see Figure 1).

Proposition 5. [5]. *For a uninorm U, one of the following two cases always holds:*

(i) *U is conjunctive (or and-like) uninorm:* $U(0,1)=U(1,0)=0$;

(ii) *U is disjunctive (or or-like) uninorm:* $U(0,1)=U(1,0)=1$.

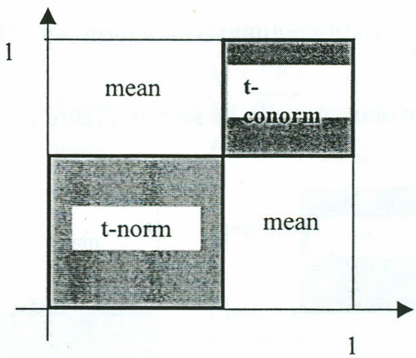

Figure 1. The structure of uninorms

For the given t-norm T and t-conorm S, the following propositions show the construction of conjunctive and disjunctive uninorms that have T and S as underlying t-norm and t-conorm.

Proposition 6. [2]. *A binary operator U is a conjunctive uninorm with the neutral element* $e \in [0,1[$ *such that* $U(\cdot,1)$ *is continuous on* $[0,e[$ *if and only if there exists a t-conorm S such that*

$$U(x,y) = \begin{cases} \varphi_e^{-1}(T(\varphi_e(x), \varphi_e(y))), & \text{if } (x,y) \in [0,e]^2 \\ \psi_e^{-1}(S(\psi_e(x), \psi_e(y))), & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{35}$$

The uninorm characterized by the above proposition is denoted by $U_{\min}$.

Proposition 7 [2]. *A binary operator U is a disjunctive uninorm with the neutral element $e \in [0,1[$ such that $U(\cdot,0)$ is continuous on $]e,1]$ if and only if there exists a t-norm T such that*

$$U(x,y)=\begin{cases}\varphi_e^{-1}(T(\varphi_e(x),\varphi_e(y))), & \text{if } (x,y)\in[0,e]^2\\ \psi_e^{-1}(S(\psi_e(x),\psi_e(y))), & \text{if } (x,y)\in[e,1]^2\\ \max(x,y), & \text{elsewhere}\end{cases} \tag{36}$$

The uninorm characterized by the above proposition is denoted by $U_{\max}$.

If the underlying t-norm and t-conorm are the min and max operators, then the first uninorms were given by Yager and Rybalov [15], using:

$$U_c(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[e,1]^2\\ \min(x,y) & \text{elsewhere}\end{cases} \tag{37}$$

and

$$U_d(x,y)=\begin{cases}\min(x,y), & \text{if } (x,y)\in[0,e]^2\\ \max(x,y), & \text{elsewhere}\end{cases} \tag{38}$$

U_c is a conjunctive right-continuous uninorm and U_d is a disjunctive left continuous uninorm.

The structure of these uninorms can be seen in Figure 2.

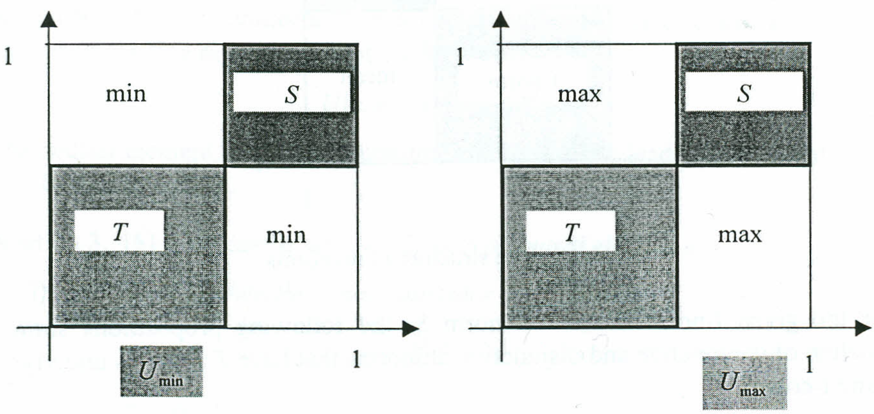

Figure 2. The structure of $U_{\min}$ and $U_{\max}$.

8. AVERAGING OPERATORS

Averaging operators represent a wide range of aggregation operators [6].

Definition 10. An averaging operator M is a mapping

$$M:[0,1]\times[0,1]\rightarrow[0,1] \tag{39}$$

and it satisfies the following properties:

A4.a $M(x,x)=x, \forall x\in[0,1]$; *idempotency,*

A4.b $M(x,y)=M(y,x), \forall x,y\in[0,1]$; *commutativity,*

A4.c $M(0,0)=0, \quad M(1,1)=1$,; *boundary conditions,*

A4.d $M(x,y)\le M(z,w)$, if $x\le z$ and $y\le w$,; *monotonicity,*

A4.e *M is continuous.*

The next proposition shows that for any averaging operator M, the global evaluation of an action will lie between the worst and the best local rating [6].

Proposition 8. [6] If *M is an averaging operator, then*

$$\min(x,y)\le M(x,y)\le\max(x,y),\ \forall(x,y)\in[0,1]^2. \tag{40}$$

The best known averaging operators are shown in the following table:

Name	M
Harmonic mean	$2xy/(x+y)$
Geometric mean	$\sqrt{xy}$
Arithmetic mean	$(x+y)/2$
Dual of geometric mean	$1-\sqrt{(1-x)(1-y)}$
Dual of harmonic mean	$(x+y-2xy)/(2-x-y)$
Median	$\text{med}(x,y,\alpha)=\begin{cases} y, & \text{if } x\le y\le\alpha \\ \alpha, & \text{if } x\le\alpha\le y \\ x, & \text{if } \alpha\le x\le y\end{cases}$
Generalized p-mean	$((x^p+y^p)/2)^{1/p},\ p\ge 1$

9. COMPENSATIVE OPERATIONS

We have see that for any t-norm and t-conorm the inequality $T\le T_M\le S_M\le S_W$ holds (see (25)), which means that there are no t-operators lying between the minimum and the maximum operators. This could be a disadvantage of the application of t-operators as aggregation operators in several intelligent systems, where fuzzy set theory is used to handle uncertain information.

A union operator produces a high output whenever at least one of the input values representing the degrees of satisfaction of different features or criteria is high. An intersection operator only produces a high output when all of the inputs are high. In real applications, for example in decision making, it would be necessary to a certain extent for a higher degree of satisfaction for one of the criteria to be compensated by a lower degree of satisfaction for other criteria. In this sense, the union provides full compensation, while the intersection provides no compensation at all.

To handle the problem Zimmermann and Zysno [17] have introduced the so–called γ-operator as the first compensatory operator. Since then, compensative operators have been studied by several authors.

Definition 11. An operator M is said to be a *compensative* if and only if

$$\min(x,y) \le M(x,y) \le \max(x,y),\ \forall (x,y) \in [0,1]^2 . \tag{41}$$

It can be seen that M includes the class of averaging operators and uninorms.

The **γ-operator** [17]. The parameter γ takes its values from the [0,1] interval and indicates the degree of compensation.

$$\Gamma_\gamma(x,y) = (xy)^{1-\gamma}\left(1-(1-x)(1-y)\right)^\gamma . \tag{42}$$

10. FUZZY ENTROPY AND ENTROPY BASED OPERATIONS

10.1. Fuzzy entropy and the entropy function

Definition 12. Let X be a universal set and *let* A be a fuzzy subset of X defined as:

$$A = \{(x, \mu_A(x)) | x \in X, \mu_A(x) \in [0,1] .$$

The *fuzzy entropy* is a function

$$e : X^F \to \Re^+$$

which satisfies the following axioms:

AE 1. $e(A) = 0$ if A is a crisp set.

AE 2. If A p B then $e(A) \le e(B)$; where A p B means that *A is sharper than B*, i.e.,

$$\mu_A(x) \le \mu_B(x) \text{ for } \mu_B(x) \le \frac{1}{2}$$

and

$$\mu_A(x) \ge \mu_B(x) \text{ for } \mu_B(x) > \frac{1}{2}, \text{ for all } x \in X .$$

AE 3. $e(A)$ assumes its maximum value if and only if A is *maximally fuzzy*. A is defined maximally fuzzy when $\mu_A(x) = \frac{1}{2}\ \forall x \in X$.

AE 4. $e(A) = e(\overline{A}),\ \forall A \in X^F$.

Definition 13. Let A be a fuzzy subset of X. The following function is said to be an elementary entropy function of A:

$$\varphi_A : X \to [0,1]; \quad \varphi_A : x \text{ a } \begin{cases} \mu_A(x), & \text{if } \mu_A(x) \le 0.5 \\ 1-\mu_A(x), & \text{if } \mu_A(x) > 0.5 \end{cases} \tag{43}$$

The special functions of the elementary entropy function are useful tools to construct the entropy of fuzzy sets [13]. It can be shown, for example, that the cardinality of the fuzzy sets

$$\Phi_A = \{(x, \varphi_A(x)) | x \in X, \varphi_A(x) \in [0,1]$$

is an entropy of A. It is easy to verify that this entropy is equivalent to the Hamming-entropy which is generated by the Hamming-distance of A from the nearest crisp set [9].

10.2. Entropy based operations

Definition 14. Let A and B be two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B *as* their elementary entropy functions, respectively. The *minimum fuzziness minimum* is defined as $T_{\min} = T_{\min}(A,B) = \{(x, \mu_{T_{\min}}(x)) | x \in X,\ \mu_{T_{\min}}(x) \in [0,1]$,

where
$$\mu_{T_{\min}} : x \text{ a } \begin{cases} \mu_A(x), & \text{if } \varphi_A(x) < \varphi_B(x) \\ \mu_B(x), & \text{if } \varphi_B(x) < \varphi_A(x) \\ \min(\mu_A(x), \mu_B(x)), & \text{if } \varphi_A(x) = \varphi_B(x) \end{cases}. \tag{44}$$

The geometrical representation of the minimum fuzziness generalized intersection can be seen in Figure 3.

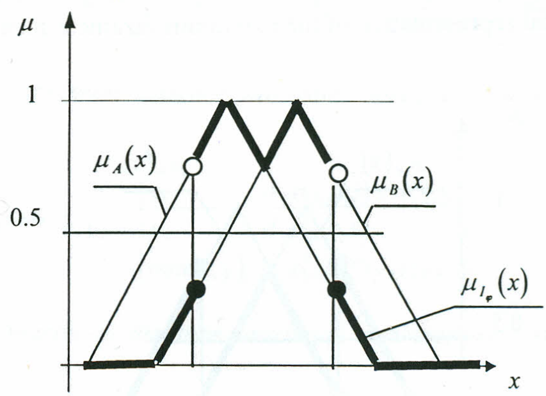

Figure 3. The minimum fuzziness minimum

Definition 15. Let A and B be two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B as their elementary entropy functions, respectively. The *minimum fuzziness maximum* is defined as $S_{\min} = S_{\min}(A,B) = \{(x, \mu_{S_{\min}}(x)) | x \in X,\ \mu_{S_{\min}}(x) \in [0,1]$, where

$$\mu_{S_{\min}} : x \mapsto \begin{cases} \mu_A(x), & \text{if } \varphi_A(x) < \varphi_B(x) \\ \mu_B(x), & \text{if } \varphi_B(x) < \varphi_A(x) \\ \max(\mu_A(x), \mu_B(x)), & \text{if } \varphi_A(x) = \varphi_B(x) \end{cases} . \tag{45}$$

Definition 16. Let A and B be two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B *as* their elementary entropy functions, respectively. The *maximum fuzziness minimum* is defined as $T_{\max} = T_{\max}(A,B) = \{(x, \mu_{T_{\max}}(x)) | x \in X, \mu_{T_{\max}}(x) \in [0,1]\}$, where

$$\mu_{T_{\max}} : x \mapsto \begin{cases} \mu_A(x), & \text{if } \varphi_A(x) > \varphi_B(x) \\ \mu_B(x), & \text{if } \varphi_B(x) > \varphi_A(x) \\ \min(\mu_A(x), \mu_B(x)), & \text{if } \varphi_A(x) = \varphi_B(x) \end{cases} . \tag{46}$$

Definition 17. Let A and B be two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B *as* their elementary entropy functions, respectively. The *maximum fuzziness maximum* is defined as $S_{\max} = S_{\max}(A,B) = \{(x, \mu_{S_{\max}}(x)) | x \in X, \mu_{S_{\max}}(x) \in [0,1]\}$, where

$$\mu_{S\max} : x \mapsto \begin{cases} \mu_A(x), & \text{if } \varphi_A(x) > \varphi_B(x) \\ \mu_B(x), & \text{if } \varphi_B(x) > \varphi_A(x) \\ \max(\mu_A(x), \mu_B(x)), & \text{if } \varphi_A(x) = \varphi_B(x) \end{cases} . \tag{47}$$

The geometrical representation of the maximum fuzziness maximum is given in Figure 4.

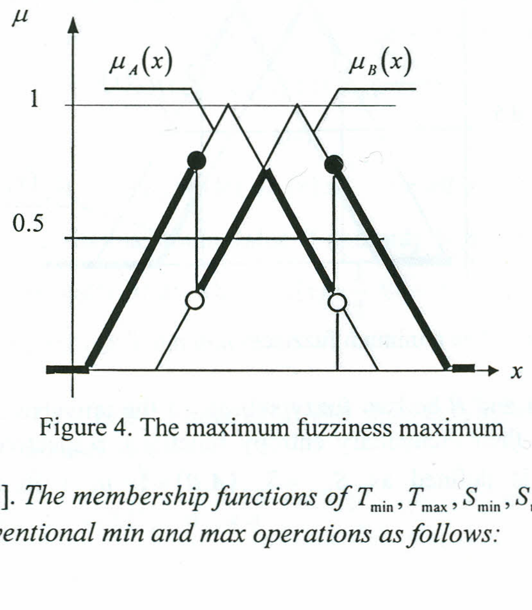

Figure 4. The maximum fuzziness maximum

Proposition 9. [13]. *The membership functions of* $T_{\min}, T_{\max}, S_{\min}, S_{\max}$ *can be expressed in terms of the conventional min and max operations as follows:*

$$\mu_{T_{\min}} = \begin{cases} \min(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) \le 1 - \mu_B(x) \\ \max(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) > 1 - \mu_B(x) \end{cases} \tag{48}$$

$$\mu_{S_{\min}} = \begin{cases} \min(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) < 1 - \mu_B(x) \\ \max(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) \ge 1 - \mu_B(x) \end{cases} \tag{49}$$

$$\mu_{T_{\max}} = \begin{cases} \max(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) < 1 - \mu_B(x) \\ \min(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) \ge 1 - \mu_B(x) \end{cases} \tag{50}$$

$$\mu_{S_{\max}} = \begin{cases} \max(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) \le 1 - \mu_B(x) \\ \min(\mu_A(x), \mu_B(x)) & \text{if } \mu_A(x) < 1 - \mu_B(x) \end{cases} \tag{51}$$

11. DISTANCE BASED BINARY OPERATIONS WITH RESPECT TO 0.5

The definition of the entropy-based fuzzy operators can be generalized to binary operators defined on $[0,1]^2$. The basic idea of extension as reformulated is the following: If the entropy of an element is less than the entropy of another element, it means that its distance from 0.5 is greater than the distance from 0.5 of another element.

Definition 18. The *maximum distance minimum operator with respect to 0.5* is defined as

$$T_{0.5}^{\max}(x,y) = \begin{cases} x, & \text{if } d(x,0.5) > d(y,0.5) \\ y, & \text{if } d(x,0.5) < d(y,0.5) \\ \min(x,y), & \text{if } d(x,0.5) = d(y,0.5) \end{cases}. \tag{52}$$

Definition 19. The *maximum distance maximum operator with respect to 0.5* is defined as

$$S_{0.5}^{\max}(x,y) = \begin{cases} x, & \text{if } d(x,0.5) > d(y,0.5) \\ y, & \text{if } d(x,0.5) < d(y,0.5) \\ \max(x,y), & \text{if } d(x,0.5) = d(y,0.5) \end{cases}. \tag{53}$$

Definition 20. The *minimum distance minimum operator with respect to 0.5* is defined as

$$T_{0.5}^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,0.5) < d(y,0.5) \\ y, & \text{if } d(x,0.5) > d(y,0.5) \\ \min(x,y), & \text{if } d(x,0.5) = d(y,0.5) \end{cases}. \tag{54}$$

Definition 21. The *minimum distance maximum operator with respect to 0.5* is defined as

$$S_{0.5}^{\min}(x,y)=\begin{cases} x, & \text{if } d(x,0.5)<d(y,0.5) \\ y, & \text{if } d(x,0.5)>d(y,0.5) \\ \max(x,y), & \text{if } d(x,0.5)=d(y,0.5) \end{cases}. \tag{55}$$

Proposition 3. *The distance based operators have the following properties:*

$T_{0.5}^{\max}$

- $T_{0.5}^{\max}(0,\mathrm{x})=0, \forall x\in[0,1]$ *,that is,* 0 *is an absorbing element and* $T_{0.5}^{\max}$ *is a conjunctive like operator,*
- $T_{0.5}^{\max}(\mathrm{x},\mathrm{x})=\mathrm{x}, \forall x\in[0,1]$ *, that is,* $T_{0.5}^{\max}$ *is idempotent,*
- $T_{0.5}^{\max}(1,\mathrm{x})=1, \forall x\in]0,1]$,
- $T_{0.5}^{\max}(0.5,x)=x$ *,that is, 0.5 is the neutral element,*
- $T_{0.5}^{\max}$ *is commutative and associative,*
- $T_{0.5}^{\max}$ *is increasing in each place of* $[0,1]\times[0,1]$.

$S_{0.5}^{\max}$

- $S_{0.5}^{\max}(1,\mathrm{x})=1, \forall x\in[0,1]$ *, that is,* 1 *is an absorbing element and* $S_{0.5}^{\max}$ *is a disjunctive like operator,*
- $S_{0.5}^{\max}(\mathrm{x},\mathrm{x})=\mathrm{x}, \forall x\in[0,1]$ *, that is,* $S_{0.5}^{\max}$ *is idempotent,*
- $S_{0.5}^{\max}(0,x)=0, \forall x\in[0,1[$,
- $S_{0.5}^{\max}(0.5,x)=x$ *, that is, 0.5 is the neutral element,*
- $S_{0.5}^{\max}$ *is commutative and associative,*
- $S_{0.5}^{\max}$ *is increasing in each place of* $[0,1]\times[0,1]$.

$T_{0.5}^{\min}$

- $T_{0.5}^{\min}(1,x)=x, \forall x\in[0,1]$ *,that is,* 1 *is a neutral element and* $T_{0.5}^{\min}$ *is a conjunctive like operator,*
- $T_{0.5}^{\min}(\mathrm{x},\mathrm{x})=\mathrm{x}, \forall x\in[0,1]$ *, that is,* $T_{0.5}^{\min}$ *is idempotent,*
- $T_{0.5}^{\min}(0,\mathrm{x})=\mathrm{x}, \forall x\in[0,1[$,
- $T_{0.5}^{\min}(0.5,x)=0{,}5$ *,that is, 0.5 is an absorbing element,*
- $T_{0.5}^{\min}$ *is commutative and associative.*

$S_{0.5}^{\min}$

- $S_{0.5}^{\min}(0,x)=x, \forall x\in[0,1]$ *,that is,* 0 *is the neutral element and* $S_{0.5}^{\min}$ *is a disjunctive like operator,*
- $S_{0.5}^{\min}(\mathrm{x},\mathrm{x})=\mathrm{x}, \forall x\in[0,1]$ *, that is,* $S_{0.5}^{\min}$ *is idempotent,*
- $S_{0.5}^{\min}(1,x)=x, \forall x\in]0,1]$,
- $S_{0.5}^{\min}(0.5,x)=0.5$ *,that is, 0.5 is the absorbing element,*
- $S_{0.5}^{\max}$ *is commutative and associative.*

Proof. The proof directly follows from the proof given by Rudas and Kaynak [13] for the properties of the entropy-based operators. ■

Corollary 1.

Consider the operators $T_{0.5}^{\max}$ *and* $S_{0.5}^{\max}$. Then

(i) *both are uninorms,*

(ii) *both are compensative operators.*

Proposition 11. The pairs ($T_{0.5}^{\max}$, $S_{0.5}^{\max}$) *and* ($T_{0.5}^{\min}$, $S_{0.5}^{\min}$) *are dual operators in the sense of equation(22).*

Proof.

(I) Suppose first that $d(x,0.5)=d(y,0.5)$, and assume that $x \le y$. This implies that $d(1-x,0.5)=d(1-y,0.5)$ and $1-x \ge 1-y$,

1. $NT_{0.5}^{\max}(Nx,Ny)=1-T_{0.5}^{\max}(1-x,1-y)=1-(1-y)=y=S_{0.5}^{\max}(x,y)$.
2. $NT_{0.5}^{\min}(Nx,Ny)=1-T_{0.5}^{\min}(1-x,1-y)=1-(1-y)=y=S_{0.5}^{\min}(x,y)$.

(II) Suppose now that $d(x,0.5)<d(y,0.5)$, and assume that $y<x<0.5$. This implies that $d(1-x,0.5)<d(1-y,0.5)$ and $1-x<1-y$,

1. $NT_{0.5}^{\max}(Nx,Ny)=1-T_{0.5}^{\max}(1-x,1-y)=1-(1-y)=y=S_{0.5}^{\max}(x,y)$.
2. $NT_{0.5}^{\min}(Nx,Ny)=1-T_{0.5}^{\min}(1-x,1-y)=1-(1-x)=x=S_{0.5}^{\min}(x,y)$.

(III) Suppose now that $d(x,0.5)<d(y,0.5)$, and assume that $0.5<y<x$. This implies that $d(1-x,0.5)<d(1-y,0.5)$ and $1-x<1-y$,

1. $NT_{0.5}^{\max}(Nx,Ny)=1-T_{0.5}^{\max}(1-x,1-y)=1-(1-y)=y=S_{0.5}^{\max}(x,y)$.
2. $NT_{0.5}^{\min}(Nx,Ny)=1-T_{0.5}^{\min}(1-x,1-y)=1-(1-x)=x=S_{0.5}^{\min}(x,y)$.

(IV) Suppose now that $d(x,0.5)<d(y,0.5)$, and assume that $x<0.5<y$. This implies that $d(1-x,0.5)<d(1-y,0.5)$ and $1-x>1-y$,

1. $NT_{0.5}^{\max}(Nx,Ny)=1-T_{0.5}^{\max}(1-x,1-y)=1-(1-y)=y=S_{0.5}^{\max}(x,y)$.
2. $NT_{0.5}^{\min}(Nx,Ny)=1-T_{0.5}^{\min}(1-x,1-y)=1-(1-x)=x=S_{0.5}^{\min}(x,y)$.

All other cases are direct consequences of the commutativity of the operators. ■

Proposition 4. *The pairs* ($T_{0.5}^{\max}$, $S_{0.5}^{\min}$) *and* ($T_{0.5}^{\min}$, $S_{0.5}^{\max}$) *satisfy the absorption law given by equations(28) and (29).*

Proof.

1. Suppose that $d(x,0.5)<d(y,0.5)$. Then

$$T_{0.5}^{\max}\left(S_{0.5}^{\min}(x,y),x\right)=T_{0.5}^{\max}(x,x)=x, \forall x \in [0,1],$$

$$S_{0.5}^{\min}\left(T_{0.5}^{\max}(x,y),x\right)=S_{0.5}^{\min}(y,x)=x, \forall x \in [0,1].$$

2. Suppose now that $d(x,0.5)>d(y,0.5)$. Then

$$T_{0.5}^{\max}\left(S_{0.5}^{\min}(x,y),x\right)=T_{0.5}^{\max}(y,x)=x,\ \forall x\in[0,1],$$

$$S_{0.5}^{\min}\left(T_{0.5}^{\max}(x,y),x\right)=S_{0.5}^{\min}(x,x)=x,\ \forall x\in[0,1].$$

3. Finally, let be $d(x,0.5)=d(y,0.5)$ and $x<y$. Than

$$T_{0.5}^{\max}\left(S_{0.5}^{\min}(x,y),x\right)=T_{0.5}^{\max}(y,x)=x,\ \forall x\in[0,1],$$

$$S_{0.5}^{\min}\left(T_{0.5}^{\max}(x,y),x\right)=S_{0.5}^{\min}(x,x)=x,\ \forall x\in[0,1].$$

For the pair ($T_{0.5}^{\min}$, $S_{0.5}^{\max}$), the proof can be carried out analogously. ∎

12. GENERAL DEFINITION OF DISTANCE BASED OPERATIONS

The generalization is based on the simle notion that 0.5 can be replaced by any number e from the unit interval [0, 1].

Definition 22. The *maximum distance minimum operator with respect to* $e\in[0,1]$ is defined as

$$T_e^{\max}(x,y)=\begin{cases}x, & \text{if } d(x,e)>d(y,e)\\ y, & \text{if } d(x,e)<d(y,e)\\ \min(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{56}$$

Definition 23. The *maximum distance maximum operator with respect to* $e\in[0,1]$ is defined as

$$S_e^{\max}(x,y)=\begin{cases}x, & \text{if } d(x,e)>d(y,e)\\ y, & \text{if } d(x,e)<d(y,e)\\ \max(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{57}$$

Definition 24. The *minimum distance minimum operator with respect to* $e\in[0,1]$ is defined as

$$T_e^{\min}(x,y)=\begin{cases}x, & \text{if } d(x,e)<d(y,e)\\ y, & \text{if } d(x,e)>d(y,e)\\ \min(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{58}$$

Definition 25. The *minimum distance maximum operator with respect to* $e\in[0,1]$ is defined as

$$S_e^{\min}(x,y)=\begin{cases}x, & \text{if } d(x,e)<d(y,e)\\ y, & \text{if } d(x,e)>d(y,e)\\ \max(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{59}$$

The min, max and the entropy-based operators are obtained as special cases. Analoguous propositions, like the ones in Section 11, can be formulated and proved for the generalized distance-based operators defined above.

13. CONCLUSIONS

In this paper a new approach to generalizing the t-operators is given. These distance-based operators are the reformulation of the entropy-based operators introduced by Rudas and Kaynak[13]. Two new pairs of dual operators are introduced, namely the maximum distance minimum and maximum, and the minimum distance minimum and maximum measured between an element and the most fuzzy. It was shown that the maximum distance minimum and maximum are uninorms.

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PRILOZI POOPĆENJU T–OPERATORA: PRISTUP TEMELJEN NA UDALJENOSTI

Sažetak

Agregiranje je jedan od ključnih problema u razvoju inteligentnih sustava, kao što su to neuralne mreže, sustavi temeljeni na neizrazitom znanju, sustavi za raspoznavanje oblika i sustavi za odlučivanje. Važan dio dizajniranja takvih sustava je izbor najprikladnijeg operatora za agregiranje. U radu se daje prikaz dobro poznatih operatora, kao što su to t-norme, t-konorme, uninorme, operatori usrednjavanja i kompenzacije, te se izlažu njihova najvažnija svojstva. Polazeći od pristupa entropije operatora neizrazite logike, uvode se dva nova para binarnih operacija temeljenih na udaljenosti, daju se i njihova poopćenja, te se izlažu njihova svojstva.

Ključne riječi: t-operator, uninorme, operatori kompenzacije, operatori agregiranja, operatori temeljeni na entropiji.