Ivan Lončar
Fakultet organizacije i informatike
Varaždin
e-mail:Ivan.Loncar@foi.hr

UDK: 513
Original Scientific Paper

A note on approximate limits

The main theorems of this paper are the following theorems.
THEOREM 2.7. *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate sequence of non - empty Čech-complete paracompact spaces* X_n *such that each* $p_{nm}(X_m)$ *is dense in* X_n*, then* $\lim\mathbf{X}$ *is non-empty and Čech-complete. Moreover,* $p_n(\lim\mathbf{X})$ *is dense in* X_n *for each* $n\in\mathbb{N}$ *.*
THEOREM 2.11. *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse sequence of absolute* G_δ *- space. Then there exist:*
a) *a cofinal subset* $M = \{n_i : i\in\mathbb{N}\}$ *of* $\mathbb{N}$*,*
b) *a usual inverse sequence* $\mathbf{Y} = \{Y_i , q_{ij} , M\}$ *such that* $Y_i = X_{n_i}$ *and* $q_{ij} = p_{ii+1}\ p_{i+1i+2} \cdots p_{j-1j}$ *for each* $i,j\in\mathbb{N}$*,*
c) *a homeomorphism* $H : \lim\mathbf{X}\rightarrow\lim\mathbf{Y}$*.*

Keywords: Approximate inverse system and limit.
Classification: 54B25, 54D30.

1 Preliminaries

A space means a Tychonoff space and *a mapping* means a continuous (not necessarily surjective) mapping.

Cov(X) is the set of all normal coverings of a topological space X.For other details see [1].

In this paper we study the approximate inverse system in the sense of S. Mardešić [12].

DEFINITION 1.1 *An approximate inverse system is a collection* **X** $= \{X_a, p_{ab}, A\}$*,where* $(A,\leq)$ *is a directed preordered set,*$X_a, a\in A$*,is a topological space and* $p_{ab}:X_b \rightarrow X_a, a\leq b$*,are mappings such that* $p_{aa}=id$ *and the following condition (A2) is satisfied:*

(A2) For each $a\in A$ *and each normal cover* $\mathcal{U}\in Cov(X_a)$ *there is an index* $b\geq a$ *such that* $(p_{ac}p_{cd}, p_{ad})\prec \mathcal{U}$*,whenever* $a\leq b\leq c\leq d$.

The inverse system in the sense of [6, p. 135.] we will call *usual inverse system.*

DEFINITION 1.2 An *approximate map* $\mathbf{p} = \{p_a : a \in A\}:X\rightarrow \mathbf{X}$ into an aproximate system $\mathbf{X}=\{X_a,p_{ab},A\}$ is a collection of maps $p_a:X\rightarrow X_a$, $a\in A$,such that the following condition holds

(AS) For any $a\in A$ and any $\mathcal{U} \in Cov(X_a)$ there is $b\geq a$ such that $(p_{ac}p_c, p_a)\prec \mathcal{U}$ for each $c\geq b$.(See [14]).

DEFINITION 1.3 Let $\mathbf{X} = \{X_a, p_{ab}, A\}$be an approximate system and let $\mathbf{p} = \{p_a : a \in A\}:X\rightarrow \mathbf{X}$ be an approximate map.We say that **p** is a *limit* of **X** provided it has the following universal property:

(UL) For any approximate map $\mathbf{q} = \{q_a : a \in A\}:Y\rightarrow \underline{\mathbf{X}}$ of a space Y there exists a unique map $g:Y\rightarrow X$ such that $p_a g=q_a$.

DEFINITION 1.4 Let $\mathbf{X}=\{X_a,p_{ab},A\}$ be an approximate system.A point $x=(x_a)\in \prod\{X_a : a \in A\}$ is called a *thread* of **X** provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall \mathcal{U} \in Cov(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in st(x_a,\mathcal{U})$.

REMARK 1.5 If X_a is a $T_{3.5}$ space,then the sets $st(x_a,\mathcal{U}), \mathcal{U} \in Cov(X_a)$, form a basis of the topology at the point x_a.Therefore,for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L)* $(\forall a \in A) \lim\{p_{ac}(x_c):c\geq a\} = x_a$.

The existence of the limit of any approximate system was proved in [14, (1.14)Theorem].

THEOREM 1.6 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system. Let* $X\subseteq \Pi\{X_a : a \in A\}$ *be the set of all threads of* $\mathbf{X}$ *and let* $p_a:X\to X_a$ *be the restriction* $p_a=\pi_a|X$ *of the projection* $\pi_a:\Pi X_a \to X_a, a\in A$. *Then* $\mathbf{p} = \{p_a:a \in A\}:X\to\mathbf{X}$ *is a limit of* $\mathbf{X}$.

We call this limit the *canonical limit* of $\mathbf{X} = \{X_a, p_{ab}, A\}$.

We say that a statement T on elements of a directed set D is fulfiled [16]:

1. For *almost all* n∈D if there exists an element n_0 ∈D such that T is fulfiled for every $n\geq n_0$.
2. For *arbitrarily large* n∈D if the set of all n∈D for which T is fulfiled is cofinal with D.

A *net* $\{A_n, n\in D\}$ is a function [16] defined on a directed set D. If $\{A_n, n\in D\}$ is a net of subsets of X,then:

3. A *limit inferior* LiA_n is the set of all point x∈X such that every neighbouhood of x intersect A_n for *almost all* n∈D.
4. A *limit superior* LsA_n is the set of all point x∈X such that every neighbouhood of x intersect A_n for *arbitrarily large* n∈D.
5. A net $\{A_n, n\in D\}$ is said to be *topologically convergent* (to a set A) if $LsA_n=LiA_n$ (=A) and in this case the set A will be denoted by $LimA_n$.

2 Approximate limit of paracompact Čech-complete spaces

We start with the following theorem.

LEMMA 2.1 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of non-empty compact Hausdorff spaces with limit X. If A′ is a cofinal subset of A, then for each family* $\mathcal{N}=\{x_a : x_a \in X_a, a\in A'\}$ *the set* $Ls\{p_a^{-1}(x_a) : a\in A'\}$ *is non-empty and* $p_a(Ls\{p_a^{-1}(x_a) : a\in A'\}) \subseteq Ls\ \{p_{ab}(x_b) : b\in A', b\geq a\}$.

Proof. For each $a\in A$ we consider the net $\mathcal{N}_a=\{p_{ab}(x_b) : b\in A', b\geq a\}$. From the compactness of X_a it follows that the set C_a of all cluster points of $\mathcal{N}_a$ is non-empty. Clearly, each C_a is closed and compact in X_a. First, we prove

(a) *For each* $a\in A$, C_a *is a subset of* $p_a(X)$.

If we suppose that some $c_a \in C_a\backslash p_a(X)$, then c_a and $p_a(X)$ respectively, have disjoint neighborhoods U and V. By virtue of the property (B3) [14, pp. 606,615] there is a $b\geq a$ such that $p_{ac}(X_c)\subseteq V$ for each $c\geq b$, $c\in A'$. This is impossible since there exists $c\geq b$ such that $p_{ac}(x_c)\in U$ (c_a is a cluster point of the net $\mathcal{N}_a$).

From (a) it easily follows that

(b) *For each* $a\in A$, *the set* $p_a^{-1}(C_a)$ *is non-empty.*

By (b) there is $y^a \in p_a^{-1}(C_a)\subseteq \lim\mathbf{X}$, $a\in A'$. Since $\lim\mathbf{X}$ is compact, there is a cluster point $y\in\lim\underline{\mathbf{X}}$ of the net $\mathcal{Y} = \{y^a : a\in A'\}$. Let us prove

(c) $p_a(y)\in C_a$, $a\in A$.

It suffices to prove that for each neighborhood U_a of $p_a(y)$ and each b_0 there exists a $d\geq b_0$ such that $p_{ad}(x_d)\in U_a$. Let $\mathcal{U}$ be a normal cover of X_a such that

$$st^2(p_a(y),\mathcal{U}) \subseteq U_a. \tag{1}$$

Let $U_1 \in \mathcal{U}$ be such that $p_a(y)\in U_1$. Then $p_a^{-1}(U_1)$ is a neighborhood of y. The set B of all $b\in A'$ with $y^b \in p_a^{-1}(U_1)$ is cofinal in A′ since y is a cluster point of $\mathcal{Y}$. By virtue of (AS) the set $B' \subseteq B$ of all $b \in B$, $b \geq b_0$, such that

$$(p_a, p_{ab}p_b) \prec \mathcal{U} \tag{2}$$

is cofinal in A. Similarly, by (A2), the set $B'' \subseteq B'$ of all $b \in B'$ such that

$$(p_{ac}, p_{ab}p_{bc}) \prec \mathcal{U}, \quad c \geq b \tag{3}$$

is cofinal in A. Let $b \in B''$.Then $y^b \in p_a^{-1}(U_1)$. Thus

$$p_a(y), p_a(y^b) \in U_1. \tag{4}$$

By virtue of (2) it follows

$$p_a(y^b), p_{ab}p_b(y^b) \in U_2 \in \mathcal{U}. \tag{5}$$

This and (4) imply

$$p_{ab}p_b(y^b) \in St(p_a(y), \mathcal{U}) \tag{6}$$

Now , $p_b(y^b) \in C_b$ since $y^b \in p_b^{-1}(C_b)$. We infer that $p_{ab}^{-1}(St(p_a(y), \mathcal{U}))$ is a neighborhood of $p_b(y^b)$. Since $p_b(y^b)$ is a cluster point of $\mathcal{N}_a = \{x_a : a \in A'\}$ there is a $d \geq b \geq b_0, d \in A'$ such that $p_{bd}(x_d) \in p_{ab}^{-1}(St(p_a(y), \mathcal{U}))$. This means that $p_{ab}(p_{bd}(x_d)) \in St(p_a(y), \mathcal{U})$. Using (3), $p_{ad}(x_d) \in St^2(p_a(y), \mathcal{U})$. Thus , by (1)

$$p_{ad}(x_d) \in U_a. \tag{7}$$

We infer that $p_a(y) \in C_a$, i.e. , $y \in p_a^{-1}(C_a)$ for each $a \in A$.■

In the sequel we shall use

LEMMA 2.2 *Let cX and cY be extensions of Tychonoff spaces X and Y and let f,g:X→Y be a pair of continuous mappings which have the extensions cf:cX→cY and cg:cX→cY. If $\mathcal{U}, \mathcal{V}$ is a pair of normal covers of cY such that $st\mathcal{V} < \mathcal{U}$, then if f and g are $\mathcal{V}|Y$-near, cf and cg are $\mathcal{U}$-near.*

Proof. Consider the normal cover $\mathcal{W}=(cf)^{-1}(\mathcal{V})\wedge(cg)^{-1}(\mathcal{V})$. For each $x\in cX$ there is a $W\in \mathcal{W}$ such that $x\in W$. Moreover,there is a point $y\in X\subseteq cX$ such that $y\in W$.Now, $cf(x)\in V_1 \in \mathcal{V}$ and $cg(x)\in V_2 \in \mathcal{V}$. Furthermore,$f(y)\in V_1 \in \mathcal{V}$ and $g(y)\in V_2 \in \mathcal{V}$. There exists a $V_3 \in \mathcal{V}$ such that $\{f(y),g(y)\}\subseteq V_3$ since f and g are $\mathcal{V}|Y$-near. We infer that $\{cf(x),cg(x)\}\subseteq st(V_3,\mathcal{V})$. This means that there is an $U\in \mathcal{U}$ such that $\{cf(x),cg(x)\}\subseteq U$ since $st\mathcal{V}< \mathcal{U}$. We infer that cf and cg are $\mathcal{U}$-near.The proof is completed.■

LEMMA 2.3 *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$*be an aproximate inverse system of Tychonoff spaces. If* c_aX_a, $a\in A$, *are Hausdorff extensions of the spaces* X_a *such that the mappings* p_{ab} *have the extensions* $c_{ab}p_{ab}$, *then* $\mathbf{cX} = \{c_aX_a,\ c_{ab}p_{ab},\ A\}$*is an approximate inverse system.*

Proof. It suffices to verify the condition (A2) for **cX.** Let $a\in A$ be fixed and let $\mathcal{U}$ be any normal cover of c_aX_a.Choose a normal cover $\mathcal{V}$ such that $st\mathcal{V}\prec \mathcal{U}$.By virtue of (A2) for **X** there is an index $b\geq a$ such that p_{ad} and $p_{ac}p_{cd}$ are $\mathcal{V}|X_a$-near.By virtue of the above Lemma we infer that $c_{ad}p_{ad}$ and $c_{ad}(p_{ac}p_{cd})$ are $\mathcal{U}$-near.Finally, from $c_{ad}(p_{ac}p_{cd}) = c_{ac}p_{ac}c_{cd}p_{cd}$ it follows that $c_{ad}p_{ad}$ and $c_{ac}p_{ac}c_{cd}p_{cd}$ are $\mathcal{U}$-near.The proof is complete.■

If X is a Tychonoff space, then by βX the Čech-Stone compactification of X is denoted.

COROLLARY 2.4 *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$*be an approximate inverse system of Tychonoff spaces. Then* $\beta\mathbf{X} = \{\beta X_a,\ \beta p_{ab},\ A\}$*is also an approximate inverse system.*

In the sequel we shall denote by P_n the natural projection $P_n:\lim\beta\mathbf{X}\rightarrow \beta X_n$.

LEMMA 2.5 *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$*be an approximate inverse system of non-empty Tychonoff spaces. If for some family* $\mathcal{N}=\{x_a:x_a \in X_a,a\in A\}$ *the set* $C_a = Ls\{p_{ab}(x_b) : b\geq a\}$ *is non-empty and compact, then* lim**X** *is non-empty.*

Proof. Consider the approximate system $\beta\mathbf{X} = \{\beta X_a, \beta p_{ab}, A\}$. By virtue of 2.1 there exists $y \in \beta \lim \mathbf{X}$ such that $y_a = P_a(y) \in D_a$, where D_a is a limit superior of $\{p_{ab}(x_b) : b \geq a\}$ in βX_a. On the other hand $C_a = Ls\{p_{ab}(x_b) : b \geq a\}$ in X_a is compact. This means that $D_a = C_a$. We infer that $y_a \in X_a$. Thus, $y \in \lim \mathbf{X}$. The proof is complete.■

LEMMA 2.6 *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of non-empty Tychonoff topologically complete spaces. If there exists a cofinal subset* A' *of* A *such that for some family* $\mathcal{N} = \{x_a : x_a \in X_a, a \in A'\}$ *the set* $C_a = Ls\{p_{ab}(x_b) : b \in A'\}$ *is non-empty and compact, then* $\lim\mathbf{X}$ *is non-empty.*

Proof. Consider the approximate system $\mathbf{Y} = \{X_a, p_{ab}, A'\}$. By virtue of Theorem 2.5 $\lim\mathbf{Y}$ is non-empty. Theorem (2.14) of [15] completes the proof.■

We give the following application of Lemma 2.6.

We say that a space X is *Čech - complete* if X is a Tychonoff space which is a G_δ - set in βX [6, p. 251.]. We shall say that the *diameter of a subset* Y of topological space X *is less than a cover* $\mathcal{A} = \{A_s : s \in S\}$ of the space X, and we shall write $\delta(Y) < \mathcal{A}$, provided there exists $s \in S$ such that $Y \subseteq A_s$ [6, p. 252.]. A Tychonoff space X is Čech - complete iff there exists a countable family $\{\mathcal{A}_i : i \in \mathbb{N}\}$ of open covers of the space X with the property:

(Č) Any family $\mathcal{F}$ of subsets of X, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for i = 1, 2, ..., has non - empty intersection $\bigcap\{\mathrm{Cl}F : F \in \mathcal{F}\}$ ([3, p. 183.], [6, p. 252.]).

THEOREM 2.7 *Let* $\mathbf{X} = \{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate sequence of non - empty Čech-complete paracompact spaces* X_n *such that each* $p_{nm}(X_m)$ *is dense in* X_n*, then* $\lim\mathbf{X}$ *is non-empty and Čech-complete. Moreover,* $p_n(\lim\mathbf{X})$ *is dense in* X_n *for each* $n \in \mathbb{N}$.

Proof. The proof is broken into several steps.

Step 1. Let us prove that for each point $x_{i_0} \in X_{i_0}$ and for each open set $U_{i_0} \ni x_{i_0}$ there exists a point $x \in \lim \mathbf{X}$ such that $p_{i_0}(x) \in U_{i_0}$. Let $\{\mathcal{A}_{n,i} : i \in \mathbb{N}\}$ be a family of open covers of X_n with the property (Č). There exists a normal cover $\mathcal{U}$ of X_{i_0} such that $\mathrm{st}^2(x_{i_0}, \mathcal{U}) \subseteq U_{i_0}$. Moreover, there exists a normal cover $\mathcal{V}$ of X_{i_0} such that $\mathrm{st}\mathcal{V}$ refines both $\mathcal{A}_{i_0,1}$ and $\mathcal{U}$. Now, we denote $\mathcal{V}$ again by $\mathcal{A}_{i_0,1}$.

Step 2. By induction, for each $i \in \mathbb{N}$, we will choose $n_i \in \mathbb{N}, n_i \geq i_0$, and the normal covers $\mathcal{U}_{j,n_i}$, $j \leq n_i$, such that :

(P1) $\mathrm{st}\mathcal{U}_{j,n_i}$, $j < n_i$, is a refinement of the covers $\mathcal{A}_{j,k}$ for $k < n_i$,

(P2) $\mathrm{st}\mathcal{U}_{n_i,n_i}$ is a refinement of the covers $q_{ji}^{-1}(\mathcal{U}_{n_j,n_i})$, $j \leq i$, the covers $\mathcal{A}_{n_i,k}$, $k \leq n_i$, and the cover $p_{jn_i}^{-1}(\mathcal{U}_{j,n_i})$, whenever $j < n_i$,

(P3) $(p_{jn}, p_{jm}p_{mn}) \prec \mathcal{U}_{j,n_i}$, $\quad n \geq m \geq n_{i+1}$, $j \leq n_i$.

Let $n_1 = i_0$ and let $\mathcal{U}_{n_1,n_1}$ be a normal cover of X_{n_1,n_1} such that $\mathrm{st}\mathcal{U}_{n_1,n_1}$ is a refinement of the cover $\mathcal{A}_{n_1,n_1}$. By (A2), there is a number $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that (P3) is satisfied for $j = n_1$ and n_2, i.e. ,

$$(p_{n_1 n}, p_{n_1 m}p_{mn}) \prec \mathcal{U}_{n_1,n_1}, \qquad n \geq m \geq n_2. \tag{8}$$

In each space X_j, $j < n_2$, there is a normal cover $\mathcal{U}_{j,n_2}$ such that (P1) is satisfied (i.e., $\mathrm{st}\mathcal{U}_{j,n_2}$ is a refinement of the covers $A_{j,k}$, $k \leq n_2$) since X_j is a paracompact space. Similarly, one can define a normal cover $\mathcal{U}_{n_2,n_2}$ such that (P2) is satisfied.

Suppose that $n_1, \ldots, n_{i-1}$, $i > 2$, and the covers $\mathcal{U}_{j,n_{i-1}}$, $j \leq n_{i-1}$, with (P1) - (P3) are defined. Let us define n_i. Firstly, we define the covers $\mathcal{U}_{j,n_{i-1}}$, $j < n_{i-1}$, $\mathcal{U}_{n_{i-1},n_{i-1}}$ such that (P1) and (P2) are satisfied. This is possible since X_j, $j \leq n_{i-1}$, is paracompact and since any two normal coverings admit a normal covering which refines both. By (A2) there is a number $n_i > n_{i-1}$ such that (P3) is satisfied. This completes the construction of an usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$ and $q_{ij} = p_{ii+1}\, p_{i+1i+2} \ldots p_{j-1j}$ for each $i,j \in \mathbb{N}$, i.e., the sequence

$$X_{n_1} \xleftarrow{p_{n_1 n_2}} \ldots \xleftarrow{p_{n_{i-1} n_i}} X_{n_i} \xleftarrow{p_{n_i n_{i+1}}} \ldots \tag{9}$$

Step 3. By virtue of Michael's theorem for usual inverse sequences [6, p. 257.], lim**Y** is non-empty. Moreover, there exists $y = (y_{n_i})$ in lim**Y** such that

$$(p_{i_0}(y), x_{i_0}) \prec \mathcal{A}_{i_0,1}. \tag{10}$$

By (P3) it follows

$$(p_{n_{j-2}n_j}, p_{n_{j-2}n_{j-1}}p_{n_{j-1}n_j}) \prec \mathcal{U}_{n_{j-2},n_{j-2}}. \tag{11}$$

This means that

$$(p_{n_{j-2}n_j}(y_{n_j}), y_{n_{j-2}}) \prec \mathcal{U}_{n_{j-2},n_{j-2}}. \tag{12}$$

since $p_{n_{j-2}n_{j-1}}p_{n_{j-1}n_j}(y_{n_j} = y_{n_{j-2}}$. By (P2) we have

$$(p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j}), p_{n_{j-3}n_{j-2}}(y_{n_{j-2}})) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{13}$$

or

$$(p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j}), y_{n_{j-3}}) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{14}$$

since $p_{n_{j-3}n_{j-2}}(y_{n_{j-2}}) = y_{n_{j-3}}$. Using (P3) for n_{j-3} , n_{j-2} , n_j and y_{n_j} we obtain

$$(p_{n_{j-3}n_j}(y_{n_j}), p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j})) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{15}$$

We infer that

$$p_{n_{j-3}n_j}(y_{n_j}) \in st(y_{n_{j-3}}, \mathcal{U}_{n_{j-3},n_{j-3}}) \tag{16}$$

Repeating this , we infer that for fixed $i \in \mathbb{N}$ and each $k \geq i + 3$

$$p_{n_i n_k}(y_{n_k}) \in st(y_{n_i}, \mathcal{U}_{n_i,n_i}). \tag{17}$$

Step 4. We see that $\{p_{in_k}(y_{n_k}) : n_k \geq i\}$ has the diameter less than $\mathcal{U}_{n_i,n_i}$. It is obvious that $Ls\{p_{in_k}(y_{n_k}) : n_k \geq i\}$ is compact. By virtue of Lemma 2.6 there exists $x \in \lim\beta\mathbf{X}$ such that $P_i(x) \in Ls\{p_{in_k}(y_{n_k}) : n_k \geq i\}$. This means that $x \in \lim\mathbf{X}$. From (17) it follows that $p_{i_0}(x) \in st(p_{i_0}(y), \mathcal{A}_{i_0,1})$. Moreover, (10) implies $(p_{i_0}(y) , x_{i_0}) \prec \mathcal{A}_{i_0,1}$. By virtue of Step 1.

st $^2(x_{i_0},\mathcal{A}_{i_0,1})\subseteq U_{i_0}$. We infer that $p_{i_0}(x)\in U_{i_0}$. This means that $p_{i_0}(\lim\mathbf{X})$ is dense in X_{i_0}. The proof is completed.■

QUESTION. Let $\mathbf{X}=\{X_n,\ p_{mn},\ \mathbb{N}\}$ be an approximate inverse sequence of non-empty Čech-complete spaces X_n. Does it follow that lim**X** is non-empty?

A Tychonoff space X is called *locally Čech - complete* if every point $x\in X$ has a Čech - complete neighbourhood [6, p. 297.]. Every locally Čech - complete paracompact space is Čech - complete. From [6, p. 423.] it follows that if $\mathbf{X}=\{X_n,\ p_{mn},\ \mathbb{N}\}$ is an approximate sequence of paracompact Čech - complete spaces , then lim**X** is paracompact and Čech - complete. By virtue of Theorem 2.7 it follows

COROLLARY 2.8 *Let $\mathbf{X}=\{X_n,\ p_{mn},\ \mathbb{N}\}$ be an approximate inverse sequence of non-empty locally Čech - complete paracompact spaces such that $p_{ij}(X_j)$ is dense in $X_i, i\leq j$. Then $p_i(\lim\mathbf{X})$ is dense in X_i. Moreover, lim**X** is paracompact and Čech - complete.*

From Theorem 2.7 it follows the approximate version of Theorem of Arens [2, Theorem 2.4.]. (See also [6, p. 257,Exercise 3.9.H.]).

COROLLARY 2.9 *Let $\mathbf{X}=\{X_n,\ p_{mn},\ \mathbb{N}\}$ be an approximate sequence of non - empty complete metric spaces X_n. If $p_{nm}(X_m)$ is dense in X_n , $m\geq n$, then lim**X** is non - empty complete metric space and $p_n(\lim\mathbf{X})$ is dense in X_n.*

A metric space X is said to be *locally complete* if for each $x\in X$ there exists an open set $U\ni x$ such that ClU is complete.

Let $X = \bigcup\{R\times\{\frac{1}{n}\}:n=1,2,...\}$ be the subspace of the space R^2. Then X is locally complete,but not complete since the sequence $\{(1,\frac{1}{n}) : n=1,2,3,..\}$ is a Cauchy non-convergent (in X) sequence.Similarly,the subspace Y = $\{(x,y):x>0,\ y=\sin\frac{1}{x}\}$ of R^2 is non-complete locally complete space.

COROLLARY 2.10 *Let* $\mathbf{X}=\{X_n, p_{mn}, I\!N\}$ *be a usual inverse sequence of non-empty locally complete metric spaces such that* $p_{ij}(X_j)$ *is dense in* $X_i, i\leq j$. *Then* $p_i(\lim\mathbf{X})$ *is dense in* X_i.

A metric space X is an absolute G_δ - space [6, p. 342] if X is a G_δ - set in any metrizable space in which it is embedded. A metrizable space X is a G_δ - space iff it is completely metrizable.

The main theorem of this Section is the following theorem.

THEOREM 2.11 *Let* $\mathbf{X}=\{X_n, p_{mn}, I\!N\}$ *be an approximate inverse sequence of absolute* G_δ *- spaces . Then there exist:*

a) *a cofinal subset* $M = \{n_i : i\in I\!N\}$ *of* $I\!N$,

b) *a usual inverse sequence* $\mathbf{Y} = \{Y_i , q_{ij} , M\}$ *such that* $Y_i = X_{n_i}$ *and* $q_{ij} = p_{ii+1}\, p_{i+1i+2} \dots p_{j-1j}$ *for each* $i,j\in I\!N$,

c) *a homeomorphism* $H : \lim\mathbf{X}\to\lim\mathbf{Y}$.

Proof. Let $\{\mathcal{A}_{n,i} : i\in I\!N\}$ be a family of open covers of X_n with the property:

(UN0) the members of $\mathcal{A}_{n,i}$ are sets of diameter less than 1/i.

By induction , for each $i\in I\!N$, we will choose $n_i \in I\!N$ and the normal covers $\mathcal{U}_{j,n_i}$, $j\leq n_i$, of X_j such that :

(UN1) $st\mathcal{U}_{j,n_i}$, $j<n_i$, is a refinement of the covers $\mathcal{A}_{j,k}$ for $k<n_i$,

(UN2) $st\mathcal{U}_{n_i,n_i}$ is a refinement of the covers $q_{ji}^{-1}(\mathcal{U}_{n_j,n_i})$,$j\leq i$, the covers $\mathcal{A}_{n_i,k}$, $k\leq n_i$,and the cover $p_{jn_i}^{-1}(\mathcal{U}_{j,n_i})$, whenever $j<n_i$,

(UN3) $(p_{jn} , p_{jm}p_{mn})\prec \mathcal{U}_{j,n_i}$, $n\geq m\geq n_{i+1}$, $j\leq n_i$,

(UN4) $(p_j , p_{jm}\, p_m)\prec \mathcal{U}_{j,n_i}$, $j\leq n_i$, $m\geq n_i$.

Let $n_1 = 1$ and let $\mathcal{U}_{n_1,n_1}$ be a normal cover of X_{n_1,n_1} such that $st\mathcal{U}_{n_1,n_1}$ is a refinement of the cover $\mathcal{A}_{n_1,n_1}$. By (A2) and (AS) ([12, p. 242.] , [15, p. 113.]) there is a number $n_2 \in I\!N$, $n_2 > n_1$, such that

$$(p_{n_1 n}, p_{n_1 m}p_{mn}) \prec \mathcal{U}_{n_1,n_1}, \qquad n \geq m \geq n_2, \tag{18}$$

and

$$(p_{n_1}, p_{n_1 m} p_m) \prec \mathcal{U}_{n_1,n_1}, \qquad m \geq n_2. \tag{19}$$

In each space X_j, $j<n_2$, there is a normal cover $\mathcal{U}_{j,n_2}$ such that (UN1) is satisfied (i.e., st$\mathcal{U}_{j,n_2}$ is a refinement of the covers $A_{j,k}$, $k \leq n_2$) since X_j is a paracompact space. Similarly, one can define a normal cover $\mathcal{U}_{n_2,n_2}$ such that (UN2) is satisfied.

Suppose that $n_1, \dots , n_{i-1}$, $i>2$, and the covers $\mathcal{U}_{j,n_{i-2}}$, $j \leq n_{i-2}$, with (UN1) - (UN4) are defined. Let us define n_i. Firstly , we define the covers $\mathcal{U}_{j,n_{i-1}}$, $j<n_{i-1}$, $\mathcal{U}_{n_{i-1},n_{i-1}}$ such that (UN1) and (UN2) are satisfied. This is possible since X_j , $j \leq n_{i-1}$, is paracompact and since any two normal coverings admit a normal covering which refines both. By (A2) and (AS) there is a number $n_i > n_{i-1}$ such that (UN3) and (UN4) are satisfied. This completes the construction of an usual inverse sequence $\mathbf{Y} = \{Y_i , q_{ij} , M\}$ such that $Y_i = X_{n_i}$ and $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \dots p_{n_{j-1} n_j}$ for each $i,j \in \mathbb{N}$, i.e., the sequence

$$X_{n_1} \xleftarrow{p_{n_1 n_2}} \dots \xleftarrow{p_{n_{i-1} n_i}} X_{n_i} \xleftarrow{p_{n_i n_{i+1}}} \dots . \tag{20}$$

Now we shall define a homeomorphism H : lim$\mathbf{X} \to$ lim$\mathbf{Y}$. Let x be any point of lim$\mathbf{X}$. We shall prove that for each cover $\mathcal{A}_{n_j,\ell}$ of X_{n_j} there is a $m \in \mathbb{N}$ such that the diameter

$$\delta(\{q_{jk} p_{n_k}(x) : n_k \geq m\}) < \mathcal{A}_{n_j,k}. \tag{21}$$

From the above construction it follows that there is a cover $\mathcal{U}_{n_j,n_i}$, $j<i$, which satisfies (UN1), (UN3) and (UN4).We set $m = n_i$. Let $n_k > m$. By virtue of (UN3) and (UN4) it follows

$$(p_{n_{k-2} n_k} p_{n_k}(x), p_{n_{k-2} n_{k-1}} p_{n_{k-1} n_k} p_{n_k}(x)) \prec \mathcal{U}_{n_{k-2},n_{k-2}}, \tag{22}$$

and

$$(p_{n_{k-2}}(x), p_{n_{k-2} n_k} p_{n_k}(x)) \prec \mathcal{U}_{n_{k-2},n_{k-2}}. \tag{23}$$

We infer that

$$(p_{n_{k-2}}(x), p_{n_{k-2}n_{k-1}}p_{n_{k-1}n_k}p_{n_k}(x)) \prec st(p_{n_{k-2}n_k}p_{n_k}(x), \mathcal{U}_{n_{k-2},n_{k-2}}). \tag{24}$$

From (UN2) it follows that st $^2\mathcal{U}_{n_{k-2},n_{k-2}}$ refines the cover $p^{-1}_{n_{k-3}n_{k-2}}(\mathcal{U}_{n_{k-3},n_{k-3}})$. Thus, (24) implies

$$(p_{n_{k-3}n_{k-2}}p_{n_{k-2}}(x), p_{n_{k-3}n_{k-2}}p_{n_{k-2}n_{k-1}}p_{n_{k-1}n_k}p_{n_k}(x)) \prec \mathcal{U}_{n_{k-3},n_{k-3}}. \tag{25}$$

Moreover , (UN4) implies

$$(p_{n_{k-3}}(x), p_{n_{k-3}n_{k-2}}p_{n_{k-2}}(x)) \prec \mathcal{U}_{n_{k-3},n_{k-3}}. \tag{26}$$

Hence

$$(p_{n_{k-3}}(x) , q_{k-3\,k}p_{n_k}(x)) \in st(p_{n_{k-3}n_{k-2}}p_{n_{k-2}}(x), \mathcal{U}_{n_{k-3}\,,\,n_{k-3}}). \tag{27}$$

Repeating this , we infer that

$$(p_{n_i}(x) , q_{ik}p_{n_k}(x)) \in st(p_{n_in_{i+1}}p_{n_{i+1}}(x) , \mathcal{U}_{n_i,n_i}), k \geq i. \tag{28}$$

Using (UN2) for the cover $\mathcal{U}_{n_j,n_i}$ we have that st $^2\mathcal{U}_{n_i,n_i}$ refines $q^{-1}_{ji}\mathcal{U}_{n_j,n_i}$. Now, (28) implies

$$\{q_{jk}p_{n_k}(x) : k \geq i\} \prec \mathcal{U}_{n_j,n_i}. \tag{29}$$

Hence , the relation (21) is proved. We infer that $\{q_{jk}p_{n_k}(x) : k \in I\!N\}$ is a Cauchy sequence. Set

$$y_{n_j} = lim\{q_{jk}p_{n_k}(x) : k \in I\!N\}. \tag{30}$$

This is possible since each X_i is completely metrizable. Moreover , we have

$$q_{ij}(y_{n_j}) = y_{n_i}, \tag{31}$$

since $q_{ij}(y_{n_j}) = q_{ij}(\lim\{q_{jk}p_{n_k}(x) : k\in \mathbb{N}\} = \lim\{q_{ij}q_{jk}p_{n_k}(x) : k\in \mathbb{N}\}$ $= \lim\{q_{ik}p_{n_k}(x) : k\in \mathbb{N}\}) = y_{n_i}$. Hence $y = (y_{n_i})$ is a point of $\lim\mathbf{Y}$. We define a maping $H_{n_i} : \lim\mathbf{X}\to X_{n_i}$ by

$$H_{n_i}(x) = y_{n_i}. \tag{32}$$

Claim 1. *The mappings H_{n_i} , $i\in\mathbb{N}$, induce a mapping $H:\lim\mathbf{X}\to\lim\mathbf{Y}$ such that $q_{n_i}H = H_{n_i}$, $i\in\mathbb{N}$.*

Claim 2. *H and H_{n_i} , $i\in \mathbb{N}$, are continuous.*

Let x be any point of $\lim\mathbf{X}$ and let $H(x) = y$. Consider any open neighbourhood U of $y\in\lim\mathbf{Y}$. There exists an open set U_{n_i} such that $y\in q_i^{-1}(U_{n_i})\subseteq U$. This means that $q_i(y)\in U_{n_i}$. By virtue of (UN0) there exist $V_\ell \in \mathcal{A}_{n_i,j}$, such that $V = V_\ell \subseteq U_{n_i}$. Let $n_j = \max\{n_i , j\}$.Consider the cover $\mathcal{U}_{n_j,n_j}$. By virtue of (UN2) $W = st^3(q_j(y) , \mathcal{U}_{n_j,n_j})$ is contained in $q_{ij}^{-1}(V)$. Hence, $q_j^{-1}(W)$ is a neighbourhood of y contained in U. Let W_1 be any member of $\mathcal{U}_{n_j,n_j}$ containing $q_j(y)$. There is an $m_0 \in \mathbb{N}$ such that $q_{jm}p_{n_m}(x)\in W_1$. By virtue of (28)

$$(p_{n_j}(x), q_{jm}p_{n_m}(x)) \in st(p_{n_jn_{j+1}}p_{n_{j+1}}(x), \mathcal{U}_{n_j,n_j}).$$

We infer that $p_{n_j}(x)\in st^2(q_j(y) , \mathcal{U}_{n_j,n_j})$. Let W_2 be any member of $\mathcal{U}_{n_j,n_j}$ which contains $p_{n_j}(x)$. There is an open set W_3 containing x such that $p_{n_j}(W_3)\subseteq W_2$. By virtue of (UN4) we have $p_{n_j}(z)\in st^3(q_j(y) , \mathcal{U}_{n_j,n_j})$ for each $z\in W_3$. This means $H(z)\in q_j^{-1}(W)\subseteq U$. The proof of the continuity of H is completed. The continuity of H_{n_i} follows from $q_{n_i}H = H_{n_i}$, $i\in \mathbb{N}$.

Claim 3. *H is one - to - one.*

Let x_1 , x_2 be any pair of distinct points of $\lim\mathbf{X}$. There exists an indeks $i\in \mathbb{N}$ such that $p_m(x_1)\neq p_m(x_2)$ for all $m\geq n_i$. There exists a cover $\mathcal{U}_{n_i,j}$ of X_{n_i} such that $p_{n_i}(x_1)$ and $p_{n_i}(x_2)$ are in the members W_1 , W_2 of $\mathcal{U}_{n_i,j}$ with disjoint closures. By virtue of (29) we have

$$\{q_{jk}p_{n_k}(x_1)\} : k \geq i\} \subseteq W_1, \tag{33}$$

and

$$\{q_{jk}p_{n_k}(x_2) : k \geq i\} \subseteq W_2. \tag{34}$$

We infer that $lim\{q_{jk}p_{n_k}(x_1)\} : k \geq i\} \neq lim\{q_{jk}p_{n_k}(x_1)\} : k \geq i\}$. By virtue of (32) we infer that $H_{n_i}(x_1) \neq H_{n_i}(x_2)$ and $H(x_1) \neq H(x_2)$. Hence , H is one - to - one.

Claim 4. *H is onto.*

Let $y = (y_{n_i})$ be any point of lim**Y**. We will define a point $x \in$ lim**X** such that H(x) = y. By (UN3) it follows

$$(p_{n_{j-2}n_j}, p_{n_{j-2}n_{j-1}}p_{n_{j-1}n_j}) \prec \mathcal{U}_{n_{j-2},n_{j-2}}. \tag{35}$$

This means that

$$(p_{n_{j-2}n_j}(y_{n_j}), y_{n_{j-2}}) \prec \mathcal{U}_{n_{j-2},n_{j-2}}. \tag{36}$$

By (UN2) we have

$$(p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j}), p_{n_{j-3}n_{j-2}}y_{n_{j-2}}) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{37}$$

or

$$(p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j}), y_{n_{j-3}}) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{38}$$

Using (UN3) for n_{j-3} , n_{j-2} , n_j and y_{n_j} we obtain

$$(p_{n_{j-3}n_j}(y_{n_j}), p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j})) \prec \mathcal{U}_{n_{j-3},n_{j-3}}, \tag{39}$$

We infer that

$$(p_{n_{j-3}n_j}(y_{n_j}), y_{n_{j-3}}) \in st(p_{n_{j-3}n_{j-2}}p_{n_{j-2}n_j}(y_{n_j}), \mathcal{U}_{n_{j-3},n_{j-3}}) \tag{40}$$

Repeating this , we infer that for fixed $i \in \mathbb{N}$ and each $k \geq i + 3$

$$p_{n_in_k}(y_{n_k}) \in st(y_{n_i}, \mathcal{U}_{n_i,n_i}). \tag{41}$$

Arguing as in (28) and (29) we see that $\{p_{in_k}(y_{n_k}) : n_k \geq i\}$ is a Cauchy sequence in X_i. Let

$$x_i = lim\{p_{in_k}(y_{n_k}) : n_k \geq i\}. \tag{42}$$

By virtue of Lemma 2.5 there exists $x \in \lim \mathbf{X}$ such that $p_i(x) = x_i$. It remains to prove that $H(x) = y$. From (41) and (42) it follows that

$$x_{n_i} \in st(y_{n_i}, \mathcal{U}_{n_i,n_i}).$$

Arguing as in (35)-(41) we infer that

$$y_{n_i} = lim\{q_{ij}(x_{n_j}) : n_j \geq i\} = lim\{q_{ij}p_{n_j}(x) : n_j \geq i\}. \tag{43}$$

By virtue of (32) and Claim 1. it follows that $H_{n_i}(x) = y_{n_i}$, $i \in \mathbb{N}$, and $H(x) = y$. The proof of the surjectivity of H is completed.

Claim 5. *H is open.* We shall prove that $G = H^{-1}$ is continuous. Let y be any point of $\lim \mathbf{Y}$ and let U be any open neighbourhood of $x = G(y)$. By virtue of the definition of a base in $\lim \mathbf{X}$, there is an open set $U_i \subseteq X_i$ such that $x \in p_i^{-1}(U_i) \subseteq U$. We infer that $x_i = p_i(x) \in U_i$. Let V_i be an open set such that $x_i \in V_i \subseteq \mathrm{Cl} V_i \subseteq U_i$. There exists a cover $\mathcal{A}_{i,j}$ such that $st(x_i , \mathcal{A}_{i,j})$ is contained in V. Moreover, by (UN1), there exists a normal cover $\mathcal{U}_{i,n_k}$ such that

$$st^3 \mathcal{U}_{i,n_k} \prec \mathcal{A}_{i,j}. \tag{44}$$

By virtue of (42) and (41) it follows that there exists a $n_\ell \geq i$ such that

$$p_{in_m}(y_{n_m}) \in st(x_i , \mathcal{U}_{i,n_k}), n_m \geq n_\ell. \tag{45}$$

and

$$p_{n_\ell n_m}(y_{n_m}) \in st(y_{n_\ell}, \mathcal{U}_{n_\ell,n_\ell}), n_m \geq n_\ell. \tag{46}$$

Let $V_{n_\ell} = st(y_{n_\ell} , \mathcal{U}_{n_\ell,n_\ell})$. Then $q_{n_\ell}^{-1}(V_{n_\ell})$ is a neighbourhood of y.For each $z \in q_{n_\ell}^{-1}(V_{n_\ell})$ we have $z_{n_\ell} = q_{n_\ell}(z) \in st(y_{n_\ell}, \mathcal{U}_{n_\ell,n_\ell})$. By virtue of (41)

$$p_{n_\ell n_m}(z_{n_m}) \in st(z_{n_\ell}, \mathcal{U}_{n_\ell,n_\ell}), n_m \geq n_\ell. \tag{47}$$

This means that

$$p_{n_\ell n_m}(z_{n_m}) \in st^2(y_{n_\ell}, \mathcal{U}_{n_\ell, n_\ell}), \; n_m \geq n_\ell. \tag{48}$$

By (UN2) we infer that

$$p_{in_\ell} p_{n_\ell n_m}(z_{n_m}) \in st^2(x_i, \mathcal{U}_{i,n_k}), \; n_m \geq n_\ell. \tag{49}$$

Using (UN3) for i and $\mathcal{U}_{i,n_k}$ we infer that

$$p_{in_m}(z_{n_m}) \in st^3(x_i, \mathcal{U}_{i,n_k}), \; n_m \geq n_\ell. \tag{50}$$

From (44) it follows that

$$p_{in_m}(z_{n_m}) \in V_i, \; n_m \geq n_\ell. \tag{51}$$

We infer that $lim\{p_{in_m}(z_{n_m}) : n_m \geq n_\ell\} \in \mathrm{Cl}V_i \subseteq U_i$. By virtue of (42) we infer that $p_i G(z) \in U_i$, i.e. , $G(z) \in U$. Thus, $G = H^{-1}$ is continuous. This means that H is is open. The proof of Theorem 2.11 is completed.■

COROLLARY 2.12 *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse sequence of compact metric spaces . Then there exist:*

a) *a cofinal subset* $M = \{n_i : i \in \mathbb{N}\}$ *of* $\mathbb{N}$*,*

b) *a usual inverse sequence* $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ *such that* $Y_i = X_{n_i}$ *and* $q_{ij} = p_{ii+1}\, p_{i+1i+2} \cdots p_{j-1j}$ *for each* $i,j \in \mathbb{N}$*,*

c) *a homeomorphism* $H : lim\mathbf{X} \rightarrow lim\mathbf{Y}$.

Proof. Each compact metric space is complete. Apply Theorem 2.11.■

REMARK 2.13 An alternate proof of the above Corollary can be found in Proposition 8. of [4] since each normal cover of a compact metric space X has a Lebesgue number [6, p. 344.]. Thus, each approximate inverse system of compact metric spaces is an approximate inverse system in the sense of M.G. Charalambous [4]. This is not true for non - compact metric spaces. M.G. Charalambous [4, Proposition 8] has a more general result for the inverse sequences of complete metric spaces and uniform bonding mappings.

3 Applications

We start with the following theorem.

THEOREM 3.1 *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse sequence of non-empty complete metric spaces and let* $\mathcal{P}$ *be a topological property which satisfies the following condition:*

(C) If $\mathbf{Z} = \{Z_n, f_{mn}, \mathbb{N}\}$ *is an inverse sequence of spaces having property* $\mathcal{P}$*, then* $\lim\mathbf{Z}$ *has property* $\mathcal{P}$.
Then $\lim\mathbf{X}$ *has the property* $\mathcal{P}$.

Proof. Let $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ be an usual inverse sequence from Theorem 2.11. From (C) it follows that $\lim\mathbf{Y}$ has the property $\mathcal{P}$. By virtue of Theorem 2.11 it follows that $\lim\mathbf{X}$ has property $\mathcal{P}$ since $\lim\mathbf{X}$ is homeomorphic to $\lim\mathbf{Y}$.■

In the sequel we shall give some application of Theorem 3.1.

THEOREM 3.2 *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse system of complete metric spaces. If* $\dim X_n \leq k$*, then* $\dim(\lim\mathbf{X})\leq k$.

Proof. Apply Theorem 3.1 and the usual inverse limit theorem of Nagami [17].■

If the spaces X_n are separable metric spaces, then we have the next theorem which is an approximate version of Theorem 1.13.4 of [7, p. 149.].

THEOREM 3.3 *Let* $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ *be an approximate inverse sequence of separable metric spaces such that* $\dim X_i \leq k, i\in N$.
Then $\dim(\lim\mathbf{X})\leq k$.

Proof. By virtue of Lemma 1.13.3. there are compact metric spaces cX_n which are the extensions of X_n such that $\dim(cX_n)\leq\dim X_n$ and such that each p_{nm} is extendable to a continuous mapping cp_{nm}. Thus, we have the approximate inverse sequence $c\mathbf{X} = \{cX_n, cp_{nm}, \mathbb{N}\}$. By virtue of the above Theorem dim(lim c**X**)$\leq$k. It follows that dim(lim**X**)$\leq$k. The proof is complete.■

A space X is *locally connected (semi-locally connected)* if for each x$\in$X and each open subset U of X such that x$\in$U there is an open subset V with x$\in$V$\subseteq$U and V is connected(X\V has only a finite number of components).

THEOREM 3.4 *Let* $\mathbf{X}=\{X_n, p_{mn}, I\!N\}$ *be an approximate inverse system of complete metric spaces. If the spaces* X_i*,*$i\in N$*,are connected (locally connected) and if the mappings* p_{ij} *are hereditarily quotient monotone surjections,then lim***X** *is connected (locally connected).*

Proof. Consider the inverse system $\mathbf{Y}=\{X_{m_i},q_{ij},M\}$ as in Theorem 2.11. By virtue of [6, p. 134.] and [6, Theorem 6.1.28.] the system **Y** has the hereditarily quotient monotone surjective bonding mappings.If the spaces X_i are connected,then lim**Y** is connected [18, Theorem 11.]. By virtue of Theorem 11. [18] lim**X** is connected.Moreover,by virtue of [18, Theorem 9.] and [18, p. 71.,Corollary] it follows that the projections q_i:lim$\mathbf{Y}\rightarrow X_{m_i}$ are hereditarily quotient and monotone.If the spaces X_{m_i} are locally connected,then from the definition of a base of topology of lim**Y** and [6, Theorem 6.1.28.] it follows that lim**Y** is locally connected.Clearly,lim**X** is locally connected.■

ACKNOWLEDGEMENT. The author is grateful to the referee for his help and valuable suggestions.

Literatura

[1] Aló R.A. and Shapiro H.L.,*Normal topological spaces*, Cambridge Univ. Press,1974.

[2] Arens R.,*Dense inverse limit rings*, Michigan Math. Jorn. 5(1958), 169-182.

[3] Arhangel'skij A. V. , *Otobraženija otkrytye i blizkie k otkrytym. Svjazi meždu prostranstvami* , Trudy moskovskogo matematičeskogo obščestva 15 (1966) , 181 - 223.

[4] M.G. Charalambous,*Approximate inverse systems of uniform spaces and an application of inverse systems*,Comment. Math. Univ. carolinae, 32, 33(1991), 551-565.

[5] Dugundji J.,*Topology*,Wm. C. Brown Publishers,Iowa,1989.

[6] Engelking R.,*General Topology*,PWN,Warszawa,1977.

[7] Engelking R.,*Dimension Theory*,Polish Scientific Publishers, Warszawa 1978.

[8] Fort M.K. and Segal J.,*Local connectednes of inverse limit spaces*, Duke Math. J.,28(1961),253-260.

[9] Ginsburg J. and Saks V.,*Some applications of ultrafilters in topology*,Pacific J. Math. 57 (1975), 403-417.

[10] Lončar I.,*Monotonicity of the projections of the approximate limit*, Glasnik matematički 28 (48) (1993) , 297 - 313.

[11] Lončar I.,*A note on approximate systems of compact spaces*, Zbornik radova FOI Varaždin 17 (1993),135-156.

[12] Mardešić S.,*On approximate inverse systems and resolutions*, Fund. Math. 142(1993),241-255.

[13] Mardešić S. and Rubin L.R.,*Approximate inverse system of compacta and covering dimension*,Pacific J. Math. 138(2):129-144,1989.

[14] Mardešić S. and Watanabe T.,*Approximate resolutions of spaces and mappings*,Glasnik Mat.,24(3):587-637,1989.

[15] Matijević V.,*A note on limit and resolution of (nongauged) approximate inverse systems*, Glasnik Matematički 28 (48) (1993) ,111 - 122.

[16] Mrowka S.,*On the convergence of nets of sets*,Fund. math.,45(2):237-246,1958.

[17] K. Nagami , *Dimension Theory,* Academic Press, New York and London, 1970.

[18] Puzio E.,*Limit mappings and projections of inverse systems,* Fund. math. 80 (1973), 57 - 73.

Primljeno: 1995-02-06

Lončar I. Bilješka o aproksimativnim limesima

SAŽETAK

U radu su izučavani limesi aproksimativnih inverznih sistema $\mathbf{X} = \{X_a, p_{ab}, A\}$u smislu S. Mardešića [10].Glavni rezultati rada su slijedeći:

(a) Ako je $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ aproksimativni inverzni niz nepraznih Čech - kompletnih parakompaktnih prostora,tada je lim$\mathbf{X}$ neprazan Čech - kompletan prostor (Teorem 2.7.).

(b) Ako je $\mathbf{X}=\{X_n, p_{mn}, \mathbb{N}\}$ aproksimativni niz apsolutno G_δ - prostora, tada postoji obični inverzni podniz koji ima limes homeomorfan limesu polaznog aproksimativnog niza (Teorem 2.11.).

(c) U trećem odjeljku dane su neke primjene rezultata (b).