Ivan Lončar
Faculty of Organization and Informatics
Varaždin
E-mail : Ivan.Loncar@foi.hr

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Inverse Limit of Continuous Images of Arcs

Abstract. *The main purpose of this paper is to study the inverse limits of continuous image of arcs. We shall prove:*

a) *If* X = $\{X_a, p_{ab}, A\}$is a monotone well - ordered inverse system of continuous image of arcs such that cf(A)$\neq \omega_1$, then X = limX is the continuous image of an arc (Theorem 2.17).

b) Let X = $\{X_a, p_{ab}, (A,\leq)\}$ be an inverse system of continuous image of arcs with monotone surjective bonding mappings. Then X = limX is the continuous image of an arc if and only if for each cyclic element Z of X and the points x, y, z∈Z there exists a countable directed subset (B, ≤) of (A, ≤) such that for each countable directed subset (C, ≤) of (A, ≤) with C⊇B the restriction $h_{BC} = p_{BC}|lim\{W_d(x,y,z), p_{dd_1}, D\}$ of the canonical projection p_{BC} is a homeomorphism

$$h_{BC} : lim\{W_d(x,y,z), p_{dd_1}, D\} \rightarrow lim\{W_c(x,y,z), p_{cc_1}, C\}$$

(Theorem 2.22).

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1 Preliminaries

The cardinality of a set X will be denoted by card(X). The cofinality of a cardinal number m will be denoted by cf(m). Cov(X) is the set of all normal coverings of a topological space X.For other details see [1]. A basis of (open) normal coverings of a space X is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U}\in$Cov(X) admits a refinement $\mathcal{V}\in \mathcal{C}$. We denote by cw(X) (*covering weight*) the minimal cardinal of a basis of normal coverings of X [9, p. 181].

LEMMA 1.1 *[9, Example 2.2]. If X is a compact Hausdorff space, then* $cw(X) = w(X)$.

In the sequel we shall use the following theorem [16, Theorem 1.1].

THEOREM 1.2 *Let X be a regular space. For each cardinal number* $\lambda \leq w(X)$ *there exists a subspace* $M_\lambda \subseteq X$ *such that* $card(M_\lambda) \leq \lambda$ *and* $w(M_\lambda) \geq \lambda$.

LEMMA 1.3 *Let X be a regular space and let* $\mathcal{F} = \{F_\alpha : \alpha < \omega_{\mu+2}\}$, *where* μ *is a fixed ordinal number, be an increasing transfinite sequence of subspaces of X with* $w(F_\alpha) = \aleph_0$ *and let* $Y = \bigcup\{F_\alpha : \alpha < \omega_{\mu+2}\}$. *Then* $w(Y) = \aleph_0$. *Moreover, if each* F_α *is closed, then there exists an* α_0 *such that* $F_{\alpha_0} = F_\beta$, *for each* $\beta \geq \alpha_0$, *and* $Y = F_{\alpha_0}$ *is closed.*

Proof. Suppose that $w(Y) \geq \aleph_1$. By virtue of Theorem 1.2, for $\lambda = \aleph_1$, there exists a subspace M_λ of Y such that $card(M_\lambda) \leq \aleph_1$ and $w(M_\lambda) \geq \aleph_1$. It is clear that

$$M_\lambda = (M_\lambda \bigcap F_0) \bigcup [\bigcup\{M_\lambda \bigcap (F_{\alpha+1} \setminus F_\alpha) : 1 \leq \alpha < \omega_{\mu+2}\}]. \qquad (1)$$

If each $Z_\alpha = M_\lambda \bigcap (F_{\alpha+1} \setminus F_\alpha)$ is non empty, then we have $card(M_\lambda) = \aleph_{\mu+2}$. This contradicts $card(M_\lambda) \leq \aleph_1$. We infer that there exists a $\beta < \omega_{\mu+2}$ such that $Z_\gamma = \emptyset$ for each $\gamma \geq \beta$. Hence, $M_\lambda \subseteq F_\beta$. Thus, $w(M_\lambda) \leq w(F_\beta) = \aleph_0$. This contradicts $w(M_\lambda) \geq \aleph_1$. Now, by virtue of [4, Problem 2.7.9 (e), p. 155] it follows that there exists an α_0 such that $F_{\alpha_0} = F_\beta$ for each $\beta \geq \alpha_0$. It is clear that $Y = F_{\alpha_0}$. Thus, Y is closed.■

THEOREM 1.4 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a well-ordered inverse system such that* $w(X_a) < \tau$, $a \in A$. *If* p_{ab} *are perfect (*p_{ab} *are open or* $\mathbf{X}$ *is continuous), then* $w(\lim \mathbf{X}) \leq \tau$.

Proof. See [16, Teorema 2.2].■

LEMMA 1.5 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a well-ordered inverse system of compact spaces such that* $w(X_a) < \tau$ *and* $card(cf(A)) > \tau$. *Then there exists an* $a \in A$ *such that the projection* $p_c : \lim \mathbf{X} \to X_c$ *is a homeomorphism for every* $c \geq a$.

Proof. By virtue of Theorem 1.4 $w(\lim \mathbf{X}) \leq \tau$. This means that there exists a family $\mathcal{U} = \{\mathcal{U}_\alpha\}$ of normal coverings of $X = \lim \mathbf{X}$ such that $\mathcal{U}$ is a basis of normal coverings of X and $card(\mathcal{U}) \leq \tau$. For each $\mathcal{U}_\alpha$ there exists an a^* such that for each b$\geq$a* there exists a normal cover $\mathcal{V}$ of X_b such that $p_b^{-1}(\mathcal{V})$ refines $\mathcal{U}_\alpha$. The cardinality of the set {a*} is $\leq \tau$. Since card(cf(A))$> \tau$, there exists an a$\in$A such that a$\geq$a* for each a*. It follows that for each $\mathcal{U}_\alpha$ and each c$\geq$a there exists a normal cover $\mathcal{V}$ of X_c such that $p_c^{-1}(\mathcal{V})$ refines $\mathcal{U}_\alpha$. Let us prove that p_c is a homeomorphism. It suffices to prove that p_c is 1 - 1 since p_c is onto and X is compact. Let x,y be a pair of distinct points of X. There is a pair of disjoint open subsets U,V of X such that x$\in$U and y$\in$V. Consider the normal cover $\mathcal{W} = \{U, V, X\backslash\{x,y\}\}$. Let $\mathcal{U}_\alpha$ be a member of $\mathcal{U}$ which refines $\mathcal{W}$. There exists a normal cover $\mathcal{V}$ of X_c such that $p_c^{-1}(\mathcal{V})$ refines $\mathcal{U}_\alpha$. Suppose that $p_c(x) = p_c(y)$. There is a member W of $\mathcal{V}$ such that $p_c(x)\in W$. It follows that x, y$\in p_c^{-1}(W)$. We infer that (x, y)$\prec \mathcal{U}_\alpha$. This is impossible since x$\in$U, y$\in$V and $\mathcal{U}_\alpha$ refines $\mathcal{W}$. Hence, $p_c(x)\neq p_c(y)$ for each pair x, y$\in$X. Thus, p_c is 1 - 1. ■

Let $X = \{X_a, p_{ab}, A\}$be an inverse system. For each subset Δ_0 of (A, $\leq$) we define sets Δ_n, n = 0, 1, ..., by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y): x,y\in \Delta_n\}$, where m(x,y) is a member of A such that x,y $\leq$m(x,y). Let $\Delta = \bigcup\{\Delta_n: n\in \mathbb{N}\}$. It is clear that card($\Delta$) = card($\Delta_0$). Moreover, Δ is directed by $\leq$ [11, Lemma 9.2]. For each directed set (A,$\leq$) we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, card(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Then A_σ is σ - directed by inclusion [11, Lemma 9.3]. If $\Delta \in A_\sigma$, let $X^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim X^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma}: X_\Gamma \to X_\Delta$ denote the mapp induced by the projections $p_\delta^\Gamma: X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system X^Γ. Now, we have [11, Theorem 9.4].

THEOREM 1.6 *If* $X = \{X_a, p_{ab}, A\}$*is an inverse system, then* $X_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ *is a* σ *- directed inverse system and* limX *and* limX_σ *are canonically homeomorphic.*

THEOREM 1.7 *Let* $X = \{X_a, p_{ab}, A\}$*be a* σ *- directed inverse system of compact metrizable spaces and surjective bonding mappings. Then* X = limX *is metrizable if and only if there exists an* a$\in$A *such that* $p_b:X\to X_b$ *is a homeomorphism for each* b$\geq$a.

Proof. If there exists an a$\in$A such that p_b is a homeomorphism for each b$\geq$a, then X is metrizable. Conversely, if X is metrizable, then cw(X) = $\aleph_0$ (see Lemma 1.1). Let $\mathcal{B} = \{\mathcal{U}_n: n\in \mathbb{N}\}$ be a basis of normal coverings of X. For each $\mathcal{U}_n$ there exists an $a(n)\in A$ such that for each $b\geq a(n)$ there exists a normal cover $\mathcal{V}$ of X_b such that $p_b^{-1}(\mathcal{V})$ refines $\mathcal{U}_n$. Since A is σ - directed there exists an $a\in A$ such that $a\geq a(n)$ for each $n\in\mathbb{N}$. It follows that for each $\mathcal{U}_n$ and each $b\geq a$ there exists a normal cover $\mathcal{V}$ of X_b such that $p_b^{-1}(\mathcal{V})$ refines $\mathcal{U}_n$. Let us prove that p_b is a homeomorphism. It suffices to prove that p_b is 1 - 1 since p_b is onto and X is compact. Let x,y be a pair of distinct points of X. There is a pair of disjoint open subsets U,V of X such that $x\in U$ and $y\in V$. Consider the normal cover $\mathcal{U} = \{U, V, X\backslash\{x,y\}\}$. Let $\mathcal{U}_n$ be a member of $\mathcal{B}$ which refines $\mathcal{U}$. There exists a normal cover $\mathcal{V}$ of X_b such that $p_b^{-1}(\mathcal{V})$ refines $\mathcal{U}_n$. Suppose that $p_b(x) = p_b(y)$. There is a member W of $\mathcal{V}$ such that $p_b(x)\in W$. It folows that x, $y\in p_b^{-1}(W)$. It follows that $(x, y)\prec \mathcal{U}_n$. This is impossible since $x\in U$, $y\in V$ and $\mathcal{U}_n$ refines $\mathcal{U}$. Hence, $p_b(x)\neq p_b(y)$ for each pair x, $y\in X$. Thus, p_b is 1 - 1. ■

THEOREM 1.8 *Let* $X = \{X_a, p_{ab}, A\}$ *be an inverse system of compact metrizable spaces* X_a *and surjective bonding mappings. Then* $X = \lim X$ *is metrizable if and only if there exists a countable subset B of A which is directed by* $\leq$ *and such that the natural projection* $p:X\to\lim\{X_b, p_{bc}, B\}$ *is a homeomorphism.*

Proof. If there exists such subset B of A, then X is metrizable. Conversely, if X is metrizable then we may assume that X is homeomorphic with $\lim X_\sigma$ from Theorem 1.6. Applying Theorem 1.7 we complete the proof.■

The following theorem is Theorem 5.1 of [11].

THEOREM 1.9 *Let* $X=\{X_n, p_{mn}, \mathbb{N}\}$ *be an inverse sequence with monotone surjective bonding mappings. If each* X_n *is the continuous image of an arc, then* $X = \lim X$ *is the continuous image of an arc.*

From Theorems 1.6 and 1.9 it follows the following theorem.

THEOREM 1.10 *Let* $X = \{X_a, p_{ab}, A\}$ *be an inverse system of continuous images of arcs with monotone surjective bonding mappings. Then* $X_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ *is an inverse system of continuous images of arcs.*

2 Inverse systems of cyclic images of arcs

An *arc* (or ordered continuum) is a Hausdorff continuum with exactly two non - separating points. Each separable arc is homeomorphic to the closed interval I = [0,1].

A continuum X is a *dendron* if each pair of distinct points of X can be separated by a third point of X. A continuum X is a dendron if it is locally connected and hereditarily unicoherent. A dendron is an arc if it is atriodic.

Let X be a non - degenerate locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of a locally connected continuum is again a locally connected continuum. We let

$$\mathrm{L}_X = \{Y \subset X : \text{Y is a non- degenerate cyclic element of X}\}.$$

LEMMA 2.1 *A continuum X is a dendron iff it is locally connected and has no non - degenerate cyclic elements.*

LEMMA 2.2 *If C is a connected subset of X and* $Y \in \mathrm{L}_X$, *Then* $C \bigcap Y$ *is connected (possibly void).*

LEMMA 2.3 *If* $f : X \to X'$ *is a monotone surjection, then for each* $Y' \in \mathrm{L}_{X'}$ *there exists* $Y \in \mathrm{L}_X$ *such that* $Y' \subseteq f(Y)$. *In particular,* L_X *is non - empty if* $\mathrm{L}_{X'}$ *is non - empty.*

LEMMA 2.4 *Let Z be a cyclic element of a locally connected continuum X. If J is a component of* $X \backslash Z$, *then* $|Bd(J)| = 1$.

Proof. See [11, p. 5.].■

LEMMA 2.5 *If Z and W are cyclic elements of a locally connected continuum X, then either* $card(Z \bigcap W) \leq 1$ *or* $Z = W$.

Proof. See [6, p. 316 , Teorema 4.].■

Let Z be a cyclic element of X. For each component J of X\Z, let Bd(J) = z_J. We define [11, p. 5] $\rho : X \to Z$ such that $\rho(x) = x$ if $x \in Z$ and $\rho(x) = z_J$ if $x \in J$. Then ρ is a monotone continuous retraction. It is called the *canonical retraction* of X onto Z.

We shall say that X is *cyclic* if it is the only cyclic element of itself, equivalently, if it has no separating point.

Let $X = \{X_a, p_{ab}, A\}$ be an inverse system and $Y \subseteq X = \lim X$. We shall denote $p_a(Y)$ by Y_a, $a \in A$.

LEMMA 2.6 *Let* X $= \{X_a,\ p_{ab},\ A\}$*be an inverse system of locally connected continua* X_a *with monotone bonding mappings and let Z be a non - degenerate cyclic element of* $X = lim$X. *There exists an* a_0 *such that* $L_b(Z_b) \neq \emptyset$*, for each* $b \geq a_0$.

Proof. Suppose that $L_{Z_b} = \emptyset$ for each b∈B (i.e., all Z_b are dendrons), where B is cofinal in A. By virtue of [10, Theorem 3] Z is hereditarily unicoherent. This means that Z is a dendron since it is locally connected. Thus, $L_Z = \emptyset$. This contradicts the assumption that Z∈L_X.■

In the sequel we shall use the following theorem [11, Theorem 2.7.].

THEOREM 2.7 *Let* X $= \{X_a,\ p_{ab},\ A\}$*be an inverse system such that* p_{ab} *are monotone surjection and Y is a cyclic element of the locally connected continuum* $X = lim$X. *For each* $a \in A$*, let* Z_a *be either a cyclic element of* X_a *or a one - point subset of* X_a*. Let* $\rho_a : X_a \rightarrow Z_a$ *denote the canonical retraction if* Z_a *is non - degenerate, and otherwise, let* $\rho_a : X_a \rightarrow Z_a$ *be the constant map. Suppose that some* Z_{a_0} *is non - degenerate, and that* $Z_a \subseteq p_{ab}(Z_b) \subseteq p_a(Y)$ *for all* $b \geq a$*. Let* $g_{ab} = \rho_a \circ (p_{ab}|Z_b)$*,* $a \leq b$*. Then each* $g_{ab} : Z_b \rightarrow Z_a$ *is a monotone surjection, the collection* Z $= \{Z_a,\ g_{ab},\ A\}$ *is an inverse system and there exists a continuous mapping* $H: X \rightarrow Z = lim$Z *such that* $H|Y : Y \rightarrow Z$ *is a homeomorphism.*

THEOREM 2.8 *A Hausdorff locally connected continuum S is the continuous image of an arc if and only if each cyclic element of S is the continuous image of an arc*

Proof. See [3, Theorem 1].■

If Y is a closed subset of X, we let K(X\Y) denote the family of all components of X\Y. Let X be a locally connected continuum. A subset Y of X is said to be a *T - set* if it is closed and |Bd(J)| = 2, for each J∈K(X\Y).

The following theorem is a part of Theorem 4.4. [11].

THEOREM 2.9 *If X is a locally connected continuum, then the following conditions are equivalent:*

1. *X is a continuous image of an arc,*

2. *X is a continuous image of an ordered compactum,*

3. *for each* $Y \in L_X$ *and any* $p,q,r \in Y$ *there exists a metrizable* T *- set* Z *in* Y *such that* $p,q,r \in Z$.

4. *For each* $Y \in L_X$ *and each closed metrizable subset* M *of* Y *there exists a metrizable* T *- set* A *in* Y *such that* $M \subseteq A$.

THEOREM 2.10 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ *- directed inverse system of continuous image of arcs such that the cyclic elements of each* X_a *are hereditarily locally connected. If the bonding mappings are monotone surjections, then* $X = \lim \mathbf{X}$ *is the continuous image of an arc.*

Proof. By virtue of [2] the projections p_a are monotone. It follows that X is locally connected. Let Y be a cyclic element of X. By virtue of Theorem 2.7 there exists the inverse system $\mathbf{Z} = \{Z_a, g_{ab}, A\}$ such that Y and $\lim\mathbf{Z}$ are homeomorphic. Moreover, each Z_a is hereditarily locally connected and each g_{ab} is monotone surjection. By virtue of Corollary 3 [5] $\lim\mathbf{Z}$ is hereditarily locally connected. From Theorem 3.4 [14] it follows that $\lim\mathbf{Z}$ is the continuous image of an arc. Theorem 2.8 completes the proof.∎

A mapping $f : X \to Y$ is said to be hereditarily monotone if for each subcontinuum $K \subseteq X$ the restriction $f|K : K \to f(K)$ is monotone [7, p. 16.].

If $f:X \to Y$ and $g:Y \to Z$ are hereditarily monotone mappings, then $gf:X \to Z$ is hereditarily monotone [7, p. 29, (5.3)].

LEMMA 2.11 *If* Z *is a cyclic element of a locally connected continuum* X*, then the canonical retraction* $\rho:X \to Z$ *is hereditarily monotone.*

Proof. Let K be any subcontinuum of X and let $\rho_K = \rho|K$. Then $\rho_K^{-1}(z) = z$ or $\rho_K^{-1}(z) = Cl(J) \cap K$, where J is a component of $X \backslash Z$ with non - empty $J \cap K$. It remains to prove that $Cl(J) \cap K$ is connected. Suppose that $Cl(J) \cap K$ is not connected. Let $z_J = Bd(J)$. There exists a component L of $Cl(J) \cap K$ such that $z_J \notin L$. Moreover, $z_J \in K$ since K is connected and J is closed and open in $X \backslash \{z_J\}$. By virtue of the normality of Cl(J) it follows that there exists a pair U,V of disjoint open (in Cl(J)) sets such that $L \subseteq U$ and $z_J \in V$.It follows that U is open and closed in X since $z_J \notin Cl(U)$. This is impossible since K is connected and $z_J \in K$.∎

THEOREM 2.12 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of hereditarily locally connected continua and hereditarily monotone bonding mappings. Then* $X = \lim\mathbf{X}$ *is hereditarily locally connected and the projections* p_a *are hereditarily monotone.*

Proof. Let Y be any subcontinuum of X. Then $\mathcal{Y} = \{p_a(Y), p_{ab}|Y_b, A\}$ is an inverse system. The bonding mappings $p_{ab}|Y_b$ are monotone. By virtue of Capel's theorem, the mappings $p_a|Y$ are monotone. Thus, the mappings p_a, $a \in A$, are hereditarily monotone. Moreover, Y is locally connected. Thus, X is hereditarily locally connected.■

Now, we have the following theorem.

THEOREM 2.13 *Let* $\mathrm{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of continuous images of arcs such that the cyclic elements of each* X_a *are hereditarily locally connected. If the bonding mappings are hereditarily monotone surjections, then* $X = \lim \mathrm{X}$ *is the continuous image of an arc.*

Proof. Let Y be a cyclic element of X. From Theorem 2.7 it follows that there exists an inverse system $\mathrm{Z} = \{Z_a, g_{ab}, A\}$ such that Z_a, $a \in A$, is a cyclic element of X_a and $Z = \lim \mathrm{Z}$ is homeomorphic to Y. Moreover, $g_{ab} = \rho_a \circ (p_{ab}|Z_b)$ is hereditarily monotone since $p_{ab}|Z_b$ and ρ_a are hereditarily monotone (Lemma 2.11). From Theorem 2.12 it follows that Z is hereditarily locally connected. Thus, Y is hereditarily locally connected. By virtue of Theorem 3.4 [14] Y is the continuous image of an arc. Theorem 2.8 completes the proof.■

From Theorems 2.7 and 2.8 it follows that it suffices to consider inverse systems $\mathrm{X} = \{X_a, p_{ab}, A\}$ of cyclic continuous images of arcs with monotone surjective bonding mappings and with cyclic $X = \lim \mathrm{X}$. Such systems will be called CMC - systems. Let X be the limit of an CMC - system, let x, y and z be distinct points of X and let $a \in A$ such that $x_a = p_a(x)$, $y_a = p_a(y)$, $z_a = p_a(z)$ are distinct points of X_a. By virtue of Theorems 3.2 and 3.6. of [11] there exists a minimal metrizable T - set T_a containing $x_a = p_a(x)$, $y_a = p_a(y)$, $z_a = p_a(z)$.

LEMMA 2.14 *Let* $\mathrm{X} = \{X_a, p_{ab}, A\}$ *be a CMC - system. The family* $\mathcal{T}_a = \{p_{ab}(T_b): b \geq a\}$ *is directed by inclusion. Moreover,* $W_a(x,y,z) = Cl(\bigcup\{T: T \in \mathcal{T}_a\})$ *is a T - set in* X_a.

Proof. For each pair $p_{ab}(T_b)$, $p_{ac}(T_c)$ of elements of $\mathcal{T}_a$ there exists a $d \in A$ such that $p_{bd}(T_d) \supseteq T_b$ since T_b is minimal. This means that $p_{ad}(T_d) \supseteq p_{ab}(T_b)$. Similarly, $p_{ad}(T_d) \supseteq p_{ac}(T_c)$. By virtue of Theorem 3.1 [11], $W_a(x,y,z)$ is a T - set.■

LEMMA 2.15 *Let* $\mathrm{X} = \{X_a, p_{ab}, A\}$ *be a CMC - system. Then* $\mathrm{W}(x,y,z) = \{W_a(x,y,z), p_{ab}| W_b(x,y,z), A\}$ *is an inverse system and* $W(x,y,z) = \lim \mathrm{W}(x,y,z)$ *is a minimal T - set in X containing x,y and z.*

Proof. It is clear that $p_{ab}(W_b(x,y,z)) = W_a(x,y,z)$. By virtue of [11, Theorem 3.13] W(x,y,z) is a T - set. Let us prove that W(x, y, z) is minimal. Suppose, on the contrary, that W(x, y, z) is not minimal. Then there exists a T - set $T \subseteq W(x, y, z)$. There exists an $a \in A$ such that $p_a(T) \subseteq W_a(x, y, z)$. On the other hand, for each $b \geq a$ we have $p_b(T) \supseteq T_b$. Thus, $p_{ab}p_b(T) \supseteq p_{ab}(T_b)$, i.e., $p_a(T) \supseteq p_{ab}(T_b)$. This means that $p_a(T) \supseteq Cl(\bigcup\{p_{ab}(T_b): b \geq a\}) = W_a(x, y, z)$, a contradiction.■

THEOREM 2.16 *Let* $X = \{X_a, p_{ab}, A\}$ *be a CMC - system. Then* $X = limX$ *is the continuous image of an arc iff for any choice of distinct points* $x,y,z \in X$ *the sets* $W_a(x,y,z)$, $a \in A$, *and* $W(x,y,z)$ *are metrizable.*

Proof. If X is a continuous image of an arc, then there exists a metrizable T - set T containing x,y and z [11, Theorem 4.4]. Clearly, $W(x,y,z) \subseteq T$ since W(x,y,z) is minimal. Hence, W(x,y,z) is metrizable. It follows that each $W_a(x,y,z)$ is metrizable since $p_a(W(x,y,z)) = W_a(x,y,z)$ [4, Theorem 4.4.17]. Conversely, if W(x,y,z) is metrizable, then X is a continuous image of an arc [11, Theorem 4.4].■

THEOREM 2.17 *Let* $X = \{X_a, p_{ab}, A\}$ *be a well - ordered inverse system such that* $cf(A) \neq \omega_1$. *If the mappings* p_{ab} *are monotone surjections and if the spaces* X_a *are continuous images of arcs, then* $X = limX$ *is the continuous image of an arc.*

Proof. If $cf(A) = \aleph_0$, then X is the continuous image of an arc (see Theorem 1.9). Suppose that $cf(A) \geq \omega_2$. Let Y be a cyclic element of X and let Z be as in Theorem 2.7. This means that Y and Z = limZ are homeomorphic. Let x, y and z be distinct points of Z. By virtue of Theorem 1.3 each $W(x_a, y_a, z_a)$ is metrizable. Moreover, by virtue of Theorem 1.5 (for $\tau = \aleph_1$) W(x, y, z) is metrizable. Theorems 2.8 and 2.9 complete the proof. ■

REMARK 2.18 Theorem 2.17 is not true if $cf(A) = \aleph_1$. This shows the following example of Nikiel [12]. Let L denote the long interval [4, p. 297]. For each ordinal number α, $0 < \alpha < \omega_1$, let $f_\alpha:[0,1] \times L \rightarrow [0,1] \times [0,\alpha]_L$ be defined by

$$f(s,t) = \begin{cases} (s,t) & \text{if } t \leq_L \alpha, \\ (s,\alpha) & \text{if } \alpha \leq_L t. \end{cases}$$

Each $X_\alpha = [0,1]\times[0,\alpha]_L$ is homeomorphic to $[0,1]\times[0,1]$ and it is a continuous image of an arc. Moreover, $w(X_\alpha) = \aleph_0$. Let $f_{\alpha\beta} = f_\alpha|[0,1]\times[0,\beta]_L$, $\beta < \alpha$. We obtain an inverse system $\{X_\alpha, f_{\alpha\beta}, \alpha < \omega_1\}$ whose limit is $[0,1]\times L$ which is not a continuous image of an arc.■

THEOREM 2.19 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ *- directed CMC - system. Then* $X = \lim\mathbf{X}$ *is the continuous image of an arc if and only if there exists an* $a\in A$ *such that* $p_{ab}| W_b(x, y, z)$: $W_b(x, y, z)\rightarrow W_a(x, y, z)$ *is a homeomorphism for each* $b\geq a$.

Proof. Apply Theorems 1.7 and 2.16.■

LEMMA 2.20 *Let X be a locally connected continuum, Y a cyclic locally connected continuum and* $f{:}X\rightarrow Y$ *a monotone surjection. Let* $W_X \subseteq X$, $W_Y \subseteq Y$ *be a pair of T - sets such that* $g = f| W_X$ *is a homeomorphism. If* $B\subseteq W_Y$ *is a T - set, then* $A = g^{-1}(B)$ *is a T - set.*

Proof. Let J be any component of $X\setminus A$. Then there exists a component K of $X\setminus W_X$ such that $K\subseteq J$. Let $Bd(K) = \{a, b\}$. By virtue of Theorem 3.12 [11] there are finitely many components $J_1, ..., J_n$ of $Y\setminus W_Y$ such that

$$J_1\bigcup...\bigcup J_n \subseteq f(K) \subseteq Cl(J_1\bigcup...\bigcup J_n)$$

and $Bd(J_1) = ... = Bd(J_n) = f(Bd(K)) = \{f(a), f(b)\}$. It is clear that $f(a)\neq f(b)$ since g is a homeomorphism and $a, b\in W_X$. The continuum $L = Cl(J_1 \cup ... \cup J_n)$ is contained in some component I of $Y\setminus B$ with $Bd(I) = \{c, d\}$, $c\neq d$. Then $f^{-1}(I)$ is a continuum containing K. Thus, $f^{-1}(I)$ is contained in J. The points $c' = g^{-1}(c)$ and $d' = g^{-1}(d)$ are distinct points and are the members of $Bd(J)$. It is clear that $card(Bd(J)) = 2$ since g is a homeomorphism. ■

THEOREM 2.21 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ *- directed CMC - system. Then* $X = \lim\mathbf{X}$ *is the continuous image of an arc if and only if there exists an* $a\in A$ *such that* $p_{ab}| T_b$: $T_b \rightarrow T_a$ *is a homeomorphism for each* $b\geq a$.

Proof. *Sufficiency.* If there exists an $a\in A$ such that $p_{ab}|T_b : T_b \rightarrow T_a$ is a homeomorphism, then we have the inverse system $\mathbf{T} = \{T_c, p_{cd}|T_d, a\leq c\leq d\}$ of metrizable T - set such that $p_{cd}|T_d$ are homeomorphisms. This means that $T = \lim\mathbf{T}$ is metrizable. By virtue of Theorem [11, Theorem 3.13] T is a T -

set containing x, y, z. From Theorem 2.9 it follows that X is the continuous image of an arc.
Necessity. By virtue of Theorem 2.19 X is the continuous image of an arc if and only if there exists an $c \in A$ such that $W_d(x, y, z)$ is metrizable for each $d \geq c$. From Theorem 1.7 it follows that W(x, y, z) is metrizable if and only if there exists an $a \in A$, $a \geq c$, such that $p_b|W(x, y, z) : W(x, y, z) \to W_b(x, y, z)$ is a homeomorphism. It is clear that $p_{ab}|W_b(x, y, z)$ is a homeomorphism. By virtue of Theorem 2.20 the set $T = (p_{ab}|W_b(x, y, z))^{-1}(T_a)$ is a T - set containing x_b, y_b and z_b. Since T_b is the minimal T - set containing x_b, y_b, z_b, we infer that $T_b \subseteq T$. Thus $p_{ab}(T) \supseteq p_{ab}(T_b)$. This means that $p_{ab}(T_b) \subseteq T_a$. In fact, $p_{ab}(T_b) = T_a$ since T_a is the minimal T - set containing x_a, y_a, z_a. Since $p_b|W(x, y, z)$ is a homeomorphism, we infer that $p_{ab}|T_b : T_b \to T_a$ is a homeomorphism. ■

We close this section with the following theorem.

THEOREM 2.22 *Let* $\mathbf{X} = \{X_a, p_{ab}, (A, \leq)\}$ *be an inverse system of continuous image of arcs with monotone surjective bonding mappings. Then* $X = \lim\mathbf{X}$ *is the continuous image of an arc if and only if for each cyclic element* Z *of* X *and the points* $x, y, z \in Z$ *there exists a countable directed subset* $(B, \leq)$ *of* $(A, \leq)$ *such that for each countable directed subset* $(C, \leq)$ *of* $(A, \leq)$ *with* $C \supseteq B$ *the restriction* $h_{BC} = p_{BC}|\lim\{W_d(x, y, z), p_{dd_1}, D\}$ *of the canonical projection* p_{BC} *is a homeomorphism*

$$h_{BC} : \lim\{W_d(x, y, z), p_{dd_1}, D\} \to \lim\{W_c(x, y, z), p_{cc_1}, C\}.$$

Proof. By virtue of Theorem 1.6, X is homeomorphic to $\lim\mathbf{X}_\sigma$, where $\mathbf{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$. We assume that $X = \lim\mathbf{X}_\sigma$. Let Z be any cyclic element of X and let $\mathbf{Z}$ be the inverse system from Theorem 2.7. Let $x, y, z \in Z$. By virtue of Theorem 2.21 there exists a $C \in A_\sigma$ such that for each $D \in A_\sigma$ with $D \supseteq C$ the canonical projection p_{CD}: $T_D \to T_C$ is a homeomorphism. Let us recall that X_C and X_D are inverse limits of the inverse sequences $\{X_c, p_{cc_1}, C\}$ and $\{X_d, p_{dd_1}, D\}$ respectively. From Lemma 2.15 it follows that T_C is the limit of $\{W_c(x, y, z), p_{cc_1}|W_{c_1}(x, y, z), C\}$. Similarly, T_D is the limit of $\{W_d(x, y, z), p_{dd_1}|W_{d_1}(x, y, z), D\}$. Hence $h_{BC} = p_{BC}|\lim\{W_d(x, y, z), p_{dd_1}, D\}$,

$$h_{BC} : \lim\{W_d(x, y, z), p_{dd_1}, D\} \to \lim\{W_c(x, y, z), p_{cc_1}, C\}$$

is a homeomorphism.■

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Lončar I. Inverzni limes neprekidnih slika lukova

SAŽETAK

U radu su izučavani inverzni limesi neprekidnih slika lukova. Dokazano je: a) Ako je X = $\{X_a, p_{ab}, A\}$dobro uređeni inverzni sistem neprekidnih slika lukova s monotonim surjektivnim veznim preslikavanjima i ako je cf(A)$\neq \aleph_1$, tada je X = limX neprekidna slika luka (Teorem 2.17).
b) Neka je X = $\{X_a, p_{ab}, (A,\leq)\}$ inverzni sistem neprekidnih slika lukova s monotonim surjektivnim veznim preslikavanjima. Tada je X = limX neprekidna slika luka onda i samo onda ako za svaku trojku točaka x, y, z bilo kojeg cikličkog elementa Z limesa X postoji prebrojiv usmjeren podskup (B, $\leq$) skupa (A, $\leq$) sa svojstvom da je za svaki prebrojiv usmjeren skup (C, $\leq$)skupa (A, $\leq$), C$\supseteq$B, restrikcija $h_{BC} = p_{BC}|lim\{W_d(x,y,z), p_{dd_1}, D\}$ homeomorfizam (Teorem 2.22).

Ključne riječi: inverzni sistem i limes, neprekidna slika luka.