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A note on approximate systems of compact spaces

In this paper we define a space $\sigma(\underline{\mathbf{X}})$ for approximate system of compact spaces. The construction is due to H.Freudenthal for usual inverse sequence [4,pp. 153-156]. We establish the following properties of this space: (1) The space $\sigma(\underline{\mathbf{X}})$is a paracompact space,(2) Moreover,if $\underline{\mathbf{X}}$ is an approximate sequence of compact (metric) spaces,then $\sigma(\underline{\mathbf{X}})$is a compact (metric) space (Lemma 2.4.). We give the following applications of the space $\sigma(\underline{\mathbf{X}})$: (3) If $\underline{\mathbf{X}}$ is an approximate system of continua,then X=lim$\underline{\mathbf{X}}$ is a continuum (Theorem 3.1),(4) If $\underline{\mathbf{X}}$ is an approximate system of hereditarily unicoherent spaces,then X=lim$\underline{\mathbf{X}}$ is hereditarily unicoherent (Theorem 3.6.),(5) If $\underline{\mathbf{X}}$ is an approximate system of the arboroids (generalized trees,trees,arcs) with monotone onto bonding mappings,then X=lim$\underline{\mathbf{X}}$ is an arboroid (generelized tree,tree,arc) (Theorems 3.18.,3.20.,3.21.,3.25.).

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1 Introduction

Let $\mathcal{U}$ be any covering of a space X. For any subset Y of X we define St(Y,$\mathcal{U}$)=$\bigcup\{$U$\in \mathcal{U}$:U$\bigcap$ Y $\neq \emptyset\}$.

Similarly,we define St$\mathcal{U}$ = $\{$St(U,$\mathcal{U}$):U$\in \mathcal{U}\}$. Inductively,for each positive integer n,$St^n\mathcal{U} = St(St^{n-1}\mathcal{U})$,where $St^1\mathcal{U}$=St$\mathcal{U}$.

We say that a cover $\mathcal{V}$ is a *star refinement* of a cover $\mathcal{U}$ if the cover St$\mathcal{V}$ is a refinement of $\mathcal{U}$.

An open cover $\mathcal{W}$ of a space X is *normal* [3,pp. 379] if there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, ...$ of open covers of a space X such that $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{W}_{i+1}$ is a *star refinement* of $\mathcal{W}_i$ for i=1,2,... A T_1 space X is paracompact iff each open cover of X is normal [3,Theorem 5.1.12.]. A T_1 space X is normal iff each locally finite open cover of X is normal [3,pp. 379].The set of all normal covers of X is denoted by Cov(X).

If $\mathcal{U}$, $\mathcal{V}\in$Cov(X) and $\mathcal{V}$ refines $\mathcal{U}$,we write $\mathcal{V}\prec\mathcal{U}$.If f,g:Y→X are $\mathcal{U}$-*near* mappings,i.e. if for any y∈Y there exists U∈ $\mathcal{U}$ with f(y),g(y)∈U,we write (f,g)$\prec\mathcal{U}$.

The approximate inverse systems were introduced by S. Mardešić and L.R. Rubin [12] for compacta and by S. Mardešić and Watanabe [13] for general topological spaces.

DEFINITION 1.1 An *approximate inverse system* $\underline{X}=\{X_a,\mathcal{U}_a,p_{ab},A\}$ consists of the following data: A preordered set (A,≤) which is directed and has no maximal element; for each a∈A,a topological space X_a and a normal covering $\mathcal{U}_a$ of X_a (called the mesh of X_a) and for each pair a≤b from A,a mapping $p_{ab}:X_b\to X_a$.Moreover the following three conditions must be satisfied:

(A1) The mappings $p_{ab}p_{bc}$ and p_{ac} are $\mathcal{U}_a$- near,a≤b≤c,i.e. $(p_{ab}p_{bc},p_{ac})\prec\mathcal{U}_a$.

(A2) For each a∈A and each normal cover $\mathcal{U}\in$Cov(X_a) there is an index b≥a such that $(p_{ac}p_{cd},p_{ad})\prec\mathcal{U}$,whenever a≤b≤c≤d.

(A3) For each a∈A and each normal cover $\mathcal{U}$ ∈Cov(X_a) there is an index b≥a such $\mathcal{U}_c\prec p_{ac}^{-1}(\mathcal{U})=\{p_{ac}^{-1}(U):U\in\mathcal{U}\}$ for each c≥b.

In the case of metric compact spaces we replaces the normal coverings by real numbers [12].

If the spaces X_a are T_1 paracompact,then in the above definition one can use all open coverings on a spaces X_a,a∈A since in this case each open cover is normal.

Let $\underline{X}=\{X_a,\mathcal{U}_a,p_{ab},A\}$ be an approximate system.A point $x=(x_a)\in\prod\{X_a:a\in A\}$ is called a *thread* of $\underline{X}$ provided it satisfies the following condition:

(L) $(\forall a\in A)(\forall\mathcal{U}\in \mathrm{Cov}(X_a))(\exists b\geq a)(\forall c\geq b)p_{ac}(x_c)\in \mathrm{st}(x_a,\mathcal{U})$.

The *canonical limit* of $\underline{X}$ is a subset of the product of the spaces X_a [12,pp. 592].

If X_a is a $T_{3.5}$ space,then the sets $\mathrm{st}(x_a,\mathcal{U}),\mathcal{U}\in\mathrm{Cov}(X_a)$, form a basis of the topology at the point x_a.Therefore,for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L)* $(\forall a\in A)\lim\{p_{ac}(x_c):c\geq a\}=x_a$.

THEOREM 1.2 *For any approximate inverse system $\underline{X}$ the canonical limit $\lim\underline{X}$ is closed in $\prod X_a$.Moreover,if all X_a are compact and non-empty,then $\lim\underline{X}$ is compact and non-empty.*

Proof. See Lemma (1.16) and Theorem (4.1) of [13].

LEMMA 1.3 *Let $\underline{X}=\{X_a,\mathcal{U}_a,p_{ab},A\}$be an approximate system of Tychonoff spaces,let X be the canonical limit of $\underline{X}$ and let B⊆A be a cofinal subset of A.Then the collection $\mathcal{B}$ of all sets of the form $p_b^{-1}(U_b)$,where b∈B and $V_b\subseteq X_b$ is open,is a basis of the topology for X.*

Proof. See [13,(1.18) Lemma.]■

THEOREM 1.4 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces with limit X.For each closed* $F \subseteq X$ *we have*

$$F = \bigcap\{p_a^{-1}(p_a(F)) : a \in A\}.$$

Proof. It is obvious that $F \subseteq p_a^{-1}(p_a(F))$ for each $a \in A$.Thus,$F \subseteq \bigcap\{p_a^{-1}(p_a(F)) : a \in A\}$.If $x \notin F$,then,by the property (B1)* [13,pp. 614.] we infer that there exists an $a \in A$ and an open set $U_a \subseteq X_a$ such that $x \in p_a^{-1}(U_a) \subseteq X - F$.This means that $p_a(x) \notin p_a(F)$ and $x \notin p_a^{-1}(p_a(F))$.■

THEOREM 1.5 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an inverse system of compact spaces with limit X.For each pair of disjoint closed sets* $F, G \subseteq X$ *there exists an* $a \in A$ *such that for each* $b \geq a$ $p_b(F) \bigcap p_b(G) = \emptyset$.

Proof. Let U,V be disjoint open sets about F,G.There exists a pair U_0, V_0 of open sets such that $F \subseteq U_0 \subseteq ClU_0 \subseteq U$ and $G \subseteq V_0 \subseteq ClV_0 \subseteq V$. By virtue of the property (B2) there exists an $a \in A$ and an open cover $\mathcal{U}_a$ of X_a such that $p_a^{-1}\mathcal{U}_a$ refines $\mathcal{U} = \{U, V, U_0, V_0, X-(ClU_0 \bigcup ClV_0)\}$. Let $\mathcal{V}_a = \{V_b : b \in B\}$ be the star refinement of $\mathcal{U}_a$. Consider the sets $B_0 = \{b \in B : \emptyset \neq p_a^{-1}(V_b) \subseteq U_0\}$ and $B_1 = \{b \in B : \emptyset \neq p_a^{-1}(V_b) \subseteq V_0\}$. Clearly B_0 and B_1 are disjoint and non-empty. Now we consider the sets $W_0 = \bigcup\{V_b : b \in B_0\}$ and $W_1 = \bigcup\{V_b : b \in B_1\}$.Let us prove $W_0 \bigcap W_1 = \emptyset$.If we assume that $W_0 \bigcap W_1 \neq \emptyset$,then $V_{b_0} \bigcap V_{b_1} \neq \emptyset$ for some $b_0 \in B_0$ and $b_1 \in B_1$.Since $st\mathcal{V}_a$ refines $\mathcal{U}_a$,there exists a $W \in \mathcal{U}_a$ such that $V_{b_0} \bigcup V_{b_1} \subseteq W$. Therefore,$U_0 \bigcap p_a^{-1}(W) \supseteq p_a^{-1}(V_{b_0}) \neq \emptyset$ and $V_0 \bigcap p_a^{-1}(W) \supseteq p_a^{-1}(V_{b_1}) \neq \emptyset$. This induces a contradiction,because $p_a^{-1}\mathcal{U}_a$ refines $\mathcal{U}$ and $U_0 \bigcap V_0 = \emptyset$.Thus,$W_0 \bigcap W_1 = \emptyset$. Moreover,we have that $st(p_a(U_0), \mathcal{V}_a) = W_0$ and $st(p_a(U_1), \mathcal{V}_a) = W_1$.■

By total induction and by the above Theorem we obtain

THEOREM 1.6 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an inverse system of compact spaces with limit X.For each finite collection* $\{F_1, \ldots, F_n\}$ *of mutually disjoint closed subsets of X there exists an* $a \in A$ *such that for each* $b \geq a$ $\{p_b(F_1), \ldots, p_b(F_n)\}$ *is a collection of mutually disjoint sets.*

THEOREM 1.7 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an inverse system of compact spaces with limit X.For each closed* $F \subseteq X$ *and each open* $U \supseteq F$ *there exists an* $a \in A$ *such that for each* $b \geq a$ *there is an open set* $U_b \supseteq p_b(F)$ *with the property*

$$p_b^{-1}(p_b(F)) \subseteq p_b^{-1}(U_b) \subseteq U.$$

We close this section with

THEOREM 1.8 *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system. For each* $a \in A$ *the family* $\mathcal{P} = \{p_{ab}(X_b) : b \geq a\}$ *is a net in* X_a *such that* $Lim\mathcal{P} = p_a(X), X = lim\underline{\mathbf{X}}$.

Proof. From the definition of the threads it follows that $p_a(X) \subseteq Li\mathcal{P}$. On the other hand, from the property (B2) we infer that if $x \notin p_a(X)$, then $x \notin Li\mathcal{P}$. Thus, $p_a(X) \supseteq Ls\mathcal{P} \supseteq Li\mathcal{P}$. Therefore, $Lim\mathcal{P} = p_a(X), X = lim\underline{\mathbf{X}}$. ■

2 The Freudenthal space $\Sigma(\underline{\mathbf{X}})$

The following construction is similar to the construction due to H. Freudenthal [4,pp. 153.] for usual inverse sequence. For any usual inverse system see [11].

Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of compact spaces with limit X and the projections $p_a : X = lim\underline{\mathbf{X}} \to X_a$. The Freudenthal space $\sigma(\underline{\mathbf{X}})$ associated to $\underline{\mathbf{X}}$ is a set

$$\sigma(\underline{\mathbf{X}}) = X \bigcup (\bigcup \{X_a : a \in A\}) \tag{1}$$

where all X_a and their limit X are considered as being disjoint sets [11], in which a topology is defined as follows. If U_a is an open set in X_a, let

$$U_a^* = \bigcup \{p_{ab}^{-1}(U_a) : b \geq a\} \bigcup p_a^{-1}(U_a) \tag{2}$$

Now, we define a topology T on $\sigma(\underline{\mathbf{X}})$ by a base [3,pp. 27] $\mathcal{B}$ which consists of all open sets U_a in all X_a and all U_a^* for all open sets $U_a \subseteq X_a, a \in A$. Since the sets $p_a^{-1}(U_a)$ form a basis for X, it follows that $\mathcal{B}$ is a cover of $\sigma(\underline{\mathbf{X}})$. By virtue of [3,pp. 27] we need to prove that for each $x \in \sigma(\underline{\mathbf{X}})$ and each pair $B, C \in \mathcal{B}$ with $x \in B \bigcap C$ there is a $D \in \mathcal{B}$ such that $x \in D \subseteq B \bigcap C$. It suffices to prove this statement if B is some U_a^* and C is some U_b^*. If x is the point of X_c, then x is contained in a set $p_{ac}^{-1}(U_a) \bigcap p_{bc}^{-1}(U_b)$ which is open in X_c and thus belongs to $\mathcal{B}$. If x is the point of X, then

$$x \in p_a^{-1}(U_a) \bigcap p_b^{-1}(U_b) \tag{3}$$

i.e., $x_a = p_a(x) \in X_a$ and $x_b = p_b(x) \in X_b$. Choose $\mathcal{V}_a \in Cov(X_a), \mathcal{V}_b \in Cov(X_b)$ such that

$$St(x_a, \mathcal{V}_a) \subseteq U_a \quad and \quad St(x_b, \mathcal{V}_b) \subseteq U_b \tag{4}$$

Take $\mathcal{W}_a \in Cov(X_a), \mathcal{W}_b \in Cov(X_b)$ such that $St^2\mathcal{W}_a < \mathcal{V}_a, St^2\mathcal{W}_b < \mathcal{V}_a$ and an index $c \in A$ such that $c \geq a, b$, (A2) and (A3) hold for $a, b, \mathcal{W}_a, \mathcal{W}_b$ and (L) holds for $x, a, b, \mathcal{W}_a, \mathcal{W}_b$. Put

$$V_c = St(x_c, \mathcal{U}_c) \tag{5}$$

Because of (3),the proof will be completed if we show that

$$x \in V_c^* \subseteq U_a^* \bigcap U_b^* \tag{5.1}$$

We first prove

$$x \in p_c^{-1}(V_c) \subseteq p_a^{-1}(U_a) \bigcap p_b^{-1}(U_b) \tag{6}$$

Consider a point $y=(y_a) \in p_c^{-1}(V_c)$.By (5) there is an $U_1 \in \mathcal{U}_c$ such that

$$x_c, y_c \in U_1 \tag{7}$$

By the choice of c (property (A3)) $\mathcal{U}_c < p_{ac}^{-1}(\mathcal{W}_a)$ and $\mathcal{U}_c < p_{bc}^{-1}(\mathcal{W}_b)$.
This means that there is a $W_1 \in \mathcal{W}_a$ and a $W_2 \in \mathcal{W}_b$ such that $U_1 \subseteq p_{ac}^{-1}(W_1)$ and $U_1 \subseteq p_{bc}^{-1}(W_2)$.Thus,(7) implies

$$p_{ac}(x_c), p_{ac}(y_c) \in W_1 \quad and \quad p_{bc}(x_c), p_{bc}(y_c) \in W_2 \tag{8}$$

By the choice of c (property (L)),there are a $W_3 \in \mathcal{W}_a, W_4 \in \mathcal{W}_b$ such that

$$x_a, p_{ac}(x_c) \in W_3 \quad and \quad x_b, p_{bc}(x_c) \in W_4 \tag{9}$$

Since $y \in p_b^{-1}(U_b) \subseteq X$,there is a $d \geq c$ sastisfying (L) for y,a,b,$\mathcal{W}_a$,$\mathcal{W}_b$ and for y,b,$\mathcal{W}_b$.
Thus,there exist a $W_5 \in \mathcal{W}_a, W_6 \in \mathcal{W}_b$ and an $U_4 \in \mathcal{U}_c$ such that

$$p_{ad}(y_d), y_a \in W_5 \quad and \quad p_{bd}(y_d), y_b \in W_6 \tag{10}$$

and

$$p_{cd}(y_d), y_c \in U_4 \tag{11}$$

By the choice of c (property (A3)),$\mathcal{U}_c < p_{ac}^{-1}(\mathcal{W}_a$ and $\mathcal{U}_c < p_{bc}^{-1}(\mathcal{W}_b$.Hence,there exist a $W_7 \in \mathcal{W}_a$ and $W_8 \in \mathcal{W}_b$ such that $U_4 \subseteq p_{ac}^{-1}(W_7)$ and $U_4 \subseteq p_{bc}^{-1}(W_8)$.By (11) we have

$$p_{ac}p_{cd}(y_d), p_{ac}(y_c) \in W_7 \quad and \quad p_{bc}p_{cd}(y_d), p_{bc}(y_c) \in W_8 \tag{12}$$

By the choice of c (property (A2)),we also have a $W_9 \in \mathcal{W}_a$ and a $W_{10} \in \mathcal{W}_b$ such that

$$p_{ac}p_{cd}(y_d), p_{ad}(y_d) \in W_9 \quad and \quad p_{bc}p_{cd}(y_d), p_{bd}(y_d) \in W_{10} \tag{13}$$

Now,(9),(8),(12),(13),(10),$St^2\mathcal{W}_a < \mathcal{V}_a$ and $St^2\mathcal{W}_a < \mathcal{V}_b$ yield a $V' \in \mathcal{V}_a$ and a $V'' \in \mathcal{V}_b$ such that $x_a, y_a \in W_1 \bigcup W_3 \bigcup W_5 \bigcup W_7 \bigcup W_9 \subseteq V'$ and $x_b, y_b \in W_2 \bigcup W_4 \bigcup W_6 \bigcup W_8 \bigcup W_{10}$ $\subseteq V''$.

This and (4) imply $p_a(y)=y_a \in St(x_a,\mathcal{V}_a) \subseteq U_a$ and $p_b(y)=y_b \in St(x_b,\mathcal{V}_b) \subseteq U_b$.This means that $y\in p_a^{-1}(U_a)\bigcap p_b^{-1}(U_b)$, i.e. , (6) is proved. In order to prove (5.1.) it suffices to prove

$$p_{cd}^{-1}(V_c) \subseteq p_{ad}^{-1}(U_a)\bigcap p_{bd}^{-1}(U_b) \quad \forall d \geq c \tag{14}$$

Let $z_d \in p_{cd}^{-1}(V_c)$.By (A2) we infer there are $W_{11} \in \mathcal{W}_a$ and $W_{12} \in \mathcal{W}_b$ such that

$$p_{ac}p_{cd}(z_d), p_{ad}(z_d) \in W_{11} \quad and \quad p_{bc}p_{cd}(z_d), p_{bd}(z_d) \in W_{12} \tag{15}$$

Since $p_{cd}(z_d)\in V_c$ we have by (A3)

$$p_{ac}(x_c), p_{ac}(p_{cd}(z_d)) \in W_1 \quad and \quad p_{bc}(x_c), p_{bc}(p_{cd}(z_d)) \in W_2 \tag{16}$$

From (15),(16) and (L) for x,a,b,$\mathcal{W}_a$,$\mathcal{W}_b$ (choise of d) it follows $x_a, p_{ad}(z_d)\in St\mathcal{V}_a$ and $x_a, p_{ad}(z_d)\in St\mathcal{V}_b$.By (4) $p_{ad}(z_d)\in U_a$ and $p_{bd}(z_d)\in U_b$. We infer that $z_d \in p_{ad}^{-1}(U_a)\bigcap p_{bd}^{-1}$ (U_b) and (14) is proved.Hence,we have $x\in V_c^* \subseteq U_a^* \bigcap U_b^*$, i.e., (5.1.) is proved.This means that $\mathcal{B}$ is a basis for some topology T on $\sigma(\underline{\mathbf{X}})$.

A *net in a topological space* X [3,pp. 73.] is an arbitrary function from a non-empty directed set D to the space X.Nets will be denoted by $\mathcal{N} = \{x_d:d\in D\}$.A point $x\in X$ is called a *limit* of a net $\mathcal{N} = \{x_d:d\in D\}$ if for every neighborhood U of x there is an index $d_0 \in D$ such that $x_d \in U$ for each $d\geq d_0$.We say that the net $\mathcal{N}$ *converges* to x.A point is called a *cluster* point of a net $\mathcal{N} = \{x_d:d\in D\}$ if for every neighborhood U of x and every $d_0 \in D$ there exists an index $d\geq d_0$ such that $x_d \in U$.

LEMMA 2.1 *Let* $\underline{\mathbf{X}} = \{X_a,\mathcal{U}_a,p_{ab},A\}$ *be an approximate system of non-empty compact spaces with limit X.*

1. *If B is cofinal in A,then each family* $\mathcal{N}=\{x_a:x_a \in X_a, a\in B\}$ *is a net in* $\sigma(\underline{\mathbf{X}})$ *which has at least one cluster point* $x\in X\subseteq \sigma(\underline{\mathbf{X}})$.
2. *Each point* $x\in X$ *is a limit of the net* $\{p_a(x_a):a\in B\}$.

Proof. For each $a\in A$ we consider a net $\mathcal{N}_a=\{p_{ab}(x_b):b\geq a,b\in B\}$.From the compactness of X_a it follows that a set C_a of all cluster points of $\mathcal{N}_a$ is non-empty.Clearly,each C_a is closed and compact in X_a.First,we prove

(a) *For each* $a\in A$ C_a *is a non-empty subset of* $p_a(X)$.

If we suppose that some $c_a \notin p_a(X)$,then c_a and $p_a(X)$ have disjoint neighborhood U and V.By virtue of the property (B3) * [13,pp. 606,615] there is an index $b\geq a$ such that $p_{ac}(X_c)\subseteq V$ for each $c\geq b$.This is impossible since there exists an index $c\geq b, c\in B$,such that $p_{ac}(x_c)\in V$ (c_a is a cluster point of the net $\mathcal{N}$).

From (a) easy follows

(b) *For each $a \in A$ the set $p_a^{-1}(C_a)$ is non-empty.*

(c) *For each $a \in A$ and each neighborhood U_a of C_a there is an index $b \geq a$ such that $p_{ac}(C_c) \subseteq U_a$ for each $c \geq b$.*

By virtue of the normality of X_a there is an open set V_a such that $C_a \subseteq ClV_a \subseteq U_a$. If we suppose that for each $b \in A$ there is an index $c \geq b$ such that $p_{ac}(C_c)$ intersects $F_a = X_a \setminus U_a$, then we have a net $\mathcal{M} = \{p_{ac}(x_c)\}$ in the compact set F_a. This means that there is a point f_a in F_a which is a cluster point of the net $\mathcal{M}$. Now, we prove that f_a is a cluster point of the net $\mathcal{N}$. Let W_a be any neighborhood of f_a and let $\mathcal{U}$ be a cover of X_a such that $St\mathcal{U}$ is a refinement of a cover $\{U_a, X_a \setminus ClV_a, W_a\}$. There is an index $d \in A$ such that $p_{ad}(c_d) \in St^2(f_a, \mathcal{U})$ since f_a is a cluster point of $\mathcal{M}$. We may assume that there is a $x_e \in \mathcal{N}$ such that $p_{de}(x_e), c_d \in V_e \in \mathcal{U}_e$ since c_d is a cluster point of $\mathcal{N}_d$. We take such index $e \in A$ which satisfies (A2) and (A3). Thus, we have $p_{ad}p_{de}(x_e), p_{ad}c_d \in St^2\mathcal{U}$ and $p_{ad}p_{de}(x_e), p_{ae}x_e \in St\mathcal{U}$. We infer that $f_a, p_{ae}(x_e) \in U_1 \in \mathcal{U}$. Finally, $f_a, p_{ae}(x_e) \in W_a$. This means that f_a is a cluster point of $\mathcal{N}$. This is impossible since $C_a \subseteq V_a$. The proof of (b) is completed.

By the same method of proof, using the definition of the threads, we have

(d) *For each $a \in A$ and each neighborhood U_a of C_a there is an index $b \geq a$ such that $p_a(p_c^{-1}(C_c)) \subseteq U_a$ for each $c \geq b$.*

(e) *The set $C = \bigcap \{p_a^{-1}(C_a) : a \in A\}$ is non-empty subset of X.*

Let K be a set of all cluster points of nets $\{y_a : y_a \in p_a^{-1}(C_a) : a \in A\}$. Clearly, K is non-empty since X is compact. Let us prove K=C. It suffices to prove $K \subseteq C$ since the inclusion $C \subseteq K$ is obvious. Let k be any point of K. Suppose that $p_a(k)$ is not in some C_a. There are disjoint open sets U, V such that $C_a \subseteq U$ and $p_a(k) \in V$. By virtue of (d) there is an index $b \in A$ such that $p_a p_c^{-1}(C_c)$ is in U. This means that $p_a^{-1}(V)$ is a neighborhud of k which contains no points of $p_c^{-1}(C_c)$ for each $c \geq b$. This is impossible since k is a cluster point some net $\{y_a : y_a \in p_a^{-1}(C_a) : a \in A\}$. We conclude that $K \subseteq C$ and K=C.

In order to complete the proof it suffices to prove that each $k \in K$ is a cluster point of the net $\mathcal{N}$. Let U_a^* be any neighborhood of k in $\sigma(\underline{X})$. This means that $k \in p_a^{-1}U_a$ and $p_a(k) \in U_a$. Since $p_a(K) \in C_a$ (see (a)), we infer that $p_a(k)$ is a cluster point of $\mathcal{N}_a$. Thus, for each $b \in A$ there is an index $c \geq b$ such that $p_{ac}(x_c) \in U_a$, where $x_c \in \mathcal{N}$. This means that $x_c \in p_{ac}^{-1}(U_a)$, i.e., $x_c \in U_a^*$. The proof is completed since the second statement easy follows from the definition of the topology T on $\sigma(\underline{X})$. ■

LEMMA 2.2 *Let $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of compact spaces. If U is a neighborhood of $X = \lim \underline{X}$ in $\sigma(\underline{X})$, then there exists an $a \in A$ such that $X_b \subseteq U$ for each $b \geq a$.*

Proof.Since X is compact and since the sets (2) form a basis for open sets of the points of X,one can find $\{U^*_{a_i}:i=1,...,n\}$ such that

$$V = \bigcup\{U^*_{a_i} : i = 1, ..., n\} \tag{17}$$

and $X \subseteq V \subseteq U$.In order to complete the proof,it suffices to find an $a \in A$, $a \geq a_1,..., a_n$, such that

$$X_a \subseteq V \tag{18}$$

since then we have

$$X_b \subseteq V \subseteq U, b \geq a \tag{19}$$

Suppose that not exists an $a \in A$ which satisfies (18).This means that for each $a \in A$ there is a point $x_a \in X_a - V$.We obtain a net $\{x_a:a \in A\}$ in $\sigma(\underline{X})$ which has no a cluster point in $V \supseteq X$.This contradicts Lemma 2.1.The proof is completed.■

LEMMA 2.3 *Let* $\underline{X} = \{X_a,\mathcal{U}_a,p_{ab},A\}$*be an approximate system of compact spaces.Then* $\sigma(\underline{X})$ *is paracompact.Moreover,if* $\underline{X}$ *is an approximate sequence, then* $\sigma(\underline{X})$ *is compact.*

Proof. Let $\mathcal{V}=\{V_\mu\}$ be any cover of $\sigma(\underline{X})$.Since X is compact,there is a finite subcollection,consisting of sets $V_{\mu(1)},...,V_{\mu(n)}$ which cover X.Let V be the union of this subcollection.By virtue of Theorem 2.2. there is an $a \in A$ such that all X_b,$b \geq a$,are in V.Let us recal that the set $X^*_a = (\bigcup\{X_b:b \geq a\}\bigcup X$ is of type (2) with $U_a=X_a$ and it is open in $\sigma(\underline{X})$.Now consider the following collection $\mathcal{U}$ of open sets of $\sigma(\underline{X})$:take first the open sets $X^*_a \bigcap V_{\mu(1)},...,X^*_a \bigcap V_{\mu(n)}$ for members of $\mathcal{U}$.Furthermore,for each $b \in A - \{c:c \in A,c \geq a\}$ consider the open covering $\{X_b \bigcap V_\mu\}$ of X_b and take the members of finite subcovering as new members of $\mathcal{U}$.This is possible since X_b is compact and open in $\sigma(\underline{X})$.The family $\mathcal{U}$ of open sets of $\sigma(\underline{X})$ is star-finite covering of $\sigma(\underline{X})$ which refines the covering $\mathcal{V}$.Moreover,$\mathcal{U}$ is a locally finite refinement of $\mathcal{V}$.The proof of paracompactnes is completed.If $\underline{X}$ is an approximate sequence,then we obtain a finite subcovering since the set $A - \{c:c \in A,c \geq a\}$ is finite.The proof is completed.■

Let us note that from the proof of Lemma 2.3. it follows

THEOREM 2.4 *Let* $\underline{X}=\{X_n,\epsilon_n,p_{mn},N\}$*be an approximate inverse sequence of compact metric spaces* X_n*. Then* $\sigma(\underline{X})$ *is a compact metric space.*

Proof. The space $\lim\underline{X}$ is a compact space.Thus,$\lim\underline{X}$ has a countable base $\mathcal{B}_1$ since it is metrizable space [3,4.1.15. Theorem].This means that the cardinality of $\mathcal{B}$ is $\aleph_0$.Thus,the space $\sigma(\underline{X})$ is metrizable [3,pp. 351].■

We close this Section with the following theorem which is similar to the theorem for usual inverse system of compact spaces due to S. Mardešić [11, Theorem 4.].

THEOREM 2.5 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of compact spaces and let $f:X \rightarrow R$ be a mapping of its limit into a simplicial complex. Then there exists an $a \in A$ such that for each $b \geq a$ one can define a mapping $f_b : X_b \rightarrow R$ with the property that $f_b p_b$ is homotopic to f.*

Proof. We may assume that R is a finite simplicial complex since f(X) is compact and is contained in a finite subcomplex of R. There is a mapping $F:V \rightarrow R$, where V is a neighbourhood of X in $\sigma(\underline{\mathbf{X}})$, which is an extension of f since each finite simplicial complex is an ANE for normal spaces. For each point $x \in X$ there is a simplex s(x) of R such that $f(x) \in s(x)$. From the continuity of F it follows that there are a neighbourhood $V(x) \subseteq V$ of x in $\sigma(\underline{\mathbf{X}})$ such that F(V(x)) is a subset of s(x). Let W be the union of all V(x). Obviosly, W is the neighbourhood of X in $\sigma(\underline{\mathbf{X}})$. By virtue of Lemma 2.2. there is an $a \in A$ such that $X_b \subseteq W$, for each $b \geq a$. For each $x \in X$ we have $p_b(x) \in V(x)$ and $F(p_b(x)) \in s(x)$. We define f_b as the restriction $F|X_b$. The homotopy $H(x,t)=tf(x)+(1-t)f_b p_b(x)$ is the desired homotopy. The proof is completed.■

3 Applications

In this Section we give some applications of the space $\sigma(\underline{\mathbf{X}})$. We start with

THEOREM 3.1 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an aproximate system of continua. The space $X = \lim \underline{\mathbf{X}}$ is a continuum.*

Proof. By virtue of 1.2. X is compact. Suppose that X is not connected. There is a pair F,G of closed (in X) disjoint subsets of X. Since X is closed in $\sigma(\underline{\mathbf{X}})$, the sets F and G are closed in normal space $\sigma(\underline{\mathbf{X}})$ (Lemma 2.3). There are two disjoint open (in $\sigma(\underline{\mathbf{X}})$) sets U and V which contain F and G. By virtue of Lemma 2.2. there is an $a \in A$ such that X_b is contained in $U \bigcup V$ for each $b \geq a$. It is easy to prove that X_b intersects U and V. This is impossible since X_b is connected.■

In the sequel we use the notion of a net of sets in the sense of [14] or [8,pp. 343.].

A *net of sets* $\{A_n : n \in D\}$ of topological space X is a function [14] defined on a directed set D which assigns to each $n \in D$ a subset A_n of X.

If $\{A_n : n \in D\}$ is a net of subsets of X, then:

1. A *limit inferior* LiA_n is the set of all points $x \in X$ such that for every neighborhood U of x there exists $n_0 \in D$ such that U intersect A_n for each $n \geq n_0$.

2. A *limit superior* LsA_n is the set of all points $x \in X$ such that for every neighborhood U of x and each $n_0 \in N$ there is $n \geq n_0$ such that U intersect A_n.

A net $\{A_n{:}n\in D\}$ is said to be *topologically convergent* (to a set A) if $LsA_n=LiA_n$ $(=A)$ and in this case the set A will be denoted by $LimA_n$.

LEMMA 3.2 *Let $\{C_n{:}n\in D\}$ be a net of subsets of a space X such that $LiC_n\neq\emptyset$. Let U be a neighbourhood of LsC_n such that $X\backslash U$ is compact. Then there is an $m\in D$ such that $C_p\subseteq U$ for each $p\geq m$.*

Proof. Suppose, on the contrary, that for each $m\in D$ there is a $p\in D$ such that $Z_p=C_p\backslash U$ is non-empty. Let z_p be any point of Z_p and let P be a set of all such $p\in D$. A net $\{z_p{:}p\in P\}$ has a cluster point z in a compact $X\backslash U$. This is imposible since $z\in LsC_n\subseteq U$. The proof is completed.■

LEMMA 3.3 *Let $\{C_n{:}n\in D\}$ be a net of connected sets C_n of a normal space X such that $LiC_n\neq\emptyset$. If for each neighborhood U of LsC_n the set $X\backslash U$ is compact, then LsC_n is connected.*

Proof. Suppose that LsC_n is disconnected. This means that there are disjoint closed subsets F and G of LsC_n such that $LsC_n=F\bigcup G$. The sets are closed in X since LsC_n is closed in X. From the normality of X it follows that there are two disjoint open sets U and V such that $F\subseteq U$ and $G\subseteq V$. This means that $LsC_n\subseteq U\bigcup V$. We infer that either $LiC_a\bigcap U\neq\emptyset$ or $LiC_a\bigcap V\neq\emptyset$. Let $LiC_a\bigcap U\neq\emptyset$. By virtue of Lemma 3.2. there is a $m\in D$ such that $C_p\subseteq U\bigcup V$ for each $p\geq m$. Clearly, there is some $p\geq m$ such that C_p intersects U (since $LiC_a\bigcap U\neq\emptyset$) and C_p intersects V (since $V\bigcap LsC_n\neq\emptyset$). This means that $C_p\subseteq U\bigcup V$ and $U\bigcap C_p\neq\emptyset, V\bigcap C_p\neq\emptyset$. This contradicts the connectednes of C_p.■

LEMMA 3.4 *Let $\underline{X}=\{X_a,\mathcal{U}_a,p_{ab},A\}$ be an aproximate system of the compact spaces. Let $\{C_a{:}a\in A, C_a\subseteq X_a\}$ be a net of continua such that LiC_a is non-empty. Then LsC_a is a non-empty subcontinuum of X.*

Proof. It is clear that $LiC_a\subseteq LsC_a\subseteq X$. Suppose that LsC_a is disconnected. We infer that there is a pair F,G of disjoint closed subsets of LsC_a such that $LsC_a=F\bigcup G$. The sets F and G are closed in X and in $\sigma(\underline{X})$. There are disjoint open sets of $\sigma(\underline{X})$ (since $\sigma(\underline{X})$ is normal) such that $F\subseteq U$ and $G\subseteq V$. We infer that either $LiC_a\bigcap U\neq\emptyset$ or $LiC_a\bigcap V\neq\emptyset$. Let $LiC_a\bigcap U\neq\emptyset$. We claim that there is an $a\in A$ such that $C_b\subseteq U\bigcup V$ for each $b\geq a$. In the opposite case we obtain a net $\mathcal{N}=\{x_b{:}b\in A, x_a\in C_b, b\geq a\}$. By virtue of Lemma 2.1. the net $\mathcal{N}$ has a cluster point in X. As in the proof of Lemma 2.1. we see that $x\notin U\bigcup V$, which is imposible since $x\in LsC_a$. Thus, there is an $a\in A$ such that $C_b\subseteq U\bigcup V, b\geq a$. It is clear that there is an index $b\geq a$ such that C_b intersects U (since $LiC_a\bigcap U\neq\emptyset$) and V (since V contains the points of LsC_a). But, this is impossible since C_b is connected and $C_b\subseteq U\bigcup V$. The proof is completed.■

LEMMA 3.5 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of the non-empty compact spaces with limit X. For each closed $F \subseteq X$ we have a net $\mathcal{N}(F) = \{p_a(F): a \in A\}$ and, for each $a \in A$, a net $\mathcal{N}_a(F) = \{p_{ab}p_b(F): b \geq a\}$ such that*

1. $p_a(F) = Lim\mathcal{N}_a(F)$,
2. $F = Lim\mathcal{N}(F)$.

Proof. From the definition of the threads it follows that $p_a(F) \subseteq Li\mathcal{N}_a(F)$. On the other hand, from property (B2) [13, pp. 601,615] we infer that if $x \notin p_a(F)$, then $x \notin Ls\mathcal{N}_a(F)$. Thus, $p_a(F) \supseteq Ls\mathcal{N}_a(F) \supseteq Li\mathcal{N}_a(F)$. Hence, $Lim\mathcal{N}_a(F) = p_a(F)$. The second statement of Theorem it follows from 2. of Lemma 2.1. Namely, we have $F \subseteq Li\mathcal{N}(F)$. On the other hand, for each point $y \in X \setminus F$ there is an index $b \in A$ such that $p_b(y)$ and $p_b(F)$ have disjoint neighborhoods U_b and V_b. It follows that $U_b^* \bigcap p_c(F)$ for each $c \geq b$. This means that $y \notin Ls\mathcal{N}$, i.e., $Ls\mathcal{N} \subseteq F$. Finally, we have $F = Ls\mathcal{N} = Li\mathcal{N} = Lim\mathcal{N}$ and the proof is completed.■

We say that a space X is ***hereditarily unicoherent*** if for each pair C,D of closed connected subsets of X, with $C \bigcup D$ connected, the intersection $C \bigcap D$ is connected.

THEOREM 3.6 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of hereditarily unicoherent compact spaces. Then $X = lim\underline{\mathbf{X}}$ is hereditarily unicoherent.*

Proof. Let C,D be a pair of subcontinua of X such that $C \bigcup D$ is connected . We must to prove that $C \bigcap D$ is connected. By virtue of the above Lemma we have $C = Lim\mathcal{N}(C)$ and $D = Lim\mathcal{N}(D)$. Each $F_a = p_a(C) \bigcap p_a(D)$ is connected since X_a is hereditarily unicoherent. By virtue of 2. of Lemma 2.1. each point x of $C \bigcap D$ is a limit of a net $\{p_a(x): a \in A\}$. Thus, $\emptyset \neq LiF_a \supseteq C \bigcap D$. On the other hand for each $y \notin C \bigcap D$ we have $y \notin C$ or $y \notin D$. Let $y \notin C$. By virtue of the definition of a base in X there is a $b \in A$ such that $p_b(y)$ and $p_b(C)$ have the disjoint neighborhoods U_b and V_b. From 2. of the above Lemma it follows that there is an index $c \geq b$ such that $p_c(C) \subseteq V_b$. This means that $U_b^* \bigcap p_c(C) = \emptyset$. We infer that $y \notin LsF_a$. Thus, $LsF_a \subseteq C \bigcap D$. From this and the relation $LiF_a \supseteq C \bigcap D$ it follows $C \bigcap D = LiF_a$. By virtue of Lemma 3.4. LsF_a is connected. Thus, $C \bigcap D$ is connected and the proof is completed.■

By the same method of proof as in the proof of Theorem 3.6. we have

THEOREM 3.7 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an inverse system of continua. If all the spaces X_a are unicoherent and if all p_{ab} are onto, then $X = lim\underline{\mathbf{X}}$ is unicoherent.*

REMARK 3.8 Without ontoness of the bonding mappings the aproximate limit of unicoherent continua need not be unicoherent since this is not true for usual inverse

limit [15,pp. 228,REMARK.].If $\underline{X} = \{X_a, \epsilon_a, p_{ab}, A\}$is an usual inverse system of metric locally connected unicoherent continua,then the usual limit is unicoherent (without assuming the bonding maps are onto) [15,pp. 228.,REMARK.].This means that the following question is natural:
Is it true that the approximate limit of an approximate system of metric locally connected unicoherent continua and into bonding mappings is unicoherent?

Now we give the affirmative answer on the above question.Firstly,we give some necessary definitions.

Let S be the circle $|z| = 1$ in the complex plane.The space of the real numbers we denote by R.

A continuous mapping f:X→S is said to be *equivalent to 1* on a set $Y \subseteq X$, written $f \sim 1$ on Y,provided there exists a continuous mapping ϕ:Y→R such that [18,pp. 220] $f(x) = e^{i\phi(x)}$,x∈ Y.

Two mappings f_1,f_2:X→S will be said to be *exponentially equivalent* or simply equivalent on a set Y⊆X provided their ratio f_1/f_2 is ~1 on Y [18,pp. 225].

A space X will be said to have the *property (b)* provided every mapping f:X→S is ~ 1 [18,pp. 226].

A mapping f:X→S homotopic to the mapping f_0:X→S,$f_0(x)=1$ for all x∈X, is said to be *homotopic to 1*,f≃1.

In the sequel we need the following facts: (a) In order that a mapping f:X→S be ~1 it is necessary and sufficient that f be homotopic to 1 [18,pp. 226]. (b) In order that two mappings f_1,f_2:X→S be equivalent it is necessary and sufficient that they be homotopic [18,pp. 226]. (c) Every connected space X having property (b) is unicoherent [18,pp. 227]. (d) In order that a locally connected continuum have property (b) it is necessary and sufficient that it be unicoherent. (e) If X is any space and f,g:X→S^n two maps such that for each x∈X,f(x) and g(x) are not antipodal,f≃g.In particular,a nonsurjective f:X→S^n is always nullhomotopic [2,pp. 316].

THEOREM 3.9 *Let $\underline{X}=\{X_n,\epsilon_n,p_{mn},N\}$ be an approximate inverse sequence of locally connected unicoherent metric continua.The $X=\lim\underline{X}$ is unicoherent.*

Proof. Let us prove that X has the property (b).Let f:X→S be any mapping.By virtue of Lemma 2.5. there is an a∈A such that for each b≥a there is a mapping g:X_b →S such that gp_b and f are homotopic.From the proof of Lemma 2.5. it follows that there is an index b∈A such that for each f(x)∈S we choose an open set V_x such that V_x contains no antipodal points.We obtain a cover $\mathcal{V} = \{V_x$:x∈X}. We infer that f≃gp_a.From (a) and (d) it follows that g≃1.Consequently,gp_a ≃1 [9,pp. 362,Teorema 4.].Hence,f≃1.By virtue of (a) we infer that f~1,i.e.,X has the property (b).Finally,X is unicoherent (see (c)).■

A continuum X is said to be *hereditarily unicoherent at a point* p∈X if the intersection of any two subcontinua of X,each of which contains p,is connected [1]. It is proved that a continuum X is hereditarily unicoherent at a point p iff,given any point x∈X,there exists a unique subcontinuum px which is irreducible between p and x [5].

THEOREM 3.10 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an inverse system of continua. If* $z=(z_a : a \in A)$ *is a thread such that all the spaces* X_a *are hereditarily unicoherent at a point* z_a*,then* $X = \lim \underline{X}$ *is hereditarily unicoherent at a point z.*

Proof. Modify the proof Theorem 3.6.■

A mapping f:X→Y is said to be *monotone relative to a point* z∈X if for each subcontinuum Q of Y such tha f(z)∈Q the inverse image $f^{-1}(Q)$ is connected [1].

A quasi-order ≤ on a set X is a reflexive and transitive binary relation.If this relation is also antisymetric,it is called a *partial order*.It is order-dense if whenever x≺y (i.e. x≤y and y≰x),there exists a z∈X such that x≺z≺y.An element z∈X is called a zero of X if z≤x for each x∈X.A quasi-order on a topological space X is said to be closed if its graph is a closed subset of the product space XxX.

A *generalized tree* means a hereditarily unicoherent continuum which admits a closed order-dense partial order with a zero.

If a continuum X is hereditarily unicoherent at a point z,then the quasi-order $\leq_z$ on X defined by $x \leq_z y$ if and only if zx⊆zy,where zx is a unique subcontinuum which is irreducible between z and x,is said to be a *weak cutpoint order with respect z.*

Let continua X and Y be hereditarily unicoherent at a point p and q respectively and let $\leq_p$ and $\leq_q$ be weak cutpoint orders on X and Y with respec to p and q correspondingly.A mapping f:X→Y onto Y is said to be **order-preserving** (or $\leq_p$-preserving) if $a \leq_p b$ implies $f(a) \leq_q f(b)$ for every a,b∈X [1].

The following property of mappings monotone relative to a point will be needed in the sequel.

THEOREM 3.11 (1,PROPOSITION 2.) *.Let continua X and Y be hereditarily unicoherent at points p and q,respectively,and let f:X→Y be a mapping onto Y such that f(p)=q.Then the following conditions are equivalent:*

1. *f is monotone relative to p,*
2. *f(px)=f(p)f(x) for each x∈X,*
3. *f is* $\leq_p$*-preserving,*

and each of them is implied by

4. *f/px is monotone for each x∈X.*

THEOREM 3.12 *Let continua X and Y be hereditarily unicoherent at points p and q,respectively,and let f:X→Y be a mapping of X onto Y with f(p)=q.If X is arcwise connected,or if X is metric and Y is arcwise connected,then all four conditions of the above Theorem are equivalent.*

Proof. See [1].■

COROLLARY 3.13 *If a continuum X is an arc with end point p,and if a mapping f on X is monotone relative to p,then the image f(X) is an arc,and f is monotone.*

Proof. See [1].■

Now we consider the approximate limits.We start with the following theorem.

THEOREM 3.14 *Let* $\underline{X} = \{X_a,\mathcal{U}_a,p_{ab},A\}$ *be an approximate inverse system of continua with limit X.If each* X_a *is irreducible between* x_a *and* y_a *such that* $x=(x_a:a\in A)$ *and* $y=(y_a:a\in A)$ *are thread,then X is irreducible between x and y.*

Proof. Let us recall that the usual version of this Theorem was proved in [1,PROPOSITION 5.].By virtue of Theorem 3.1. X is connected.Suppose that there is a continuum Y⊂X which contains x and y.This means that there exists a point z∈X-Y.By 1.5. we obtain an a∈A such that $p_a(z)\notin p_a(Y)$.Since $p_a(Y)$ contains x_a and y_a,we infer that $p_a(Y)\subset X_a$.This is imposible since X_a is irreducible between x_a and y_a.■

LEMMA 3.15 *Leta f:X→Y be a monotone surjection.If Y is hereditarily unicoherent and I(a,b) is irreducible between a,b,then f(I(a,b)) is irreducible between f(a) and f(b),i.e.,I(f(a),f(b))=f(I(a,b)).*

Proof. Now,$f^{-1}(I(f(a),f(b)))$ is a continuum since it contains a and b and f is monotone.This means that $f^{-1}(I(f(a),f(b)))\supseteq I(a,b)$.Thus, $f(I(a,b))\subseteq I(f(a),f(b))$. On the other hand,$f(I(a,b))\supseteq I(f(a),f(b))$ since I(f(a),f(b)) is irreducible between f(a) and f(b). Thus,f(I(a,b))=I(f(a),f(b)) and the proof is completed.■

The following lemma is a generalization of Lemma 2.2. of [10].

LEMMA 3.16 *Let* $\{C_n:n\in D\}$ *be a net of subcontinua of a continuum X.If* $x,y\in LiC_n$ *and the continuum* LsC_n *is irreducible between x and y,then the net* $\{C_n:n\in D\}$ *is convergent.*

Proof. Suppose,on the contrary,that the net $\{C_n:n\in D\}$ is not convergent,i.e., there is a point $c\in LsC_n\setminus LiC_n$.It follows that there is a neighborhood U of c such that for each n∈D there is an index m∈D,m≥n,such that $C_m\cap U=\emptyset$.The collection $\{C_m:m\in M\}$ is a net in X\U and a subnet of $\{C_n:n\in D\}$.This means that $L=Ls\{C_m:m\in M\}$ is nonempty subset of X\U and $c\in U\subseteq X\setminus L$.By Lemma 3.3. L is connected,i.e.,subcontinuum

of X.From $x,y \in L$ and from the irreducibility of LsC_n,it follows that $L \supseteq LsC_n$.On the other hand,$L \subseteq LsC_n$.Thus,$L = LsC_n$.
This is impossible since $c \in LsC_n \setminus LiC_n = L \setminus LiC_n$ and $c \notin L$.The proof is completed.■

A continuum X is said to be *smooth at a point* $p \in X$ (in the sense of Gordh [5]) if X is hereditarily unicoherent at p and for each convergent net $(a_n : n \in D)$ of points of X the condition $a = \lim a_n$ implies that the net $(pa_n : n \in D)$ of subcontinua of X is convergent to the limit continuum pa.

The usual version of the following theorem was proved as Theorem 1. of [1].

THEOREM 3.17 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of continua with limit X.If there exists a thread* $y = (y_a : a \in A)$ *such that the mapping* p_{ab} *is monotone relative to* y_a *for each* $a,b \in A$ *with* $a \leq b$,*and if* X_a *is smooth and hereditarily unicoherent at* $y_a, a \in A$,*then X is smooth at the point y.*

Proof. a)By Theorem 3.1. X is a continuum.Theorem 3.10. implies that X is hereditarily unicoherent at y.
b) Let $\{x^\mu : \mu \in M\}$ be a net which converges to a point $x \in X$.There are subcontinua $I(y,x)$ and $I(y,x^\mu), \mu \in M$,since X is hereditarily unicoherent.For each $a \in A$ we have also the subcontinua $I(y_a,x_a), I(y_a,x_a^\mu), \mu \in M$,irreducible between $y_a = p_a(y)$ and $x_a^\mu = p_a(x^\mu)$.It is obvious that each net $\{x_a^\mu : \mu \in M\}$ converges to x_a.Moreover,from the smoothness of X_a at y_a and from the above Lemma it follows that a net $\{I(y_a,x_a^\mu) : \mu \in M\}$ of subcontinua converges to $I(y_a,x_a)$.
c) *$Ls\{I(y_a,x_a^\mu) : a \in A\} = K^\mu = I(y,x^\mu), \mu \in M$.*
By virtue of Lemma 3.4. each net $\{I(y_a,x_a^\mu) : a \in A\}$ has a non-empty and connected $Ls\{I(y_a,x_a^\mu) : a \in A\} = K^\mu$.Clearly,$K^\mu \supseteq I(y,x^\mu)$ since $I(y,x^\mu)$ is irreducible between y,x^μ and $\{y,x^\mu\} \subseteq K^\mu$. By virtue of Lemma 3.5. we have $I(y,x^\mu) = Lim\{p_a(I(y,x^\mu)) : a \in A\}$.Since each $p_a(I(y,x^\mu))$ contains $I(y_a,x_a^\mu)$,we infer that $K^\mu \subseteq I(y,x^\mu)$.Finally,we have $K^\mu = I(y,x^\mu)$.
d) *For each $a \in A$ and each $\mu \in M$ we have $p_a(K^\mu) = I(y_a,x_a^\mu)$.*
Clearly,$p_a(K^\mu) \supseteq I(y_a,x_a^\mu)$.Suppose that there is an $a \in A$ and a point $z_a \in p_a(K^\mu) \setminus I(y_a,x_a^\mu)$. This means that there are disjoint open sets U_a and V_a such that $z_a \in V_a$ and $I(y_a,x_a^\mu) \subseteq U_a$. From the smoothnes of X_a at y_a it follows that there is an open and connected set W_a such that $I(y_a,x_a^\mu) \subseteq ClW_a \subseteq U_a$.From the definition of the thread it follows that there is a $b \in A$ such that $p_{ac}(y_c)$ and $p_{ac}(x_c^\mu)$ are in W_a for each $c \geq b$.This means that $p_{ac}(I(y_c,x_c^\mu)) \subseteq ClW_a$ since $p_{ac}(I(y_c,x_c^\mu))$ is irreducible between $p_{ac}(x_c)$ and $p_{ac}(x_c^\mu)$ (see Lemma 3.15).It follows that U_a^* is a neighbourhood of a point $z \in K, p_a(z) = z_a$,such that $U_a^* \cap I(y_c,x_c^\mu) = \emptyset$.This means that $z \notin Ls\{I(y_a,x_a^\mu) : a \in A\} = K^\mu$. This is impossible since $z \in K^\mu$.By Theorem 1.4. it follows that $K^\mu = \bigcap\{p_a^{-1}(I(y_a,x_a^\mu)) : a \in A\}$. Similarly,we have $K = \bigcap\{p_a^{-1}(I(y_a,x_a)) : a \in A\}$,where $Ls\{I(y_a,x_a) : a \in A\} = K$.
e) *$Ls\{K^\mu : \mu \in M\} = Ls\{I(y,x^\mu) : \mu \in M\} = I(y,x)$.*

It is obvious that $Ls\{I(y,x^{\mu}):\mu \in M\} \supseteq I(y,x)$ since $Ls\{I(y,x^{\mu}):\mu \in M\}$ contains x and y and I(y,x) is irreducible between x and y.We prove that $Ls\{I(y,x^{\mu}):\mu \in M\} \subseteq I(y,x)$.Let z be any point in X-I(y,x).By virtue of the definition of a base in X,there is an $a \in A$ such that $p_a(z)=z_a \notin p_a(I(y,x))=$(by Steps 3.1. and 3.2.)$=I(y_a,x_a)$.This means that there is a neighbourhood U_a of z_a and a neighbourhood V_a of $p_a(I(y,x))$ such that $U_a \bigcap V_a = \emptyset$.By Step 3.2. $p_a(I(y,x))=I(y_a,x_a)$. Since $I(y_a,x_a)=Lim\{I(y_a,x_a^{\mu}):\mu \in M\}$,we infer that there is a $\mu_0 \in M$ such that,for each $\mu \geq \mu_0$,U_a and $I(y_a,x_a^{\mu})$ are disjoint.From 3.2. it follows that $p_a^{-1}(U_a)$ and $I(y,x^{\mu})$ are disjoint.Since $p_a^{-1}(U_a)$ is a neighbourhood of z,we infer that $z \notin Ls\{I(y,x^{\mu}):\mu \in M\}$.Thus,$Ls\{I(y,x^{\mu}):\mu \in M\} = I(y,x)$ and 3.3. is proved.

f) $I(y,x)=Lim\{I(y,x^{\mu}):\mu \in M\}$.
Apply Step 3.3. and Lemma 3.12.

By virtue of Lemma 3.10. and Step 3.4. it follows that X is smooth at y.

g)For each $a \in A$ there exists a closed weak cutpoint order $\leq_a$ with respect to the point x_a.Define a relation $\leq$ on X by $y \leq z$ iff $y_a \leq_a z_a$ for all $a \in A$.The relation $\leq$ is transitive and reflexive,i.e.,$\leq$ is quasi-order.

h)By the same method of proof as in the proof of Theorem 1. of [1] it follows that $\leq$ is closed.

i)In order to complete the proof we ought to show the quasi-order $\leq$ is the weak cutpoint order with respect x,i.e. that $y \leq z$ holds iff $xy \subset xz$.

i1)Let us prove that $y \leq z \Rightarrow xy \subset xz$.Suppose $xy \not\subset xz$.By hereditarily unicoherence of X at x we infer that $y \notin xz$.From 1.5. it follows that there is an $a \in A$ such that $y_a=p_a(y) \notin p_a(xz)$.This means that the continuum $x_a z_a$ contains no the point y_a since $p_a(xz) \supseteq x_a z_a$.On the other hand,from $y \leq z$ it follows $y_a \leq z_a$,i.e.,$x_a y_a \subset x_a z_a$.This contradiction completes the proof.

i2)Now we prove $(xy \subset xz) \Rightarrow (y \leq z)$.This easy follows from c).■

An arc (generalized) is defined as a continuum (not necessarily metrizable) with exactly two non-separating points (called the end points of the arc).Clearly,X is an arc iff X is an ordered continuum with two points 0 and 1 such that $0 \leq x \leq 1$ for each $x \in X$.A continuum X is said to be an *arboroid* if X is hereditarily unicoherent and arcwise connected (i.e. any two points of X can be joined by a generalized arc).A metrizable arboroid is called *a dendroid.*

THEOREM 3.18 *Let* $\underline{X} = \{X_a,\mathcal{U}_a,p_{ab},A\}$ *be an approximate inverse system with limit X such that:*

1. X_a *is an arboroid for each* $a \in A$*,and*
2. *there exists a thread* $x=(x_a:a \in A)$ *such that* p_{ab} *is monotone relative to* x_b *for each* $a,b \in A, a \leq b$.

Then X is an arboroid.

Proof. By Theorem 3.10. X is hereditarily unicoherent at x.In order to complete the proof it suffices to prove that xy is an arc for each $y \in X$.

Step 1. We define a linear order $\prec$ in xy [3,pp. 17] such that for each $z \in xy$ $x \prec z \prec y$.

Step 2. Let $x_a = p_a(X)$ and $y_a = p_a(y_a)$ for each $a \in A$.There is an arc $x_a y_a$ whose end points are x_a and y_a since X_a is arcwise connected.We may assume that a linear order in $x_a y_a$ is such that $y_a \prec_a x_a$.

Step 3. By c) from the proof of Theorem 3.17. it follows that $p_a(xy) = x_a y_a$ for each $a \in A$.

Step 4. Let $z, w \in xy$.We claim that either $z_a \prec_a w_a$ or $w_a \prec_a z_a$ for each $a \in A$.Suppose that $z_a \prec_a w_a$ for some $a \in A$ and that there is a set $B = \{b : b \geq a\}$ cofinal in A and such that $w_b \prec_b z_b$.Choose the disjoint neighbourhoods $U(x_a)$, $U(w_a), U(z_a), U(y_a)$ for the points x_a, w_a, z_a, y_a.By virtue of the condition (L) and the cofinality of B there is an index $b \in B$ such that the points $p_{ab}(x_b), p_{ab}(w_b)$, $p_{ab}(z_b), p_{ab}(y_b)$ lie in the neighbourhoods $U(x_a), U(w_a), U(z_a), U(y_a)$.If we suppose that $z_b \succ w_a$,then $p_{ab}(x_b z_b)$ contains $p_{ab}(w_b)$ since $p_{ab}(x_b y_b)$ is an arc.This means that there is a point $v_b \in x_b z_b$ such that $p_{ab}(v_b) = p_{ab}(w_b)$.Hence,$p_{ab}^{-1}(w_a)$ contains z_b since $p_{ab}/x_b y_b$ is monotone.We infer that $p_{ab}(z_b) = p_{ab}(w_b \in U(w_a)$ and $p_{ab}(z_b) \in U(z_a)$.This is impossible since $U(z_a)$ and $U(w_a)$ are disjoint.Now,for each $c \in A$ there is a $b \in B$ such that $b \geq a, c$.This means that $z_c \prec w_c$ since $z_b \prec w_b$.

Step 5. We set $z \leq w$ iff $z_a \leq w_a$ for each $a \in A$.

Step 6. Now we prove that xy is an arc,i.e.,that each point $z \in xy$ other than x,y is a cut point of xy.Let B be a set cofinal in A such that $z_b \notin x_b y_b$.This means that x_b is a cut point of $x_b y_b$.We set $U_b = (p_b/xy)^{-1}([x_b, z_b))$ and $V_b = (p_b/xy)^{-1}((z_b, y_a])$.Clearly,$U_b$ and V_b are disjoint open sets of xy which contain x and y respectively.Consider the sets $U = \bigcup\{U_b : b \in B\}$ and $V = \bigcup\{V_b : b \in B\}$.It follows that U and V are open sets which contain x and y respectively.In order to complete the proof of this Step it suffices to prove that U and V are disjoint and $U \bigcup V = xy - \{z\}$.For each $w \in xy$,distinct from z,there is a $b \in B$ such that $w_b \neq z_b$.Then,$w \in U_b$ or $w \in V_b$.This means that $U \bigcup V = xy - \{z\}$.Finally,suppose that there is a point $w \in U \bigcap V$.This means that w is in some U_c and in some V_d,$c, d \in B$.It follows $w_c \geq z_c$ and $w_d \leq z_d$.This contradicts Step 4.Thus,xy is an arc.

Step 7. It follows that X is an arboroid and the proof of Theorem is completed.■

Since the approximate limit of an approximate inverse sequence of metrizable spaces is metrizable (as a subspace of the space $\Pi\{X_n : n \in N\}$) we have

COROLLARY 3.19 *Let* $\underline{X} = \{X_n, \epsilon_n, p_{mn}, N\}$ *be an approximate inverse sequence with limit X such that:*

1. *X_n is a dendroid for each $n \in N$,and*

2. *there exists a thread $x=(x_n:n\in N)$ such that p_{mn} is monotone relative to x_n for each $m,n\in A, m\leq n$.*

Then X is a dendroid.

An arboroid is smooth if and only if it is a generalized tree.Thus,Theorems 3.18. and 3.19 imply

COROLLARY 3.20 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system with limit X such that:*

1. *X_a is a generalized tree with a point x_a as a zero for each $a\in A$,and*
2. *the points x_a form a thread $x=(x_a:a\in A)$ such that p_{ab} is monotone relative to x_b for each $a,b\in A, a\leq b$.*

Then X is a generalized tree with the point x as a zero.

A continuum is a *tree* if each pair of points is separated by third point [17]. A continuum X is a tree iff X is locally connected and hereditarily unicoherent [17].

Each tree is a generalized tree [17,Theorem 6.].Thus we have

COROLLARY 3.21 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system with limit X such that each X_a is a tree and each p_{ab} is monotone.Then X is a tree.*

Proof. By Theorem above X is a generalized tree.From 3.17. it follows that X is smooth in each point.By virtue of [10] or [16] we infer that X is locally connected.Thus,X is a tree.■

A point x of an arboroid X is said to be a *ramification point* if there are three arcs in X emanating from x and disjoint out of x.An arboroid having at most one ramification point is called a *fan*.Then the ramification point is called the *top* of the fan.

THEOREM 3.22 *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system with limit X such that:*

1. *X_a is a fan with the top x_a for each $a\in A$,and*
2. *the points x_a form a thread $x=(x_a:a\in A)$ such that p_{ab} is monotone relative to x_b for each $a,b\in A, a\leq b$.*

Then X is a fan with the top x.

Proof. By Theorem 3.19. we infer that X is an arboroid.It remains to prove that X has at most one ramification point.Suppose that the point $y=(y_a)$ is an ramification point of X distinct from x.This means that there is an $a_0 \in A$ such that $y_a \neq z_a, a \geq a_0$.Let K,L,M be three arcs in X (disjoint out y) with endpoints y and k,l,m respectively.By virtue of 1.5. there is an index $b_0 \geq a_0$ such that y_b, k_b, l_b, m_b are mutually distinct points for each $b \geq b_0$.There exist three arcs K_b, L_b, M_b which are mutually disjoint out y_b since X_b is an arboroid.This is impossible since X_b is a fan and $x_b \neq y_b$.■

Combining this Corollary and the last Theorem we get

COROLLARY 3.23 *Let $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system with limit X such that:*

1. *X_a is a fan with the top x_a at which it is smooth for each $a \in A$,and*
2. *the points x_a form a thread $x=(x_a : a \in A)$ such that p_{ab} is monotone relative to x_b for each $a, b \in A, a \leq b$.*

Then X is a fan which is smooth at its top x.

A continuum X with precisely two nonseparating points is called a *generalized arc*.A continuum X is said to be an *arc* if X is a metrizable generalized arc.A tree X is a generalized arc if and only if X is atriodic.

THEOREM 3.24 *Let $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system of generalized arcs.Then $X = \lim \underline{X}$ is atriodic.*

Proof. Suppose that T is a subcontinuum of X which is a triod.This means that T is the sum of three generalized arcs C_x, C_y,and C_z,such that the common part of each two of them is the common part of all three of them and is a point.Let $x \in C_x - (C_y \bigcup C_z), y \in C_y - (C_x \bigcup C_z), z \in C_z - (C_x \bigcup C_y)$ and $t = C_x \bigcap C_y \bigcap C_z$. By virtue of the definition of a basis in X,there exist $a \in A$ and open sets V_x, V_y, V_z of X_a which are pairwise mutually exclusie and which contain x_a, y_a, z_a,respectively,so that

$$p_a^{-1}(V_x) \bigcap C_y = \emptyset = p_a^{-1}(V_x) \bigcap C_z,$$

$$p_a^{-1}(V_y) \bigcap C_x = \emptyset = p_a^{-1}(V_y) \bigcap C_z,$$

$$p_a^{-1}(V_z) \bigcap C_y = \emptyset = p_a^{-1}(V_z) \bigcap C_x.$$

Now,one of x_a, y_a,or z_a lies between t_a and one of x_a, y_a,or z_a.Supose that $t_a \prec x_a \prec y_a$. Then $p_a(C_y)$ intersects t_a and y_a and hence x_a,but $p_a(C_y)$ does not intersects V_x. This is a contradiction.So,X contains no triod.■

THEOREM 3.25 *Let* $\underline{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate system of generalized arcs with a limit X.If the bonding mappings are monotone and onto, then X is a generalized arc.*

Proof. By virtue of Theorem 3.21. X is a tree.From 3.24. it follows that X is atriodic.Thus X is a generalized arc.■

COROLLARY 3.26 *Let* $\underline{X}=\{X_n, \epsilon_n, p_{mn}, N\}$ *be an approximate sequence of the arcs and monotone onto mappings.Then* $X=\lim\underline{X}$ *is an arc.*

Proof. Now,from 3.25. it follows that X is a generalized arc.Moreover,X is metrizable generalized arc.Thus,X is an arc.■

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Lončar I. O aproksimativnim sistemima kompaknih prostora

SAŽETAK

U radu je definiran prostor $\sigma(\underline{X})$za aproksimativni inverzni sistem $\underline{X} = \{X_a,\mathcal{U}_a,p_{ab},A\}$ kompaktnih prostora.

U drugom odjeljku dokazuju se osnovna svojstva prostora $\sigma(\underline{X})$,kao što su parakompaktnost i metrizabilnost za aproksimativni niz.

Treći odjeljak sadrži razne primjene prostora $\sigma(\underline{X})$.Primjenjujući poopćene nizove skupova dokazujemo da je prostor X=lim$\underline{X}$ povezan i (nasljedno) unikoherentan ako su takvi prostori sistema $\underline{X} = \{X_a,\mathcal{U}_a,p_{ab},A\}$(Teoremi 3.1. i 3.6.).Dokazano je nadalje da je lim$\underline{X}$ arboroid (poopćeno stablo,stablo,luk) ako su prostori sistema takvi a vezna preslikavanja monotona (Teoremi 3.18., 3.20.,3.21. i 3.25.).