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INFINITE-DIMENSIONALITY OF INVERSE LIMIT SPACE

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Among other results we prove: (1) If $f:X\to Y$ is a closed surjection between normal countably compact spaces such that $|Fr\ f^{-1}(y)|<k$, then a weak infinite-dimensionality of X implies a weak infinite-dimensionality of Y; (2) If X is a limit of noramalcountably compact strongly infinite-dimensional spaces and closed bonding mappings f_{ab} such that $|Fr\ f^{-1}_{ab}(x_a)|<k$, then X is countably compact and strongly infinite-dimensional; (3) Let $X=\{X_a,f_{ab},A\}$ be an inverse system of infinite-dimensional Cantor-manifolds X_a. If the mappings f_{ab} are monotone such that $|Fr\ f^{-1}_{ab}(x_a)|<k$, then $\lim X$ is an infinite-dimensional Cantor-manifold.

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0.INTRODUCTION

We say that a space X is **A-weakly (S-weakly) infinite-dimensional** [2] if for each sequence $\{(A_i,B_i):i\in N\}$ of the pairs A_i,B_i of closed disjoint subsets A_i,B_i of X there exist the partitions C_i between A_i and B_i such that $\cap\{C_i:i\in N\}=\emptyset$ ($\cap\{C_i:i=1,2,...,k\}=\emptyset$).

A space X is **A-strongly (S-strongly) infinite-dimensional** if X is not A-weakly (S-weakly) infinite-dimensional.

Frequently we use the words weakly infinite-dimensional instead of the words A-weakly infinite-dimensional.

A space X is **infinite-dimensional Cantor-manifold** if X is compact and X-F isconnected for each closed weakly infinite-dimensional subset F of X.

The cardinality of a set A is denoted by |A| or by card(A).

If A is well-ordered,then cf(A) means the cofinality of A, i.e. the smallest ordinal number which is cofinal in A.

We use the notion of the inverse system as in [3].

1. THE MAIN RESULTS

We start with the following lemma.

1.1.LEMMA.A countably compact space X is A-weakly infinite-dimensional iff X is S - weakly infinite-dimensional.

Proof. Lemma follows from the definitions of the countable compactness, A-weak and S-weak infinite-dimensionality.

1.2.LEMMA.[2:543]. If X is normal S-weakly infinite-dimensional,then the Stone-Cech'scompactification βX is weakly infinite-dimensional.

1.3.REMARK. Clearly,if βX is weakly infinite-dimensional, then X is S-weakly infinite-dimensional and A-weakly infinite-dimensional.

From 1.2. and 1.3. we infer

1.4.LEMMA.A normal countably compact space X is weakly infinite-dimensional iff βX isweakly infinite-dimensional.

In the sequel we use the following

1.5.LEMMA.[1:23].Let $f:X\rightarrow Y$ be a mapping from a weakly infinite-dimensional compact space X.If the space $Y=f(X)$ is strongly infinite-dimensional, then there exists $y\in Y$ such that $card(f^{-1}(y))\geq c=2^{\aleph_0}$.

1.6.COROLLARY. Let $f:X\rightarrow Y$ be a mapping between compact spaces such that each fiberis countable. If X is weakly infinite-dimensional, then Y is weakly infinite-dimensional.

1.7.COROLLARY. Let $f:X\rightarrow Y$ be a closed surjection between normal spaces such that for each $y\in Y$ $card(f^{-1}(y))\leq k$.If X is S-weakly infinite-dimensional,then Y is so.

Proof. Let $\beta f:\beta X\rightarrow \beta Y$ be the Stone-Cech extension of f. By Lemma 1. of [9] we infer that $card((\beta f)^{-1}(y))\leq k$.By 1.2.,1.3. and 1.6. we complete the proof.

1.8.THEOREM. Let $f:X\rightarrow Y$ be a closed mapping between normal countably compact spaces such that $card(f^{-1}(y))\leq k$ for each $y\in Y$. If X is weakly infinite-dimensional, thenY is weakly infinite-dimensional.

Proof. Let $\beta f:\beta X\rightarrow \beta Y$ be Stone-Cech's extension of f.From [9:Lemma 1.] it follows that $card((\beta f)^{-1}(y))\leq k$ for each $y\in \beta$ .Since X is weakly infinite-dimensional (1.4. Lemma) it follows that βY is weakly infinite-dimensional. Lemma 1.4. completes the proof.

1.8.1.REMARK. Let us note that it suffices to assume that card $(Fr\, f^{-1}(y))\leq k$.

1.9.THEOREM. Let $\mathbf{X}=\{X_n,f_{nm},N\}$ be an inverse sequence of normal countably compact spaces X_n and closed mappings f_{nm} such that card $(Fr\, f^{-1}_{nm}(x_n))\leq k$. If the spaces X_n are strongly infinite-dimensional, then $X=\lim\mathbf{X}$ is strongly infinite-dimensional.

Proof.From [7] (or [3:260]) it follows that X is countably compact. One readily sees that card($f^{-1}_n(x_n)$)≤ k .Theorem 1.8. completes the proof.

1.10.THEOREM. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ be an inverse system of compact spaces X_a such that card($f^{-1}_{ab}(x_a)$)≤ k.If the spaces X_a are strongly infinite-dimensional,then X=lim**X** is strongly infinite-dimensional.

Proof. Apply Lemma 1.4.

1.11.THEOREM. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ be a σ-directed inverse system of compact spaces X_a such that each fiber $f^{-1}_{ab}(x_a)$ is finite. The space lim**X** is strongly infinite-dimensional iff the spaces X_a are strongly infinite-dimensional.

Proof. Suppose that card($f^{-1}_a(x_a)$)=$\aleph_0$ for some $x_a \in X_a$. For each pair $x,y \in f^{-1}_{ab}(x_a)$ there exists $a_0 \in A$ such that $f_b(x) \neq f_b(y)$ for each $b>a_0$ .From the σ-directedness of A it follows that there is an c∈A such that card($f_c f^{-1}_a(x_a)$)=$\aleph_0$. This contradicts the fact that the fibers are finite. Thus, card($f^{-1}_a(x_a)$)=$\aleph_0$ for each a∈A and each x_a. From Lemma 1.5. it follows that limX is strongly infinite-dimensional. The converse follows from the following

1.12.THEOREM. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ bean σ-directed inverse system of compact spaces X_a. If the spaces X_a are weakly infinite-dimensional, then X=lim**X** is weakly infinite-dimensional.

Proof. Let $\{(A_i,B_i),i \in N\}$ be any sequence of pairs of disjont closed subsets of X. There exists an $a_i \in A$ such that $f_{a_i}(A_i) \cap f_{a_i}(B_i)=\emptyset$.Since **X** is σ-directed there exists $a>a_i$ such that $fa(Ai) \cap fa(Bi)=\emptyset$ for each i∈N. By virtue of the weak infinite-dimensionality of X_a we have the partitions C_i between $f_a(A_i)$ and $f_a(B_i)$ with $\cap C_i=\emptyset$. The sets $f^{-1}_a(C_i)$ are thepartitions between A_i and B_i such that $\cap f^{-1}_a(C_i)=\emptyset$. The proof is completed.

1.13.COROLLARY. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ be a σ-directed inverse system of compact spaces X_a.If the spaces X_a are weakly infinite-dimensional and the fibers $f^{-1}_{ab}(x_a)$ are finite,then X=lim**X** is weakly infinite-dimensional.

1.14.REMARK. If X in Theorems 1.11.,1.12. and 1.13. is $\aleph_1$-directed, then one can assume that card($f^{-1}_{ab}(x_a)$) ≤$\aleph_0$.

We say that $\mathbf{X}=\{X_a,f_{ab}A\}$, is a **factorizable** [10] inverse systemor **f-system** if or feach function f:X=lim**X**→[0,1] there is an a∈A and g_a:X_a→[0,1] such that $f=g_a f_a$.

For each $\mathbf{X}=\{X_a,f_{ab},A\}$ we denote by β**X** the system $\{\beta X_a,\beta f_{ab},A\}$.

1.15.LEMMA. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ be an inverse system with a limit X, then βX is homeomorphic with limβ**X**.

Proof. Apply 3.6.3. Corollary of [3].

1.16.THEOREM. Let $\mathbf{X}=\{X_a,f_{ab},A\}$ be aσ-directed inverse system of the Lindeloff spaces X_a and closed mappings f_{ab} such that all the fibers $f^{-1}_{ab}(x_a)$ have at most k points.

Then lim**X** = X is S-weakly (S-strongly) infinite-dimensione iff the spaces Xa are S-weakly (S-strongly) infinite-dimensionel.

Proof. We consider the system β**X**. From [10:28] it follows that **X** is a f-system. Hence, β**X** is homeomorphic with limβ**X**. The system β**X** satisfies the conditions of Theorem 1.11. and Corollary 1.13.By Lemma 1.2. we complete the proof.

1.17.THEOREM. Let **X**={X_a,f_{ab},A} be a well-ordered inverse system of normal spaces with hl(X_a)<ω_τ and cf(A)>ω_τ .If the mappings f_{ab} are closed such that card ($f^{-1}_{ab}(x_a)$)≤k, then X=lim**X** is S-weakly (strongly S-weakly) infinite-dimensional iff the spaces X_a are S-weakly (strongly S-strongly) infinite-dimensional.

Proof. By virtue of [8] it follows that for every closed subset F of X there is a a∈A such that F=$f^{-1}_a(F_a)$ for some closed subset F_a of X_a. This means that **X** is an f-system [10:27]. Applying the system β**X** as in the proof of Theorem 1.16. we complete the proof.

1.18.THEOREM. Let **X**={X_a,f_{ab},A} be an σ-directed inverse system of the spaces X_a suc that the Souslin number c(X_a)≤$\aleph_0$ and that the mappings f_{ab} are open and closed. A space X=lim**X** is S-weakly (S-strongly) infinite-dimensional iff the spaces X_a are S-weakly(S-strongly) infinite-dimensional.

Proof. From [10:28] it follows that **X** is an f-system. By virtue of [6] and [9] we infer that $\beta f_{ab}:\beta X_b\to\beta X_a$ are open and closed with card($(\beta f_{ab})^{-1}(x_a)$)≤k for each $x_a\in\beta X_a$. From Lemma 1.15. it follows that βX is homeomorphic with limβ**X**.Since limβ**X** is weakly (strongly) infinite-dimensional (Theorem 1.12.), we infer that βX is weakly (strongly) infinite-dimensional. Lemma 1.2. completes the proof.

We close this Section with the inverse systems of the infinite-dimensional Cantor-manifolds. Firstly we prove

1.19.LEMMA. Let f:X→Y be a mapping froma weakly infinite-dimensional compact space X onto a strongly infinite-dimensional space Y. There exists y∈Y such that |Fr f^{-1}(y)|≥c=$2^{\aleph_0}$.

Proof. Let X' be a set obtained by adjoining to the union ∪{Fr f^{-1}(y) : y∈Y} one point from each fibre f^{-1}(y) which has an empty boundary. The restriction f/X' :X'→Y (onto Y) satisfies 1.5. Lemma. The proof is completed.

From Lemma 1..2,Lemma 1.19. and [9:Lemma 1.] it follows

1.20.LEMMA. Let f:X→Y be a closed mapping such that card(Fr f-1(y))|≤k for each y∈Y. If X and Y are normal spaces, then S-weak infinite-dimensional of X implies S-weak infinite-dimensionality of Y.

1.21.LEMMA. Let f:X→Y be a mapping and F⊆X compact weakly infinite-dimensional subspace.If f(F) is strongly infinite-dimensional, then there is y∈f(F) with |Frf^{-1}(y)|≥c.

Proof. Let X_1 be a set obtained by adjoining to the set $F - \cup \{Int\ f^{-1}(y) : y \in f(F)\}$ one point from $f^{-1}(y) \cap F$ if $Fr f^{-1}(y) \cap F = \emptyset$. The restriction $f/X_1 : X1 \rightarrow f(F)$ satisfies Lemma 1.5. The proof is completed.

1.22.THEOREM. Let $\mathbf{X} = \{X_a, f_{ab}, A\}$ be an inverse system of the infinite-dimensional Cantor-manifolds X_a. If the mappings f_{ab} are monotone such that card $(Fr f^{-1}_{ab}(x_a)) \leq k$, then $X = \lim \mathbf{X}$ is an infinite-dimensional Cantor-manifold.

Proof. Let F be a weakly infinite-dimensional closed subset of X.Since card $(Fr f^{-1}_{a}(x_a)) \leq k$ we infer that $Y_a = f_a(F)$ is weakly infinite-dimensional (1.21. Lemma) .This means that $Z_a = X_a - f_a(F)$ is connected. From the fact that f_a are monotone [3:436] it follows that $f^{-1}_{a}(Z_a)$ are connected. Since $X - F = \cup \{f^{-1}_{a}(Z_a) : a \in A\}$ and $\cup \{f^{-1}_{a}(Z_a) : a \in A\}$ is connected [3:436] we infer that X - F is connected. The proof is completed.

1.23.THEOREM. Let $\mathbf{X} = \{X_a, f_{ab}, A\}$ be a σ-directed inverse system of the infinite-dimensional X_a.If the mappings f_{ab} are monotone such that the sets $Fr\ f^{-1}_{ab}(x_a)$ are finite, then $X = \lim \mathbf{X}$ is an infinite-dimensional Cantor-manifold.

1.24.THEOREM. Let $\mathbf{X} = \{X_a, f_{ab}, A\}$ be an $\aleph_1$-directed inverse system of infinite-dimensional Cantor manifolds X_a. If the mappings f_{ab} are monotone such that $|Fr f^{-1}_{ab}(x_a)| \leq \aleph_0$, then $X = \lim \mathbf{X}$ **is an infinite-dimensional Cantor-manifold.**

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