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The n-dimensional Simply Isotropic Space

Let $(I_n^{(1)}, M)$ be n-dimensional simply isotropic space as it is defined in [3]. The absolute of this Cayley-Klein's geometry consists of the hyperplane ω and the hypersphere Γ of zero radius and $\Gamma \subset \omega$. We shall first determine the group of general isotropic similarities which preserve the absolute.

Then we define the isotropic distance d between two points, the isotropic angle φ between two nonisotropic hyperplanes and the isotropic angle $\tilde{\varphi}$ between two isotropic hyperplanes. In case $d=0$ we shall define the range between points. In case $\varphi=0$ ($\tilde{\varphi}=0$) isotropic distance a (isotropic distance φ^*) between nonisotropic (isotropic) hyperplanes is defined.

All these notions are invariants of the group of isotropic motions which is subgroup of the group of general isotropic similarities.

We shall also define distance between a point and a nonisotropic hyperplane.

All these invariants for the three dimensional case are given in [2].

Key words: hyperplane, invariant, isotropic motion, simply isotropic space, transformation

Let $P_n(\mathbf{R})$ be n-dimensional real projective space and ω a hyperplane, $\omega \in P(\mathbf{R})$. Then $A_n = P_n \backslash \omega$ is affine space.

Let Γ denotes the hypersphere of radius zero in ω. $\{\omega, \Gamma\}$ is called the absolute of one Cayley-Kleine's geometry. We are looking for those projective automorphismes of P_n, which preserve the absolute. These transformations form a group, which is called the group of general isotropic similarities.

The projective coordinates in P_n are denoted as $(\xi_0:\xi_1:...:\xi_n)$. We can assume that the equation of ω is given by

$\xi_0=0$,

and let $x_1 = \frac{\xi_1}{\xi_0}, \ldots, x_n = \frac{\xi_n}{\xi_0}$ be the affine coordinates.

Now the normal form of $\Gamma \subset \omega$ is

$$\xi_0 = \xi_1^2 + \ldots + \xi_{n-1}^2 = 0. \qquad (1)$$

The vertex of Γ is F(0:0:...:1).

The projective transformations of P_n, written in projective coordinates, are given by

$$\bar{\xi}_j = \sum_{k=0}^{n} a_{ik}\xi_k,\ j=0,\dots,n\ ;\ \det(a_{jk})\neq 0. \qquad (2)$$

Then the point F will be a fixed point of the transformation (2), if

$$a_{01}=\dots=a_{0n}=0 \quad \text{and} \quad a_{1n}=a_{2n}=\dots=a_{n-1,n}=0.$$

From now on, we will use the following notations:

$$\bar{X}=(\bar{\xi}_1,\dots,\bar{\xi}_{n-1})^T,\ X=(\xi_1,\dots,\xi_{n-1})^T,$$

$$A=(a_{ik}) \quad (i,k=1,\dots,n-1),\ a=(a_{10},\dots,a_{n-1,0})^T.$$

According to the notations above, (2) can be written as

$$\begin{cases} \bar{\xi}_0 = a_{00}\xi_0 \\ \bar{X} = a\xi_0 + AX \\ \bar{\xi}_n = a_{n0}\xi_0+\dots+a_{nn}\xi_n\ . \end{cases} \qquad (3)$$

The condition that the quadratic form $\xi_1^2+\dots+\xi_{n-1}^2$ has to be invariant and from $\bar{X}=AX$, we obtain:

$$\bar{X}^T\bar{X}=(AX)^T AX$$

and

$$\lambda X^T X = X^T A^T AX.$$

By the equation above it follows that $A^TA=\lambda E$, which means T is an orthogonal matrix.

By letting $a_{00}=1$ (normalization), $\bar{x}=(\bar{x}_1,\dots,\bar{x}_{n-1})^T, x=(x_1,\dots,x_{n-1})^T$, (3) becomes

$$\begin{cases} \bar{x} = a+\lambda Tx \\ \bar{x}_n = a_{n0}+a_{n1}x_{n1}+\dots+\ a_{nn}x_{nn}. \end{cases} \qquad (4)$$

The transformations (4) are called the group of general isotropic similarities. The orthogonal matrix T (order n-1) depends on $\frac{1}{2}(n-1)(n-2)$ parameters, there are (n+1) parameters in equation for $\bar{x}_n$, and (n-1) parameters in a and there is λ too. Altogether there are $(n^2+n+4)/2$ parameters in (4).

A straight line passing through the point F is called isotropic and the hyperplane which containes the point F is called the isotropic hyperplane. We say, that two points are parallel if there is an isotropic straight line which passes through them.

Because the hyperplane $x_n=0$ (basic hyperplane) is a Euclidean hyperplane, so we can take a Euclidean orthogonal coordinate system $\{x_1, ..., x_{n-1}\}$ in $x_n=0$. Then we add an axis x_n in the isotropic direction to that system. In this way we obtain an orthogonal coordinate system $\{x_1, ..., x_{n-1}, x_n\}$, which is an isotropically orthogonal system if we define that the isotropic direction is orthogonal to all nonisotropic directions.

Let $A(a_1, ..., a_n)$, $B(b_1, ...,b_n)$ be two nonparallel points in the isotropic space $I_n^{(1)}$. By writing $\mathbf{A}=(a_1,...,a_{n-1})^T$, $\mathbf{B}=(b_1,...,b_{n-1})^T$ (it means $\mathbf{A}\neq\mathbf{B}$) define the isotropic distance d between the points A and B by the following formula.

$$d^2(A,B) := (b_1-a_1)^2+...+(b_{n-1}-a_{n-1})^2. \tag{5}$$

Clearly we can write $d^2(A,B)=(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A})$.

Now we prove that the group of general isotropic similarities preserves d. Let A,B pass through (4) to $\overline{A}, \overline{B}$. Then we have

$$\begin{aligned} d^2(\overline{A},\overline{B}) &= (\overline{\mathbf{B}}-\overline{\mathbf{A}})^T(\overline{\mathbf{B}}-\overline{\mathbf{A}}) \\ &=(a+\lambda T\mathbf{B}-(a+\lambda T\mathbf{A}))^T(a+\lambda T\mathbf{B}-(a+\lambda T\mathbf{A})) \\ &=(\mathbf{B}-\mathbf{A})\,T\lambda^2T^T(\mathbf{B}-\mathbf{A})^T \\ &=\lambda^2(\mathbf{B}-\mathbf{A})(\mathbf{B}-\mathbf{A})^T \end{aligned}$$

This implies that d is relative invariant of (4) and will be absolute invariant if $\lambda^2=1$. We obtain two transformations:

$$\begin{cases} \overline{x}=a+Tx \\ \overline{x}_n=a_{n0}+a_{n1}x_1+...+a_{nn}x_n \end{cases} \tag{6}$$

and

$$\begin{cases} \overline{x}=a-Tx \\ \overline{x}_n=a_{n0}+a_{n1}x_1+...+a_{nn}x_n \end{cases} \tag{7}$$

The transformations (7) do not form a group. But the transformations (6) form a group which preserves the distance d.

Geometrically, d is a Euclidean distance between the projections of the given points to the hyperplane x_n=const (actually $x_n=0$-basic hyperplane).

Clearly, $d=0 \Leftrightarrow \mathbf{A}=\mathbf{B}$. In case $d=0$ we define the so called range between parallel points A and B as

$$r(A,B):=b_n-a_n. \tag{8}$$

Now we look when r will be invariant under (6). Let A, B, $\overline{A}$ and $\overline{B}$ be as above. We have

$$r(\overline{A},\overline{B}) = \overline{b}_n - \overline{a}_n$$
$$= a_{nn} r(A,B).$$

As we can see, r is relative invariant of (6). If we put $a_{nn}=1$, r will become absolute invariant of (6). The transformations

$$\begin{cases} \overline{x} = a + Tx \\ \overline{x}_n = a_{n0} + a_{n1}x_1 + \ldots + a_{n,n-1}x_{n-1} + x_n \end{cases} \tag{9}$$

form a group, which is called the group of isotropic motions in $I_n^{(1)}$.

If we adopt the notation $b=(a_{n1}, \ldots, a_{n,n-1})^T$ (9) could be dispayed as

$$\begin{cases} \overline{x} = a + Tx \\ \overline{x}_n = a_{n0} + b^T x + x_n. \end{cases} \tag{M}$$

It can be seen that the first n-1 coordinates are transformed as coordinates in the (n-1)-dimensional Euclidean space.

Now, $\left(I_n^{(1)}, M\right)$ could be called n-dimensional simply isotropic space, in which we write $I_n^{(1)}$ because the absolute of I_n has one singularity.

The equation of the nonisotropic hyperplane has the following form

$$x_n = u_1x_1 + \ldots + u_{n-1}x_{n-1} + u_0 \tag{10}$$

and the isotropic hyperplane has the following equation

$$u_1x_1 + \ldots + u_{n-1}x_{n-1} + u_0 = 0. \tag{11}$$

In addition, we put $u=(u_1,\ldots,u_{n-1})^T$ and the equation (10) becomes

$$x_n = u^T x + u_0. \tag{12}$$

and (11) becomes

$$u^T x + u_0 = 0.$$

Let us see how the u and u_0 coordinates are transformed when (12) transforms under (M) into $\overline{x}_n = \overline{u}^T \overline{x} + \overline{u}_0$. Then

$$x = T^T(\overline{x} - a)$$
$$x_n = \overline{x}_n - a_{n0} - b^T T^T \overline{x} + b^T T^T a$$

follows from (M).

By substituting this back into (12) we obtain

$$\overline{x}_n = \left(u^T + b^T\right)T^T\overline{x} - \left(u^T + b^T\right)T^T a + a_{n0} + u_0.$$

Thus

$$\begin{cases} \overline{u} = T(u+b) \\ \overline{u}_0 = a_{n0} + u_0 - \left(u^T + b^T\right)T^T a. \end{cases} \tag{13}$$

The coordinate transformations (13) are called the coordinate transformations of the nonisotropic hyperplane.

We assume now that an isotropic hyperplane $u^T x + u_0 = 0$ is transformed, under (M), to $\overline{u}^T T^T(\overline{x}-a)+u_0=0$. As in the previous case the following relation could be shown

$$u^T T^T(\overline{x}-a)+u_0=0$$

and also

$$\begin{cases} \overline{u} = Tu \\ \overline{u}_0 = u_0 - u^T T^T a. \end{cases} \tag{14}$$

Let H_1 ... $x_n = u^T x + u_0$ and H_2 ... $x_n = v^T x + v_0$ be two nonisotropic hyperplanes. The isotropic angle φ between H_1 and H_2 is defined as

$$\varphi^2 := (u_1 - v_1)^2 + \ldots + (u_{n-1} - v_{n-1})^2 . \tag{15}$$

Actually, $\varphi^2 = (u-v)^T(u-v)$. It is easy to show that φ is an invariant of (13). So, φ is invariant of (M).

In case $\varphi=0$ (i.e. $u=v$), we say that the nonisotropic hyperplanes H_1 and H_2 are parallel. For such hyperplanes we can compute the isotropic distance between two isotropic hyperplanes by the formula

$$a(H_1, H_2) = v_0 - u_0 . \tag{16}$$

We can see that a is invariant of (14).

Let us consider isotropic hyperplane invariants. Let H_1 ... $u_1 x_1 + \ldots + u_{n-1} x_{n-1} + u_0 = 0$ and H_2 ... $v_1 x_1 + \ldots + v_{n-1} x_{n-1} + v_0 = 0$ be two isotropic hyperplanes. The angle $\tilde{\varphi}$ between H_1 and H_2 is defined as

$$\cos\tilde{\varphi}(H_1, H_2) = \frac{u_1 v_1 + \ldots + u_{n-1} v_{n-1}}{+\sqrt{\left(u_1^2 + \ldots + u_{n-1}^2\right)\left(v_1^2 + \ldots + v_{n-1}^2\right)}}, \tag{17}$$

consequently,

$$\cos\tilde{\varphi}(H_1, H_2) = \frac{u^T v}{|u|\,|v|}.$$

In addition, we will prove that $\tilde{\varphi}$ is invariant of (M). Actually, we have

$$\cos\tilde{\varphi}(\overline{H}_1, \overline{H}_2) = \frac{\overline{u}^T \overline{v}}{|\overline{u}|\,|\overline{v}|} = \frac{(Tu)^T Tv}{\sqrt{(Tu)^T Tu \cdot (Tv)^T Tv}} = \frac{u^T v}{|u|\,|v|} = \cos\tilde{\varphi}(H_1, H_2).$$

Obviously, when $\tilde{\varphi}=0$ (this means $\cos\tilde{\varphi}=1$) $u=\lambda v$ follows. In that case, we say that the angle between isotropic hyperplanes is zero and hyperplanes are identical or parallel , so we can define their distance as

$$\varphi^*(H_1,H_2)=\frac{v_0-u_0}{|u|} \tag{18}$$

We also assumed that for the definition above, the hyperplanes had the following equations: $H_1 \dots u_1x_1+\dots+u_{n-1}x_{n-1}+u_0=0$, $H_2 \dots u_1x_1+\dots+u_{n-1}x_{n-1}+v_0=0$. It is easy to see that φ^* is an invariant of (14).

Each isotropic straight line is orthogonal to each nonisotropic hyperplane. Moreover, a straight line p is orthogonal to the isotropic hyperplane H if its projection $\tilde{p}$ to the basic hyperplane $x_n=0$ is orthogonal to H in the Euclidean metric. Two nonisotropic straight lines are orthogonal if their projections on $x_n=0$ are orthogonal in the Euclidean metric. Each isotropic straight line is orthogonal to any nonisotropic straight line. Two isotropic hyperplanes are orthogonal if their projections on $x_n=0$ are orthogonal in Euclidean metric.

Each motion mentioned above is invariant of (M). A hyperplane orthogonal to some nonisotropic hyperplane is necessarily isotropic .

In the hyperplane ω the transformations (M) induce dual Euclidean geometry; in isotropic hyperplanes they induce isotropic geometry and in nonisotropic hyperplanes Euclidean geometry.

The distance between a point $A(a_1,a_2,\dots a_n)$ and a nonisotropic hyperplane $H\dots x_n=u^Tx+u_0=0$ is defined naturally as follows. First, we find the isotropic straight line *i* through the point A (it is necessarily orthogonal to H), and let N be the point of intersection of *i* and H. Then we set

$$d(A,H)=r(A,N).$$

In the case when the isotropic straight line *i* has the following parametrical equations $x_1=a_1,\dots,x_{n-1}=a_{n-1}$ ($a^0=(a_1,\dots,a_{n-1})^T$), N has the coordinates $N=(a_1,\dots,a_{n-1},u^Ta^0+u_0)$ and clearly

$$d(A,M)=u^Ta^0+u_0-a_n=u_1x_1+\dots+u_{n-1}x_{n-1}+u_0-a_n \tag{19}$$

follows.

The distance between a point $A(a_1,a_2,\dots a_n)$ and a isotropic hyperplane $H\dots u^Tx+u_0=0$ is the Euclidean distance between their projections $\tilde{H}$ and $\tilde{A}$ on the hyperplane $x_n=0$. So we have

$$d(A,H)=\frac{|u^Ta^0+u_0|}{|u|}. \tag{20}$$

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Divjak B. Jednostruko izotropni n-dimenzionalni prostor

Sažetak

Neka je $P_n(\mathbf{R})$ n-dimenzionalni realni projektivni prostor, a ω hiperravnina u tom prostoru. Sada je $A_n=P_n\backslash\omega$ afini prostor. Uzmimo da je Γ hipersfera radijusa nula u ω. $\{\omega,\Gamma\}$ zovemo apsolutom jedne Cayley-Kleinove geometrije.

Tražimo one projektivne automorfizme od P_n koji čuvaju apsolutu. Te transformacije čine grupu koju zovemo općom izotropnom grupom sličnosti.

Stavimo li da su $x_1, \ldots, x_n$ afine koordinate u A_n, zatim da je $x=(x_1, \ldots, x_{n-1})^T$, opća grupa sličnosti ima oblik

$$\begin{cases} \overline{x} = a + \lambda T x \\ \overline{x}_n = a_{n0} + a_{n1}x_{n1} + \ldots + a_{nn}x_{nn}. \end{cases} \tag{4}$$

gdje je sa T dana ortogonalna matrica reda (n-1).

Za neparalelne točke A i B definiramo njihovu izotropnu udaljenost d kao euklidsku udaljenost njihovih projekcija u osnovnoj hiperravnini $x_n=0$. Iako su točke A i B različite njihova udaljenost d može biti nula. U tom slučaju definiramo i raspon r zadanih točaka kao razliku njihovih n-tih koordinata.

Podgrupa grupe (4) za koju su d i r apsolutne invarijante je grupa gibanja (M).

U jednostruko izortopnom n-dimenzionalnom prostoru razlikujemo dvije vrste hiperravnina: neizotropne i izotropne.

Analogno izotropnoj udaljenosti točaka za neizotropne hiperravnine definiramo njihov kut φ i u slučaju kada je $\varphi=0$ izotropnu udaljenost hiperravnina a. Za izotropne hiperravnine te su veličine označene sa $\tilde{\varphi}$ i φ^*.

Veličine φ, a, $\tilde{\varphi}$ i φ^* su apsolutne invarijante od (M).

Ne postoji invarijanta od (M) koja uključuje neizotropne i izotropne hiperravnine, ali je još moguće definirati izotropnu udaljenost točke od neizotropne hiperravnine na prirodni način.