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A NOTE ON HEREDITARILY LOCALLY CONNECTED CONTINUA

Ivan Lončar
University of Zagreb, Faculty of Organization and Informatics, Varaždin, Croatia
E-mail : iloncar@foi.hr

The main purpose of this paper is to prove some theorems concerning nonmetric hereditarily locally connected continua using inverse systems.

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1 INTRODUCTION

The results in this paper were initiated by a question from Nikiel, Tuncali and Tymchatyn. They asked [7, Problem 9.10.] when the inverse limit of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$of hereditarily locally connected continua with monotone surjective bonding mappings p_{ab} is a hereditarily locally connected continuum (a continuous image of an arc).

We shall prove that X = lim**X** is a hereditarily locally connected continuum if and only if each countable subsystem of **X** has a hereditarily locally connected limit. Moreover, the following two statements are equivalent (Theorem 3.9): (i) The limit of an inverse system of hereditarily locally connected continua with surjective monotone bonding mappings is hereditarily locally connected, (ii) The limit of an inverse sequence of hereditarily locally connected metric continua with surjective monotone bonding mappings is hereditarily locally connected.

We say that $\mathbf{X} = \{X_a, p_{ab}, A\}$is *$\sigma$-directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

THEOREM 1.1 *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$be a σ-directed inverse system of compact spaces with surjective bonding mappings and a limit X. Let Y be a metric compact space. For each surjective mapping $f: X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b: X_b \to Y$ such that $f = g_b p_b$.*

Proof. Let $\mathcal{B}$ be a countable basis of Y and let $\mathcal{V}$ be the collection of all finite subfamilies of $\mathcal{B}$ which cover Y. Clearly, card$(\mathcal{V}) = \aleph_0$. Hence, $\mathcal{V} = \{\mathcal{V}_n : n \in \mathbb{N}\}$. For each $\mathcal{V}_n$ $f^{-1}(\mathcal{V}_n)$ is a covering of X. There exists an $a(n) \in A$ such that for each $b \geq a(n)$ there is a cover $\mathcal{V}_{nb}$ of X_b with $p_b^{-1}(\mathcal{V}_{nb}) \prec f^{-1}(\mathcal{V}_n)$. From the σ - directedness of A it follows that there is an $a \in A$ such that $a \geq a(n)$, $n \in \mathbb{N}$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ is degenerate. Suppose that there exists a pair u,v of distinct points of Y such that $u, v \in f(p_b^{-1}(x_b))$. Then there exists a pair x,y of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U,V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, X \backslash \{u,v\}\}$. There exists a covering $\mathcal{V}_n \in \mathcal{V}$ such that $\mathcal{V}_n \prec \{U, V, X \backslash \{u,v\}\}$. We infer that there is a covering $\mathcal{V}_{nb}$ of X_b such that $p_b^{-1}(\mathcal{V}_{nb}) \prec f^{-1}(\mathcal{V}_n)$. It follows that $p_b(x) \neq p_b(y)$ since x and y lie in the disjoint members of the covering $f^{-1}(\mathcal{V}_n)$. This is impossible since $x, y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define g_b: $X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is clear that $g_b p_b = f$. Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and p_b is a quotient mapping (as a closed mapping). ■

THEOREM 1.2 *Let X be a compact space. There exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to lim$\mathbf{X}$.*

Proof. See [6, pp. 152 , 164].■

THEOREM 1.3 *[6, p. 163, Theorem 2.]. If X is a locally connected compact space, then there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected compact space, each p_{ab} is a monotone surjection and X is homeomorphic to lim$\mathbf{X}$. Conversely, the inverse limit of such a system is always a locally connected compact space.*

A continuous mapping $f: X \to Y$ is said to be *confluent* [5, p. 225] if for each subcontinuum Q of Y and each component K of $f^{-1}(Q)$ we have $f(K) = Q$.

A continuous mapping $f: X \to Y$ is said to be *weakly confluent* [5, p. 226] if for each subcontinuum Q of Y there exists a component K of $f^{-1}(Q)$ such that $f(K) = Q$.

2 HEREDITARILY LOCALLY CONNECTED CONTINUA

A continuum X is said to be *hereditarily locally connected* if each subcontinuum of X is locally connected.

THEOREM 2.1 *[3, Corollary 3]. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ - directed inverse system of hereditarily locally connected continua X_a. Then $X = $ lim$\mathbf{X}$ is hereditarily locally connected.*

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$be an inverse system. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, n = 0, 1, ..., by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y): x,y \in \Delta_n\}$, where m(x,y) is a member of A such that $x,y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n: n \in \mathbb{N}\}$. It is clear that $card(\Delta) = card(\Delta_0)$. Moreover, Δ is directed by $\leq$ [7, Lemma 9.2]. For each directed set $(A,\leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, card(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \leq\}.$$

Then A_σ is σ - directed by inclusion [7, Lemma 9.3]. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If Δ, $\Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma}: X_\Gamma \to X_\Delta$ denotes the map induced by the projections $p^\Gamma_\delta: X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system X^Γ. Now, we have the following theorem.

THEOREM 2.2 [7, Theorem 9.4]. *If* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is an inverse system, then* $\mathbf{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ *is a* σ *- directed inverse system and* $\lim\mathbf{X}$ *and* $\lim\mathbf{X}_\sigma$ *are canonically homeomorphic.*

THEOREM 2.3 *If X is a hereditarily locally connected continuum, then there exists a* σ*-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *such that each* X_a *is a metrizable hereditarily locally connected continuum, each* p_{ab} *is a monotone surjection and X is homeomorphic to* $\lim\mathbf{X}$.

Proof. Apply [8, Corollary 2.9] and Theorem 2.2.■

Let X be a non - degenerate locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of a locally connected continuum is again a locally connected continuum. We let

$$\mathbf{L}_X = \{Y \subset X : \text{Y is a non- degenerate cyclic element of X}\}.$$

LEMMA 2.4 *[7, Lemma 2.2]. If C is a connected subset of X and* $Y \in \mathbf{L}_X$*, then* $C \bigcap Y$ *is connected (possibly void).*

LEMMA 2.5 *[7, Lemma 2.3]. If* $f : X \to X'$ *is a monotone surjection, then for each* $Y' \in \mathbf{L}_{X'}$ *there exists* $Y \in \mathbf{L}_X$ *such that* $Y' \subseteq f(Y)$*. In particular,* $\mathbf{L}_X$ *is non - empty if* $\mathbf{L}_{X'}$ *is non - empty.*

The following theorem is a generalization of the well-known result of G.T. Whyburn [10, p. 81] which asserts that a metric continuum X is hereditarily locally connected if and only if each cyclic element $Z \subseteq X$ is hereditarily locally connected.

THEOREM 2.6 *A locally connected continuum X is hereditarily locally connected if and only if each cyclic element of X is hereditarily locally connected.*

Proof. By virtue of Theorem 1.3 there exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$of metric locally connected spaces such that p_{ab} are monotone and X is homeomorphic to lim**X**. Let us prove that each X_a is hereditarily locally connected. It suffices to prove that each cyclic element Z_a of X_a is hereditarily locally connected. By virtue of Lemmaa 2.5 there exists a cyclic element Z of lim**X** such that $p_a(Z) \supseteq Z_a$. Since lim**X** is homeomorphic to X, Z is hereditarily locally connected. This means that $p_a(Z)$ is hereditarily locally connected. It follows that Z_a is hereditarily locally connected since $Z_a \subseteq p_a(Z)$. We infer that each X_a is hereditarily locally connected since X_a is a metric continuum. From Theorem 2.1 it follows that X is hereditarily locally connected.■

3 Σ - COHERENT CONTINUA

A space X is called σ - coherent [4] if the intersection of every descending sequence of connected subsets of X is connected.

The following facts are known.

REMARK 3.1 If X is a metric σ - coherent continuum, then:

1. X is hereditarly decomposable ([4, Lemma 1]),
2. X is hereditarily arcwise connected ([4, Lemma 2]),
3. X is locally connected,
4. X is hereditarily locally connected.

In order to prove that each non-metric σ - coherent locally connected continuum is hereditarily locally connected we shall prove the following lemma.

LEMMA 3.2 *Let X be a σ - coherent continuum.If $f:X \to Y$ is a monotone surjection, then Y is a σ - coherent continuum.*

Proof. Let

$$Y_1 \supseteq Y_2 \supseteq ... \supseteq Y_n \supseteq ...$$

be a descending sequence of connected subsets of X. By virtue of [2, Theorem 6.1.29] each $f^{-1}(Y_n)$ is a connected subset of X. Now, we have a descending sequence

$$f^{-1}(Y_1) \supseteq f^{-1}(Y_2) \supseteq ... \supseteq f^{-1}(Y_n) \supseteq ...$$

of connected sets in X. The intersection $X^* = \bigcap\{f^{-1}(Y_n)\colon n \in \mathbb{N}\}$ is connected since X is σ - coherent. If $Y^* = \bigcap\{Y_n\colon n \in \mathbb{N}\}$, then $f^{-1}(Y^*) = X^*$. Thus $f(X^*) = Y^*$. We infer that Y^* is connected since X^* is connected.■

Now we shall prove that each σ - coherent continuum is hereditarily locally connected.

THEOREM 3.3 *If X is a σ - coherent locally connected continuum, then X is hereditarily locally connected.*

Proof. If X is a metric continuum, then X is hereditarily locally connected (see 4. of 3.1). Suppose that X is non-metric. By virtue of Theorem 1.3 there exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$of metric continua X_a and monotone surjective bonding mappings p_{ab} such that X is homeomorphic to lim$\mathbf{X}$. By virtue of Lemma 3.2 each X_a is σ - coherent. By virtue of 4. of 3.1 each X_a is hereditarily locally connected. We infer that X is hereditarily locally connected (Theorem 2.1). ∎

QUESTION 1. Is each σ - coherent continuum hereditarily locally connected?

QUESTION 2. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$be an inverse system of σ - coherent continua and surjective monotone bonding mappings. Is it true that X = lim$\mathbf{X}$ is σ - coherent (hereditarily locally connected) ?

THEOREM 3.4 *The property of being σ - coherent is cyclically reducible and extensible.*

Proof. If X is σ - coherent and if Z is a cyclic element of X, then every descending sequence of connected subsets of Z is a descending sequence of connected subsets of X. Thus, the intersection of this descending sequence of connected subsets of Z is connected. We infer that Z is σ - coherent. Thus, the property of being σ - coherent is reducible. Let us prove that the property of being σ - coherent is extensible. Let $\{Y_n\colon n\in I\!N\}$ be a descending sequence of connected subsets of X and let $Y = \bigcap\{Y_n\colon n\in I\!N\}$. Let a,b be a pair of distinct points of Y. We shall prove that there exists a connected subset of Y which contains a and b. Let $E(a,b) = \{a, b\}\cup\{x\in X : \text{a and b are in distinct components of } X\backslash\{x\}\}$ and let $C(a,b) = E(a,b)\cup\{H\colon \text{H is a cyclic element of X and } card(H\cap E(a,b)) = 2\}$. If H is one such cyclic element, then x_H and y_H denote the points of $H\cap E(a,b)$. Recall that C(a,b) is called the cyclic chain from a to b in X. Recall also that E(a,b) is a compact space and $\leq$ is the natural ordering on E(a,b) from a to b which induces the original topology on E(a,b), where $\leq$ is defined by $x\leq y$ provided either $x = a\neq y$ or a and y are in distinct components of $C(a,b)\backslash\{x\}$. Consider the set $K(a,b) = E(a,b)\cup\{Y\cap H\colon \text{H is a cyclic element of X and } card(H\cap E(a,b)) = 2\}$. Let us prove that K(a,b) is a connected subset of Y. Suppose that there is a point $x\in E(a,b)$ which is not in Y. Then a and b are in the distinct components E and F of $X\backslash\{x\}$. There exists a $n\in I\!N$such that x is not in Y_m for each $m\geq n$. Now, $Y_m\cap E$ is a non-empty open and closed subset of Y_n containing no point b. This is impossible since each Y_m is connected. Thus $K(a,b)\subseteq Y$. In order to complete the proof it suffices to prove that K(a, b) is connected. Suppose it is not. Then there exists a separation $K(a,b) = K_1\cup K_2$. Let $a\in K_1$ and $b\in K_2$. There exists a first point of K_2 in E(a,b) and a last point $c\in K_1$ in E(a,b) that precedes d. Then c and d are respectively the points x_H and y_H of some cyclic element H of X such that card(H$\cap$E(a,b)) = 2 (see (2) of the proof of Theorem 2 of [1] and the proofs of (5.1) and Theorem 5.2 of [10, p. 71]). Now $K_1 \cap (H \cap Y)$ and $K_2 \cap (H \cap Y)$ is a separation of H$\cap$Y. This is impossible since each $Y_n \cap H$ is connected (Lemma 2.4) and $H \cap Y = \cap\{Y_n \cap H: n \in \mathbb{N}\}$ is connected by the assumption that each cyclic element of X is σ - coherent. ■

A surjection f: X→Y is called *locally weakly confluent* provided for each point y of Y there exists a closed neighbourhood V of y in Y such that the restriction $f|f^{-1}(V)$ is weakly confluent [5, p. 226].

THEOREM 3.5 *[5, Theorem 3.1]. Locally weakly confluent images of metric hereditarily locally connected continua are hereditarily locally connected.*

For non - metric continua we have the following theorem.

THEOREM 3.6 *[9, Theorem 7]. Weakly confluent images of hereditarily locally connected continua are hereditarily locally connected.*

We shall prove the following generalization of the above theorems.

THEOREM 3.7 *Locally weakly confluent images of hereditarily locally connected continua are hereditarily locally connected.*

Proof. The proof consists of several steps.
Step 1. Let f: X→Y be a locally confluent mapping and let X be hereditarily locally connected. By virtue of [11, Lemma 1.5, p. 70] Y is locally connected.
Step 2. By Theorem 2.3 there exists a σ - directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric hereditarily locally connected continua and monotone bonding mappings such that X is homeomorphic to lim**X**. Similarly, by Theorem 1.3 there exists a σ - directed inverse system $\mathbf{Y} = \{Y_a, q_{ab}, B\}$ of metric locally connected continua and monotone bonding mappings such that Y is homeomorphic to lim**Y**.
Step 3. There exists a $b_0 \in B$ such that each mapping $q_c f$: $X \to Y_c$ is locally weakly confluent. From the assumption that f is locally weakly confluent it follows that for each y∈Y there exists an open set V(y) such that $f|f^{-1}(\mathrm{Cl}V(y))$: $f^{-1}(\mathrm{Cl}V(y)) \to \mathrm{Cl}V(y)$ is weakly confluent. The compactness of Y implies that there exists a finite subcover $\mathcal{V} = \{V(y_1), ..., V(y_n)\}$ of the cover {V(y): y∈Y}. There exists a $b_0 \in B$ such that for each $c \geq b_0$ there exists a cover $\mathcal{V}_c$ with the property that $q_c^{-1}(\mathcal{V}_c)$ is one which refines $\mathcal{V}$. Let us prove that q_c is weakly confluent for a fixed $c \geq b_0$. Let y_c be any point of Y_c. There exists a member V_c of $\mathcal{V}_c$ such that $y_c \in V_c$. Moreover, there exists a $V(y_i) \in \mathcal{V}$ such that $q_c^{-1}(V_c) \subseteq V(y_i)$. Now, let K_c be a subcontinuum of $\mathrm{Cl}(V_c)$. Then $q_c^{-1}(K_c)$ is a subcontinuum of $\mathrm{Cl}(V(y_i))$ since q_c is monotone. We infer that there exists a subcontinuum C of X, $C \subseteq f^{-1}(q_c^{-1}(K_c))$, such that $f(C) = q_c^{-1}(K_c)$ since f is locally weakly confluent. Clearly, $q_c(f(C)) = K_c$. Hence, $q_c f$ is locally weakly confluent.

Step 4. Let b_0 be as in Step 3. For each $c \geq b_0$ there exists an $a(c) \in A$ and a mapping r_c: $X_{a(c)} \rightarrow Y_c$ such that $q_c f = r_c p_{a(c)}$. This follows from the fact that Y_c is a metric space and from Theorem 1.1. We have the following diagram

$$\begin{array}{ccc} X_{a(c)} & \stackrel{p_{a(c)}}{\longleftarrow} & X \\ \downarrow r_c & & \downarrow f \\ Y_c & \stackrel{q_c}{\longleftarrow} & Y \end{array} \tag{1}$$

Step 5. Every mapping r_c is locally weakly confluent. This follows from Theorems 2.7 and Proposition 2.1 of [5].
Step 6. Each space Y_c is hereditarily locally connected. This is a consequence of Theorem 3.5 since r_c is locally weakly confluent (Step 5.) and $X_{a(c)}$, Y_c are metric spaces.
Step 7. Theorem 2.1 implies that Y is hereditarily locally connected. The proof of this Theorem is now completed. ■

Now we consider the inverse systems of hereditarily locally connected continua. We will give a partial answer to [7, Problem 9.10.] mentioned at the beginning of the Introduction.

THEOREM 3.8 *The following two statements are equivalent:*
(i) The limit of an inverse sequence of hereditarily locally connected continua with surjective monotone bonding mappings is hereditarily locally connected,
(ii) The limit of an inverse sequence of metric hereditarily locally connected continua with surjective monotone bonding mappings is hereditarily locally connected.

Proof. It suffices to prove that the limit of an inverse sequence of hereditarily locally connected continua with surjective monotone bonding mapping is hereditarily locally connected if the limit of an inverse sequence of metric hereditarily locally connected continua with surjective monotone bonding mapping is hereditarily locally connected. The proof consists of several steps.
Step 1. By virtue of Theorem 2.3 for each X_n there exists a σ - directed inverse system

$$\mathbf{X(n)} = \{X_{(n,a_n)}, f_{(n,a_n)(n,b_n)}, A_n\} \tag{2}$$

such that each $X_{(n,a_n)}$ is a metric hereditarily locally connected continuum, each $f_{(n,a_n)(n,b_n)}$ is monotone and X_n is homeomorphic to $\lim \mathbf{X(n)}$.
Step 2. Put $B = \{(n, a_n) : a_n \in A_n, n \in \mathbb{N}\}$ and put C to be the set of all countable subsets c of B of the form

$$c = \{(1, a_1), (2, a_2), ..., (n, a_n), ...\}, \tag{3}$$

where $a_1 \in A_1$, $a_2 \in A_2$, ..., $a_{\in} A_n$, ...

Step 3. Let D be a subset of C containing all c∈C for which there exists the monotone surjective mappings

$$g_{(n,a_n)(n+1,a_{n+1})} : X_{(n+1,a_{n+1})} \to X_{(n,a_n)} \tag{4}$$

such that

$$X_{(1,a_1)} \overset{g_{(1,a_1)(2,a_2)}}{\longleftarrow} X_{(2,a_2)} \overset{g_{(2,a_2)(3,a_3)}}{\longleftarrow} \dots \overset{g_{(n-1,a_{n-1})(n,a_n)}}{\longleftarrow} X_{(n,a_n)} \longleftarrow \dots \tag{5}$$

is an inverse sequence and each diagram

$$\begin{array}{ccc} X_n & \overset{p_{nn+1}}{\longleftarrow} & X_{n+1} \\ \downarrow f_{(n,a_n)} & & \downarrow f_{(n+1,a_{n+1})} \\ X_{(n,a_n)} & \overset{g_{(n,a_n)(n+1,a_{n+1})}}{\longleftarrow} & X_{(n+1,a_{n+1})} \end{array} . \tag{6}$$

commutes, where $f_{(n,a_n)} : X_n \to X_{(n,a_n)}$ is the canonical projection. Let us note that from the commutativity of diagram (6) it follows that $g_{(n,a_n)(n+1,a_{n+1})}$ is monotone and surjective.

Step 4. *The set D is non - empty. Moreover, for each countable subset* $\{d_i : i\in I\!N\}$, $d_i = \{(j , a_i^{(j)} : j\in I\!N\}$, there exists a d∈D such that $d = \{(1 , a_1) , (2 , a_2) , \dots \}$ and for each $i\in I\!N$ $a_i \geq a_i^{(j)}$, $j\in I\!N$.

Proof of Step 4. By virtue of the σ - directedness of A_1 there exists an $a_1 \in A_1$ such that $a_1 \geq a_1^{(j)}$, $j\in I\!N$.The space $X_{(1,a_1)}$ is a metric compact space and there exists a mapping $f_{(1,a_1)}p_{12}$: $X_2 \to X_{(1,a_1)}$. By virtue of Theorem 1.1 there exists an $a_2 \in A_2$ such that for each $b \geq a_2$ there is a mapping $g_{(1,a_1)(2,b)}$: $X_{(2,b)} \to X_{(1,a_1)}$ with $f_{(1,a_1)}p_{12} = g_{(1,a_1)(2,b)}f_{(2,b)}$, i.e., the diagram

$$\begin{array}{ccc} X_1 & \overset{p_{12}}{\longleftarrow} & X_2 \\ \downarrow f_{(1,a_1)} & & \downarrow f_{(2,b)} \\ X_{(1,a_1)} & \overset{g_{(1,a_1)(2,b)}}{\longleftarrow} & X_{(2,b)} \end{array} \tag{7}$$

commutes. By virtue of the σ - directedness of A_2 we may assume that $a_2 \geq a_2^{(j)}$, $j\in I\!N$. Suppose that $(1 , a_1)$, $(2 , a_2)$, ..., (n , a_n) are defined such that the diagram (6) commutes for $n\in\{1, 2, \dots, n - 1\}$. We define $a_{n+1} \in A_{n+1}$ considering the space $X_{(n,a_n)}$ and the mapping $f_{(n,a_n)}p_{nn+1}$: $X_{n+1} \to X_{(n,a_n)}$. Again, by Theorem 1.1 there exists an $a_{n+1} \in A_{n+1}$ such that for each $b \geq a_{n+1}$ there is a mapping $g_{(n,a_n)(n+1,b)}$: $X_{(n+1,b)} \to X_{(n,a_n)}$ with $f_{(n,a_n)}p_{nn+1} = g_{(n,a_n)(n+1,b)}f_{(n+1,b)}$, i.e., the diagram

$$\begin{array}{ccc} X_n & \overset{p_{nn+1}}{\longleftarrow} & X_{n+1} \\ \downarrow f_{(n,a_n)} & & \downarrow f_{(n+1,b)} \\ X_{(n,a_n)} & \overset{g_{(n,a_n)(n+1,a_b)}}{\longleftarrow} & X_{(n+1,a_b)} \end{array} \tag{8}$$

commutes. By virtue of the σ - directedness of A_{n+1} we may assume that $a_{n+1} \geq a_{n+1}^{(j)}$, $j\in I\!N$.By induction on $n\in I\!N$,d∈D with the required property $d = \{(1 , a_1) , (2 , a_2) , \dots \}$ is defined.

Step 5. We define a partial order on D as follows. Let d_1,d_2 be a pair of members of D such that $d_1 = \{(1, a_1), (2, a_2), ...\}$ and $d_2 = \{(1, b_1), (2, b_2), ...\}$. We write $d_2 \leq d_1$ if and only if $b_i \leq a_i$ for each $i \in \mathbb{N}$. From Step 4. it follows that $(D, \leq)$ is σ - directed.
Step 6. *For each $d \in D$ the limit space X_d of the inverse sequence (5) is a metric hereditarily locally connected continuum (by the assumption of the Theorem) and the projections $g_{(n,a_n)}: X_d \to X_{(n,a_n)}$ are monotone. Moreover, there exists a mapping $F_d: X \to X_d$.* The existence of F_d follows from the commutativity of the diagram (6).
Step 7. If $d_1 = \{(1, a_1), (2, a_2), ...\}$, $d_2 = \{(1, b_1), (2, b_2), ...\}$ and $d_2 \geq d_1$, then for each $n \in \mathbb{N}$ commutes the diagram

$$\begin{array}{ccc} X_{(n,b_n)} & \xleftarrow{g_{(n,b_n)(n+1,b_{n+1})}} & X_{(n+1,b_{n+1})} \\ \downarrow f_{(n,a_n)(n,b_n)} & & \downarrow f_{(n+1,a_{n+1})(n+1,b_{n+1})} \\ X_{(n,a_n)} & \xleftarrow{g_{(n,a_n)(n+1,a_{n+1})}} & X_{(n+1,a_{n+1})} \end{array} \qquad (9)$$

This follows from the commutativity of the diagrams of the form (6) for d_1 and d_2, i.e., from the commutativity of the diagrams

$$\begin{array}{ccc} X_n & \xleftarrow{p_{nn+1}} & X_{n+1} \\ \downarrow f_{(n,a_n)} & & \downarrow f_{(n+1,a_{n+1})} \\ X_{(n,a_n)} & \xleftarrow{g_{(n,a_n)(n+1,a_{n+1})}} & X_{(n+1,a_{n+1})} \end{array} \qquad (10)$$

and

$$\begin{array}{ccc} X_n & \xleftarrow{p_{nn+1}} & X_{n+1} \\ \downarrow f_{(n,b_n)} & & \downarrow f_{(n+1,b_{n+1})} \\ X_{(n,b_n)} & \xleftarrow{g_{(n,b_n)(n+1,b_{n+1})}} & X_{(n+1,b_{n+1})}. \end{array} \qquad (11)$$

Step 8. From Step 7. it follows that for $d_1, d_2 \in D$ with $d_2 \geq d_1$ there exists a mapping $F_{d_1d_2}: X_{d_2} \to X_{d_1}$ (see [2, p. 138]) such that $F_{d_1} = F_{d_1d_2}F_{d_2}$.
Step 9. Let $d_1, d_2, d_3 \in D$ and let $d_1 \leq d_2 \leq d_3$. Then $F_{d_1d_3} = F_{d_1d_2}F_{d_2d_3}$. This follows from Step 7. and the commutativity condition in each inverse system $\mathbf{X(n)} = \{X_{(n,a_n)}, f_{(n,a_n)(n,b_n)}, A_n\}$ (see (2) of Step 1.).
Step 10. *The collection $\{X_d, F_{de}, D\}$ is an inverse system of metric hereditarily locally connected continua with monotone bonding mappings whose inverse limit X_D is a hereditarily locally connected continuum. Moreover, the projections $F_d: X_D \to X_d$, $d \in D$, are monotone and surjective.*

Apply Steps 1. - 9. and Theorem 2.1.

Step 11. *There is a homeomorphism* $F: X \to X_D$.

By Step 6. and Step 8. for each $d \in D$ there is a mapping $F_d: X \to X_d$ such that $F_{d_1} = F_{d_1d_2}F_{d_2}$ for $d_2 \geq d_1$. This means that there exists a mapping $F: X \to X_D$ [2, p. 138]. Let us prove that F is 1 - 1. Take a pair x , y of distinct points of X. There exists an $n \in \mathbb{N}$ such that $x_n = p_n(x)$ and $y_n = p_n(y)$ are distinct points of

X_n. Now, there exists an (n , a_n) such that $f_{(n,a_n)}(x_n)$ and $f_{(n,a_n)}(y_n)$ are distinct points of

$X_{(n,a_n)}$. From Step 4. it follows that there is a d∈D such that d = {(1 , a_1) , (2 , a_2) , ... (n , a_n), ...}. We infer that $F_d(x)$ and $F_d(y)$ are distinct points of X_d. Thus, F is 1 - 1. It remains to prove that the mapping F is a surjection. Let x be a point of X_D. For each d∈D, d = {(1 , a_1) , (2 , a_2) , ... }, we have a point $F_d(x)$. Now, we have the points $g_{(n,a_n)}F_d(x)$ in $X_{(n,a_n)}$ and the subsets $Y_n = f^{-1}_{(n,a_n)}(g_{(n,a_n)}F_d(x))$ of X_n. From the commutativity of the diagrams (6) it follows that $p_{mn}(Y_n) \subseteq Y_m$, $m \leq n$. This means that $\mathbf{Y} = \{Y_n, p_{nm}|Y_m, I\!N\}$ is an inverse sequence of compact spaces. Thus, $Y_d = \lim \mathbf{Y}$ is a non-empty subset of X. It is readily seen that the collection $\{Y_d\colon d \in D\}$ has the finite intersection property. Thus, $Y = \bigcap\{Y_d\colon d \in D\}$ is non - empty. For each y∈Y we have $F_d(y) = F_d(x)$, d∈D. Thus, F(y) = x.The proof of this Step is complete.

Step 12. *X is hereditarily locally connected.* From Step 10. it follows that F : X→X_D is a homeomorphism onto X_D. Thus, X is hereditarily locally connected since X_D is hereditarily locally connected (Step 9.). The proof of the Theorem is complete. ■

We close this section with the following theorem.

THEOREM 3.9 *The following two statements are equivalent:*
(i) The limit of an inverse system of hereditarily locally connected continua with surjective monotone bonding mappings is hereditarily locally connected.
(ii) The limit of an inverse sequence of hereditarily locally connected metric continua with surjective monotone bonding mappings is hereditarily locally connected.

Proof. Consider the inverse system $X_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$. By virtue of Theorem 2.1 limX is hereditarily locally connected if and only if each X_Δ is a hereditarily locally connected continuum. Theorem 3.8 completes the proof.■

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Ivan Lončar

BILJEŠKA O NASLJEDNO LOKALNO POVEZANIM KONTINUUMIMA

Sažetak

U djelu [10] detaljno su izučeni metrički nasljedno lokalno povezani kontinuumi. U radu [9] pokazano je da neke od karakterizacija iz djela [10] vrijede i u nemetričkom slučaju. U ovom radu pokazujemo da je nasljedna lokalna povezanost ciklički proširljiva i reducibilna i u nemetričkom slučaju. Analogna je situacija i u posebnom slučaju takozvanih σ - koherentnih kontinuuma. Na kraju rada izučavani su inverzni sistemi nasljedno lokalno povezanih kontinuuma i dan je djelomičan odgovor na jedno pitanje [7, Problem 9.10.].

Ključne riječi : Inverzni sistem, nasljedno lokalno povezan, σ - koherentan, slabo konfluentno preslikavanje.