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Soundness and completeness of Saxena and Tripathi's formal system

In this paper, a theory of functional dependencies is introduced (a logical approach). The theory consists of two axioms for predicate $E(X, t_1, t_2)$, where $E(X, t_1, t_2)$ means that tuples t_1 and t_2 are equal on a set of attributes X , and axioms for the theory of finite sets (implicitly). In that theory, we have proved that Saxena and Tripathi's formal system is sound and complete for functional dependencies.

Key words: completeness, formal system, soundness, theory of functional dependencies.

1 Introduction

Logical design of relational database schemas is based on the constraints. Of particular importance are the constraints called functional dependencies ([Armstrong and Delobel 80], [Ginsburg and Zaiddan 82], [Maier 83], [Ullman 88], and [Vardi 88]).

P.C. Saxena and R.C. Tripathi have introduced a formal system for functional dependencies ([Saxena and Tripathi 89]). Their system, called ST-system in this paper, consists of three rules: reflexivity, generalized union, and cancellation.

The purpose of this work is to present the theory of functional dependencies (a logical approach) with the proof of soundness and completeness of ST-system. The paper is organized as follows. In the next section, we give the basic concepts of a theory (in logic). In Section 3, we present our theory of functional dependencies. The theory consists of axioms for the theory of finite sets (implicitly) and two axioms for predicate $E(X, t_1, t_2)$, where $E(X, t_1, t_2)$ means that tuples t_1 and t_2 are equal on a set of attributes X. In Section 4 , we prove that ST-system is sound and complete.

Conclusions are discussed in Section 5.

We assume some familiarity with deductive reasoning, e.g., as described in [Manna and Waldinger 85], and dependency theory, e.g.,as described in [Ullman 88] and [Vardi 88].

2 Characterisation of a theory

A theory consists of a language and a set of sentences (called axioms).

Let T be a theory whose axioms are $A_1, .., A_k, ...$ An interpretation I is a model for the theory T if each axiom A of the theory is true under I.

A sentence F of a theory is valid in the theory if F is true under every model for the theory.

A sentence G is implied by a set of sentences $F_1, .., F_k$ in the theory T if, whenever each F_i is true under a model I for the theory T, G is also true under the model I. It is denoted by $F_1, .., F_k \overset{T}{\models} G$

A theory T_1 is an augmentation of a theory T_2 if the vocabulary of the theory T_2 is a subset of the vocabulary of the theory T_1 and each axiom of T_2 is also axiom of T_1.

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An expression of the form $F_1, .., F_m \overset{T}{\vdash} G$, where T is a theory, $m \geq 0$, and $F_1, .., F_m, G$ are sentences of the theory T, is a rule of T.

The rule $F_1, .., F_m \overset{T}{\vdash} G$ says that we generate G from $F_1, .., F_m$ (in T). A formal system for a theory T is a finite set of rules of the theory T. Let $F_T = \{R_1, .., R_m\}$ be a formal system of the theory T. We shall say that F_T is sound if each rule R_i in F_T is sound.

The rule $R_i : F_1, .., F_i \overset{T}{\vdash} G$ is sound if $F_1, .., F_i \overset{T}{\models} G$.

Now we give the characterization of completeness of a formal system.
Let $F_T = \{R_1, .., R_m\}$ be a formal system of a theory T.

A derivation of a sentence G from sentences $F_1, .., F_k$ in F_T is a sequence of sentences $G_1, .., G_m = G$, where each G_i is either F_j (for some j = 1,..,k) or is generated from preciding sentences in the sequence by a rule R_t in F_T.

If G is derivable from $F_1, .., F_k$ in a formal system F_T, we write $F_1, .., F_k \overset{F_T}{\vdash} G$.

Let F_T be a formal system of a theory T, and S be a set of sentences of T. We shal say that F_T is complete for S if

$$(\forall S_1 \subseteq S)(\forall G \in S)[S_1 \overset{T}{\models} G \Rightarrow S_1 \overset{F_T}{\vdash} G]$$

3 The theory of functional dependencies

The theory of functional dependenies, τ, is an augmentation of the theory of finite sets [Manna and Waldinger 85], whose vocabulary contains

- A ternary predicate symbol $E(X, t_1, t_2)$.

Under the intended models for the theory, the relation $E(X, t_1, t_2)$ is true if tuples t_1 and t_2 are equal on a set attributes X, where X is a subset of the relational schema R.

The Axioms The theory of functional dependencies, τ, is the theory whose axioms include those of the theory of finite sets [Manna and Waldinger 85]] and the following axioms:

A1: $(\forall X, Y)(\forall t_1, t_2)[Y \subseteq X \Rightarrow [E(X, t_1, t_2) \Rightarrow E(Y, t_1, t_2)]]$ (triviality)

A2: $(\forall X, Y)(\forall t_1, t_2)[E(X, t_1, t_2) \wedge E(Y, t_1, t_2) \Rightarrow E(XY, t_1, t_2)]$ (union)

where XY is an abbreviation for $X \cup Y$.

The sentence schema

$$F : (\forall t_1, t_2)[E(U, t_1, t_2) \Rightarrow E(V, t_1, t_2)]$$

is the functional dependency, and is denoted by $U \rightarrow V$.

4 Soundness and completeness of ST-system

ST-system for the theory of functional dependencies, τ, contains three rules:

ST1: $\overset{\tau}{\vdash} U \rightarrow V$ if $V \subseteq U$ (reflexivity)
ST2: $U \rightarrow V,\ W \rightarrow Z \overset{\tau}{\vdash} UW \rightarrow VZ$ (generalized union)
ST3: $UW \rightarrow VW,\ U \rightarrow W \overset{\tau}{\vdash} U \rightarrow V$ (cancellation)

In the following proposition we show that ST-system is sound.
Proposition (the soundness of ST-system)
ST-system is sound, that is, we have

st1: $\overset{\tau}{\models} U \rightarrow V$ if $V \subseteq U$

st2: $U \rightarrow V \overset{\tau}{\models} UW \rightarrow VW$

st3: $UW \rightarrow VW,\quad U \rightarrow W \overset{\tau}{\vdash} U \rightarrow V$

Proof

We prove st1.
Consider arbitrary tuples a and b such that E(U,a,b).
We would like to show that then E(V,a,b).

Because $V \subseteq U$ and E(U,a,b), we have (by the triviality axiom A1) E(V,a,b), as we wanted to show.
Because we have established that E(U,a,b) implies E(V,a,b) for arbitrary tuples a and b , we can conclude that $U \rightarrow V$ holds.

We prove st2.
We have to prove $U \rightarrow V,\quad W \rightarrow Z \quad \overset{\tau}{\models} UW \rightarrow VZ$.

Let a and b be arbitrary tuples such that E(UW,a,b).
We would like to conclude that then E(VZ,a,b). For $U \subseteq UW$ and $W \subseteq UW$, we obtain (by the triviality axiom A1) E(U,a,b) and E(W,a,b).

Because E(U,a,b) and $U \rightarrow V : (\forall t_1, t_2)[E(U, t_1, t_2) \Rightarrow E(V, t_1, t_2)]$, we have E(V,a,b).

Analogously, because E(W,a,b) and $W \rightarrow Z : (\forall t_1, t_2)[E(W, t_1, t_2) \Rightarrow E(Z, t_1, t_2)]$, we obtain E(Z,a,b).

Because we have established that E(V,a,b) and E(Z,a,b), we have (by the union axiom A2) E(VZ,a,b), as we wanted to show.
Because $E(UW, a, b) \Rightarrow E(VZ, a, b)$ holds for arbitrary tuples a and b, we have $UW \rightarrow VZ$.

We prove st3.
We have to prove $UW \rightarrow VW,\ U \rightarrow V \overset{\tau}{\models} U \rightarrow V$.
For arbitrary tuples a and b , suppose that E(U,a,b).
We would like to show that then E(V,a,b).
Because E(U,a,b) and
$U \rightarrow W : (\forall t_1, t_2)[E(U, t_1, t_2) \Rightarrow E(W, t_1, t_2)]$, we have E(W,a,b).

For E(U,a,b) and E(W,a,b), we obtain (by the union axiom A2) E(UW,a,b).
Because E(UW,a,b) and
$UW \rightarrow VW : (\forall t_1, t_2)[E(UW, t_1, t_2) \Rightarrow E(VW, t_1, t_2)]$, we have
E(VW,a,b).
For E(UW,a,b) and $V \subseteq VW$, we obtain (by the triviality axiom A1) E(V,a,b).

Because we have established that
$E(U, a, b) \Rightarrow E(V, a, b)$ holds for arbitrary tuples a and b , we can conclude that $U \rightarrow V$ holds.

To prove the completeness of ST-system, we shall first introduce a relation, called entailment relation, between two formal systems.

Let FS be a formal system and $R_i : F_1, .., F_m \overset{\tau}{\vdash} G$ be a rule of the theory of functional dependencies τ. We shall say that FS entails R_i, denoted FS ■— R_i, if $F_1, .., F_m \overset{F_S}{\vdash} G$

Let FS1 and FS2 be two formal systems of the theory of functional dependencies τ. We shall say that FS1 entails FS2, denoted $FS1$ □— $FS2$, if

$FS1$ ■— R_i, for each rule R_i in FS2.

Proposition (completeness + □—)

Let FS1 and FS2 be two formal systems for τ.
FS1 is complete for functional dependencies if
FS2 is complete for functional dependencies and FS1 □— FS2.

Proof

The proof follows immediately from the definitions of completeness and □—.

In the next proposition, we prove that ST-system is complete for functional dependencies.

Proposition (the completeness of ST-system)

ST-system is complete for functional dependencies.
Proof

We know that Armstrong's formal system AS:

AS1: $\overset{\tau}{\vdash} U \rightarrow V$ if $V \subseteq U$ (reflexivity)

AS2: $U \rightarrow V \overset{\tau}{\vdash} UW \rightarrow VW$ (augmentation)

AS3: $U \rightarrow V, V \rightarrow W \overset{\tau}{\vdash} U \rightarrow W$ (transitivity)

is complete for functional dependencies. For ST □— AS implies the completeness of ST-system, we would like to show that ST □— AS.
Because AS1 is ST1, we have to prove ST ■— AS2 and ST ■— AS3.

The proof of ST ■— AS2 is as follows:

(1) $U \rightarrow V$ hypothesis
(2) $W \subseteq W$ hypothesis
(3) $W \rightarrow W$ (2) and ST1
(4) $UW \rightarrow VW$ (1), (3), and ST2

Now, we give the proof of ST ■— AS3:

(1) $U \to V$ hypothesis
(2) $V \to W$ hypothesis
(3) $UV \to VW$ (1), (2), and ST2
(4) $U \to W$ (3), (1), and ST3

Because we have established ST □— AS, and because AS is complete for functional dependencies, we can conclude, by Proposition (completeness + □—), that ST is complete for functional dependencies.

5 Conclusions

We presented a theory of functional dependencies. The theory of functional dependencies is the theory whose axioms include those of the theory of finite sets (implicitly) and two axioms (the triviality axiom A1 and the union axiom A2) for the ternary predicate $E(X, t_1, t_2)$, where $E(X, t_1, t_2)$ means that tuples t_1 and t_2 are equal on a set of attributes X.

In the theory of functional dependencies, we showed that Saxena and Tripathi's formal system (ST-system) is sound and complete for functional dependencies.

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Maleković M. Korektnost i potpunost Saxena-Tripathijevog formalnog sustava

SAŽETAK

U ovom radu uveli smo teoriju funkcijskih zavisnosti. Teorija se sastoji od aksioma teorije konačnih skupova (implicitno) i dva aksioma (aksiom trivijalnosti i aksiom unije) za ternarni predikat $E(X, t_1, t_2)$*, gdje* $E(X, t_1, t_2)$ *znači jednakost slogova* t_1 *i* t_2 *na skupu atributa X. U teoriji funkcijskih zavisnosti dokazali smo da je formalni sistem ST, kojeg su predložili Saxena i Tripathi u [Saxena and Tripathi 89], korektan i potpun za funkcijske zavisnosti.*