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NEAR-COMPACTIFICATION OF INVERSE LIMIT SPACE

The main purpose of this paper is the investigation of the continuity of the near-compactification.

Section One contains the definition and basic properties of the near-compactification hX of a space X.

Section Two is the main section. We say that the near-compactification is X-continuous if h(limX) = lim hX for some inverse system X. The necessairy and sufficient conditions for X-continuity of the near-commpactification are given.

1. NEAR-COMPACTIFICATION hX OF A SPACE X

If X is a topological space, then the closure and the interior a subset $A \subseteq X$ is denoted by $Cl_X A$ and $Int_X A$ or by ClA and IntA.

The notion of near-compactification was introduced by Herrmann *[5]* for almost completely regular spaces and by Katetov extension for completely regular spaces (see *[19]*).

Now we give some exspository material concerning the Katetov kX since near-compactification is a quotient space of kX.

A Hausdorff space X is H-closed if for every open cover U of X there exists a finite subfamily $\{U_1,\dots,U_k\}$ of U such that $X = ClU_1 \cup \dots \cup ClU_k$ (*[17]*).

A continuous mapping $f:X \longrightarrow Y$ is said to be proper *[17]* if for each $y \in Y$ and each $V \ni y$ open in Y there exists a $V' \ni y$ which is open in Y and such that $Intf^{-1}(ClV') \subseteq Clf^{-1}(V)$ *[17]*.

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An inclusion $A \subset Y$ is proper if for each $y \in Y$ and each $V \ni y$ open in Y there exists a $V' \ni y$ open in Y and such that $Int_A(A \cap ClV') \subseteq Cl_A(A \cap V)$.

1.1. LEMMA. *[17]*. Let $f: X \longrightarrow Y$ be a continouos mapping. Then:

(i) f is proper, if Y is regular;

(ii) f is proper if X is H-closed and if Y is a Hausdorff space;

(iii) a closed subspace A of H-closed X is H-closed if the inclusion $A \subseteq X$ is proper;

(iv) each open and dense embedding is proper.

Let F be the family of all open free ultrafilters on a Hausdorff space X. The Katetov extension kX of X *[17]* is the set $X \cup F$ with topology consisting of all open subsets of X and all sets of the form $\{x\} \cup U$, where $X \in F$ and $U \in x$, U open in X.

1.2. LEMMA. (*[17]*,*[27]*). Let X be a Hausdorf space. Then:

(i) kX is H-closed;

(ii) X is open and dense {i.e. open} embedded in kX;

(iii) kX-X is discrete in the topology induced by the topology on kX;

(iv) a mapping $f: X \longrightarrow Y$ into H-closed space Y has a unique continuous extension $kf: kX \longrightarrow$ if and only if f is proper;

(v) if U and V are disjoint open subsets of X then $Cl_{kx}U \cap Cl_{kx}V \subset X$.

We say that an extension Y of X is majorizable if there exists an extension Z of X and a map $F: Z \longrightarrow Y$ which is an extension of the identity $i: X \longrightarrow X$.

An extension will be called r.o.-free if for each regulary open subset U of X the bondary BdxU in X is the same as the boundary BdxV of V in Y, where V is an arbitrary open subset of Y such that $U = V \cap X$ *[17:I.3.]*.

1.3. LEMMA. *[17:I.3.1]*. If an H-closed extension $X \subset Y$ is such that:

a) x is open in Y,

b) The remainder $Y - X$ is discrete in the topology induced from Y,

c) $X \subset Y$ if r.o.-free

then $X \subset Y$ is non-majorizable.

1.4. LEMMA. An extension $X \subset Y$ which satisfies a) and b) of Lemma 1.3. is r.o.-free if the folowing condition (K) is satisfied:

(K) If U,V is a pair of disjoint open subsets of X then $\mathrm{Cl}_Y U' \cap \mathrm{Cl}_Y U' \subseteq X$, where U', V' are arbitrary open subsets of Y such thet $U = U' \cap X$ and $V = V' \cap X$.

A p-cover of X is an open cover of X possessing a finite subfamily which is dense in X *[26]*. A map f:X ---> Y is p-map if for each p-cover U of Y then cover $f^{-1}(U) = \{f^{-1}(U) : U \in U\}$ is a p-cover of X *[26]*.

A continuous mapping f:X ---> Y is a p-mapping if f can be continuosly extended to kf:kX ---> kY *[26]*.

1.5. LEMMA/ (*[11]*,*[22]*). Let X_α be a non-empty space for each $\alpha \in A$. Then $k(P X_\alpha) = P kX_\alpha$ if at least one of the following two conditions is satisfied.

(a) X_α is H-closed for each $\alpha \in A$.

(b) There exists $X_{\alpha\circ}$ which is not H-closed. $X_{\alpha\circ}$ is finite for all $\alpha \neq \alpha_\circ$. MOreover, all but finitely many X_α 's have only one point.

Let kX_α be the Katetov extension of an almost completely regular space X and let $h \in C^*(X)$ be a real-valued bounded function on X. Then since h(X) is a dense subset of a compact space there exists a map $H \in C^*(kX)$ such that $H / X = h$.

We now define an equivalence relation R on kX as follows: xRy if H(x) = H(y) for each $h \in C^*(X)$. Let hX be the quotient space kX / R and p:kX ---> hX the natural projection.

1.6. THEOREM. *[5]*. Let X be almost completely regular. *[20]*. Then X is a dense subspace of a nearly-compact Hausdorff space hX with the following properties.

(1) For each nearly-compact space Y and continuous open f:X ---> Y there exists a unique and continuous map F:hX ---> Y such that F / X = f.

(2) Any nearly-compact space in which X can be densely embedded and which possesses property (1) is homeomorphic to hX.

(3) The space hX is the projective maximum in the class of all nearly-compact extensions of X.

(4) X is C*-embedded in hX.

The space hX is the for a completely regular space X has been constructed by Katetov (see [19],pp.168.,Construction 6.10.).

1.7. THEOREM. [19]. Let X be completely regular and hX the space constructed in the Construction 6.10. Then:

(a) hX is an H-closed completely Hausdorff extension of X,

(b) The semiregularization $(hX)_s$ is isomorphic with βX,

(c) hX is the projective maximum among H-closed completely Hausdorff extension of X,

(d) X is C*-embedded in hX.

Now we prove two properties of hX which are similar to the properties of kX.

1.8. LEMMA. The remainder hX-X is discrete in the topology induced by the topology of hX.

Proof. If hX-X is not discrete in the subspace topology then we define a new topology t' on hX as follows. For each $x \in hX-X$ we define a new family of neighborhoods containing a sets of the form $\{x\} \cup (U-(hX-X))$, where U is a neighborhood of x in hX. The identity i:(hX,t') ---> hX is continuous [9:68]. This means that (hX,t') is completely Hausdorff. Moreover, the space (hX,t') is an H-closed extension of X. By Lemma 1.7.(c) we infer that the identity i:(hX,t') ---> hX is a homeomorphism.

1.9. REMARK. Lemma 1.8. can be proved using the facts that hX is a quotient space of kX and that kX-X is discrete in the subspace topology.

1.10. LEMMA. If $X \subset Y$ is a completely Hausdorff extension of X with the properties:

(a) X is open in Y,

(b) Y - X is discrete in the subspace topology,

(c) if U and V are open in X that $Cl_X U \cap Cl_X V = \emptyset$ then $Cl_X U \cap Cl_X V = \emptyset$,

then Y is no-majorizable.

Proof. Assume that Z is the completely Hausdorff extension of X and H:Z ---> Y is a map such that H/X is the identity. If suffices to prove that H/(Z-X) is 1-1. For each pair x,y of distinct points in Z-X there exists a neighborhood U and V of x and y such that

$Cl_Z U \cap Cl_Z V = \emptyset$ since Z is completely Hausdorff. Set $U' = U \cap X$ and $V' = V \cap X$. By virtue of the condition c) we have $Cl_X U \cap Cl_X V = \emptyset$. If we suppose that $H(x) = H(y)$ then we obtain that $H(x) \in Cl_X U' \cap Cl_X V'$. We obtain a contradiction. The proof is completed.

1.11. LEMMA. If X is normal then a completely Hausdorff H-closed extension Y of X is near-compactification hX if Y-X is discrete in the subspace topology and the following condition (K) is satisfied:

(K') If F_1, F_2 is a pair of disjoint closed subsets of X then $Cl_{hX} U \cap Cl_{hX} V = \emptyset$.

Proof. The "only if" part. If Y is equivalent to hX then (K') follows from Lemma 1.10.

The "if" part. Since hX is the projective maximum there exists a mapping $H: Y \longrightarrow hX$. For each pair x,y of discinct points in Y-X we consider sets U' and V' as in the proof of Lemma 1.10. Using the normality of X we complete the proof in a similar way as in Lemma 1.10.

2. NEAR-COMPACTIFICATION OF INVERSE LIMIT SPACE

Now we start with the key lemma of this Section.

2.1. LEMMA. Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of Hausdorff spaces X_α, $\alpha \in A$. Then:

(i) if the mappings $f_{\alpha\beta}$ are open then there exists inverse system $h\underline{X} = \{hX_\alpha, hf_{\alpha\beta}, A\}$;

(ii) if limX is non-empty and if the projections $f_\alpha: \lim X \longrightarrow X_\alpha$, $\alpha \in A$, are onto, then there exists a continuous mapping $H: h(\lim X) \longrightarrow \lim hX$ which is an extension of the identity $i: \lim X \longrightarrow \lim hX$;

(iii) if the projections f_α are onto, then H is onto and limhX is an H-closed completely Hausdorff extensions of limX such that limX is open in limhX.

Proof. (i) Apply Theorem 1.6.(1).

(ii) Now we have the open mappings f:limX ---> hX, $\alpha \in A$. By virtue of Theorem 1.6.(1) there exist a) continuous mappings hf_α:h(limX) ---> hX_α, $\alpha \in A$. The family {hf:$\alpha \in A$} induces a continuous mapping H:h(limX) ---> limhX [2:138]. The proof is completed.

(iii) Let us prove that H is onto. For each x ∈ limhX we consider points $x_\alpha = f'_\alpha(x)$, $\alpha \in A$, where f'_α:limhX ---> hX_α, $\alpha \in A$, are the projections. For each x_α we have $\{x_\alpha\} = \cap$ {ClU_α: U_α is an open neighborhood of x_α.} The family $\{(hf_\alpha)^{-1}(U_\alpha):\alpha \in A\}$ is a centred family of open subsets in the H-closed space h(limX). This means that there exists a point Y ∈ ∩ $\{Cl(hf_\alpha)^{-1}(U_\alpha):\alpha \in A\}$. Celarly $hf_\alpha(y) = x_\alpha$ for each $\alpha \in A$. Thus, H(y) = x. This means that H is onto and that limhX is H-closed as a continuous image off the H-closed space h(limX). In order to complete the p[roof it suffices to prove that limX is dense in X. This is an immediate consequence of the definition of a base of the inverse limit space and the assumption that f_α are onto. Finally, limhX is completely Hausdorff. The proof is completed.

2.2 LEMMA. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with open and onto projections f_α. For each x ∈ h(limX)-limX there exists a $\alpha \in A$ such that $hf_\alpha(x) \in hX_\alpha - X_\alpha$.

Proof. Lemma is an immediate consequence of the fact that x is a free ultrafilter and the definition of a base on inverse limit space.

From Lemmas 1.10. and 2.1. we obtain the following

2.3. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of thecompletely Hausdorff spaces X_α, $\alpha \in A$, and open onto mappings $f_{\alpha\beta}$ such that the projections f_α lim$\underline{X}$ -> X_α, $\alpha \in A$, are onto. The mapping H:h(lim$\underline{X}$) -> limh$\underline{X}$ is homeomorphism. If limh$\underline{X}$ \ lim$\underline{X}$ is discrete the subspace topology and for each $\lim\underline{X}$ open set U.V. from $Cl_{\lim\underline{}}XU \cap Cl_{\lim\underline{}}XV = \emptyset$ it follows $Cl_{\lim\underline{}}XU \cap Cl_{\lim\underline{}}XV = \emptyset$.

2.4. LEMMA. Let $\underline{X}$ be as Theorem 2.3. If $f_{\alpha\beta}$ are p-perfect, then the subspace $\lim h\underline{X}-\lim\underline{X}$ is discrete If the following condition (D) is satisfied:

For each $x_\alpha \in kX_\alpha - X_\alpha$, $\alpha \in A$ there is a $\beta \geq \alpha$ such that $(kf_\beta)^{-1}(x_\beta)$ has a single point for each $x_\beta \in (kf_{\alpha\beta})^{-1}(x_\alpha)$ and each $\gamma \geq \beta$.

Proof. The "only if" part. Now the subspace $Y = \lim k\underline{X} - \lim\underline{X}$ of the space $\lim k\underline{X}$ is the limit of the inverse subsystem $Y = \{kX_\alpha - X_\alpha, f_{\alpha\beta}/(kX_\beta - X_\beta), A\}$. If each point $y \in Y$ is open in Y, then {Y} contains the fiber $\{kf_\alpha\}^{-1}(U_\alpha)$ for some open $U_\alpha \subset kX_\alpha - X_\alpha$. This means that this fiber has a single point. Thus (D) is satisfied.

The "if" part. The proof is similar.

We say that an inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an (RS)S-system if for each pair F_1, F_2 of disjoint (regulalry) closed of $\lim\underline{X}$ there is a $\alpha \in A$ such that $Clf_\alpha(F1) \cap Clf_\alpha(F2) = \emptyset$. Clearly, each S-system is RS system.

2.5. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of the completeluy Hausdorff spaces X_α, $\alpha \in A$, and open onto p-perfect mappings $f_{\alpha\beta}$ such that the projections $f_\alpha \lim\underline{X} \to X_\alpha$, $\alpha \in A$, are onto and the condition (D) is satisfied. Then $H: h(\lim\underline{X}) \to \lim h\underline{X}$ is the homeomorphism if $\underline{X}$ is RS system.

Proof. Necessity. Let shX_α, $\alpha \in A$, be the semi-regularization [7] of hX_α. We consider the inverse system $sh\underline{X} = \{shX_\alpha, shf_{\alpha\beta}, A\}$, where $shf_{\alpha\beta}\{x_\alpha\} = hf_{\alpha\beta}\{x_\alpha\}$ for each $x_\alpha \in hX_\alpha$, $\alpha \in A$. The mappings $shf_{\alpha\beta}$ are continuous [14. Theorem 3.1]. This means that $sh\underline{X}$ is inverse system of compact spaces and thus, $S: s(\lim h\underline{X}) \to \lim sh\underline{X}$

which is 1-1 and onto. If H is a homeomorphism, then lim $h\underline{X}$ is nearly-compact. Thus, S is the homeomorphism. If C1U, C1V are regulary closed in s(lim$h\underline{X}$) and, thus, in lim$sh\underline{X}$. Since $sh\underline{X}$ is S-system, there is a $\alpha \in A$ such that $C1f_{\alpha}(C1U) \cap C1f_{\alpha}(C1V) = \emptyset$ in shX_{α}. Clearly$C1f_{\alpha}(C1U) \cap C1f_{\alpha}(C1V) = \emptyset$ in hX_{α}. The proof is completed.

Sufficiency. Let U and V be a pair of open set in lim$\underline{X}$ such that (in lim$\underline{X}$) $C1U \cap C1V = \emptyset$. Since $\underline{X}$ is RS-system and $f_{\alpha\beta}$, $\alpha \in A$, are open we have open sets $f_{\alpha}(U)$ and $f_{\alpha}(V)$ such that, $C1f_{\alpha}(C1U) \cap C1f_{\alpha}(C1V) = \emptyset$ in hX_{α}. This means that in lim$h\underline{X}$ is C1U $\cap$C1V = $\emptyset$. The proof is completed.

2.6. COROLLARY. If $\underline{X}$ is an inverse system of a completely Hausdorff nearly compact spaces and open onto bonding mappings, then lim$\underline{X}$ is nearly-compact.

Proof. The mappimg S in the proof of Theorem 2.5. is a homeomorphism. Thus, lim$sh\underline{X}$ = lim$s\underline{X}$ = s(lim$h\underline{X}$). We infer that lim$x\underline{X}$ is nearly-compact space is nearly-compact.

2.7. remark. Corollary 2.6. holds also from the fact that H is a continuous surjection and that a completelly Hausdorff continuous image of a nearly-compact space in nearly-cpmpact.

The condition (d) is satisfied in the case of open mappings with finite fibers. We start with the following lemma.

2.8. LEMMA. Let $f:X \to Y$ be an open p-perfect surjection such that $|f^{-1}(y)| \leq k$ for each $y \in Y$ and some fixed natural number k $\in$ N. Let $kf:kX \to kY$ be the Katetov extension of f, X and Y (*[4]*, *[9]*, *[17]*, *[26]*) Then $|(kf)^{-1}(z)| \leq k$ for each $z \in kY$.

Proof. For each $z \in Y$ we have $(kf)^{-1}(z) = f^{-1}(z)$ since f is p-perfect i.e. $kf(kX-X) = kY-Y$. Now, let $z \in kY-Y$. Suppose that there exist distinct points $z_1, \ldots z_{k+1}$ of kX-X such that $(kf)(zi)=z, i=1,\ldots k+1$. Eaqch point zi is the ultrafilter U_i of open subset of X. Similarly, z is the ultrafilter U of open subsets of Y. From the fact kX is T_2 it follows that there exist disjoint open sets $U_i \subset X, i=1,\dots,k+1$, such $\{z_i\} \cup U_i, i=1,\dots,k+1$, are neighborhoods of z_i. From the constructiuon of $(kf)(z_i)$ it follows that $f(U_i) \in U, i=1,\dots,k+1$. This means that $Y' = \cap\{f(U_i): i=1,\dots,k+1) \neq \emptyset$. Thus, for each $y \in Y'$ there exists $x_i \in U_i, i=1,\dots,k+1$, such that $f(x_i)=y$. This means that $f^{-1}(y)$ has the cardinality $\geq k+1$. This contradicts the assumption $|f^{-1}| \leq k$. The proof is completed.

From the proof and the fact that $f(X) = Y$ also follows the next lemma.

2.9. LEMMA. Let $f:X \to Y$ be an open surjection. If $f^{-1}(y) \leq k$, $y \in Y$, then $|(kf)^{-1}(z)| \leq k$ for each $z \in kY-Y$.

2.10. LEMMA. Let $f:X \to Y$ be an open surjection between completelly Hausdorff spaces. If $|f^{-1}(y)| \leq k$, then $|(hf)^{-1}(y)| \leq k$ for every $y \in Y-Y$. Moreover, if f is p-perfect, then $|(hf)^{-1}(y)| \leq k$ for each $y \in Y$.

Proof. Apply lemma 2.7.,2.8.,2.9. and the fact that hy is the quotent spaces of the Katetov extension kX *[6]*.

2.11. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of completelly Hausdorff and open onto mappings $f_{\alpha\beta}$. If $|f_{\alpha\beta}^{-1}(x_\alpha)| \not\leq$ for each α, β and $x_\alpha \in X_\alpha$, then the condition (D) is satisfied.

Proof. By virtue of Lemma 2.10. $|(hf_{\alpha\beta}^{-1})(x_\alpha)| \leq k$ for each $x_\alpha \in hX_\alpha - X_\alpha$. If we suppose that for each $\beta \geq \alpha$ and each $x_\beta \in X_\beta$ $|(hf_{\beta\gamma})^{-1}(X_\beta)| \geq 2$ then for a sequence $\alpha \leq \beta \leq \beta_1 \leq \dots \leq \beta_n \leq \dots$ we have $|(hf_{\alpha\beta})^{-1}(x_\alpha)| \geq 2^n$, $n \to \Theta$. This is impossible since $|(hf_{\alpha\beta})^{-1}(X_\alpha) \leq k$ for each $\beta \geq \alpha$. The proof is completed. Now we give examples of inverse S-system.

2.12. EXAMPLES OF INVERSE S-SYSTEM

a) Each inverse system of compact spaces is S-system *[2]*.

b) Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence. If $\lim \underline{X}$ is countably compact, then $\underline{X}$ is S-system *[13]*.

c) From b) and *[13:3.1. Theorem]* it follows that if $\underline{X}$ is an inverse sequence of countably compact spaces and closed bonding mappings, then $\underline{X}$ is S-system.

d) Similary, if $\underline{X}$ is an inverse sequence of sequentially compact *[2]* (strongly countably compact, D-compact (*[13]*) spaces, then $\underline{X}$ is S-system.

e) Let m be an infinite cardinal. We say that X is m-compact if every open cover of X of the cardinality $\leq$ m has a finite subcover. Each countably compact spaces is $\aleph_0$-compact. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well-ordered inverse system (i.e. A is well-ordered) such that $cf(A) = \omega_m$. If $x_{\alpha,}$ $\alpha \in A$, are $\aleph_m$-compact and $f_{\alpha\beta}$ closed, then $\underline{X}$ is S-system.

f) Let hI(x) denotes the hereditary Lindelöff number of a space X*[2.284]*. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well-ordered inverse system such that $hI(x_\alpha) < \aleph_\tau$ and $cf(A) > \aleph_\tau$, the for each closed (open) set U $\subset \lim\underline{X}$ there exist a $\alpha \in A$ and closed (open) set $U_\alpha \subset X_\alpha$ such that $f^{-1}{}_\alpha(U_\alpha) = U$. *[16.2.3.Teorem]*. This means that $\underline{X}$ is S-system.

g) We say that $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ if f-system if for each continuous real-valued function $f: \lim\underline{X} \to I = (0,1) I$ there exist a $\alpha \in A$, and continuous real-valued function $g_\alpha : X_\alpha \to I$ such that $f = g_\alpha f_\alpha$ *[21]*. Clearly, if $\underline{X}$ is f-system and $\lim\underline{X}$ is normal, then $\underline{X}$ is S-system. In the paper *[21:28]* has been proved that $\underline{X}$ is f-system in the following cases:

g1) $\underline{X}$ is δ-directed inverse system with Lindelöt limit $\lim\underline{X}$,

g2) $\underline{X}$ is m-directed inverse system with open projections and the spaces X_α whose Souslin number $c(x) \leq$ m.

Let us recall that $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is m-directed it for each $B \subset A$ with $|B| \leq m$ there is $\alpha \in A$ such that $\alpha \geq \beta$ for each $\beta \in B$. If $\underline{X}$ is m-directed for $m = \aleph_0$, then we say that $\underline{X}$ id δ-directed.

2.13. THEOREM. Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence such that f_{nm} are open and onto. If X_α are completelly Hausdorff, $\lim\underline{X}$ is countably compact space and if (D) is satisfied, then the spaces $h(\lim\underline{X}$ and $\lim h\underline{X}$ are homeomorphic.

Proof. Apply Theorem 2.5. and Example 2.12.b).

2.14. THEOREM. If in Theorem 2.13. $|f^{-1}_{nm}(x_n)| \leq k$ for all $n \in N$ and $x_n \in X_n$, then the mapping $H: h(\lim\underline{X}) \to \lim h\underline{X})$ is the homeomorphism.

Proof. Apply Theorem 2.13. and Theorem 2.11.

2.15. THEOREM. Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of completelly Hausdorff countably compact spaces X_n such that f_{nm} are open, closed onto mappings with $|f^{-1}_{nm}(x_n)| \leq k$, then $\lim h\underline{X}$ is near-compactication of $\lim\underline{X}$.

Proof. The space $\lim\underline{X}$ is countably compact *[13]*. Theorem 2.14 completes the proof.

2.16. THEOREM. Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of completely Hausdorff sequential compact (strongly countably compact, D-compact) spaces and open onto mappings f_{nm} such that $|f^{-1}_{nm}(x_n)| \leq k$, then $\lim h\underline{X}$ is near-compactification of $\lim\underline{X}$.

Proof. The mapping $H: h(\lim\underline{X}) \to \lim h\underline{X}$ is the homeomorphism since $\lim\underline{X}$ is sequential compact (strongly countably compact, D-compact) *[13]* and the condition of Theorem 2.14. are satisfied.

2.17. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an well-ordered inverse system of completely Hausdorff $\aleph_m$-compact spaces X_α and open, closed onto mappings $f_{\alpha\beta}$ auch that for each $x_\alpha \in X_\alpha$ and each $\beta \geq \alpha$ $|f^{-1}_{\alpha\beta}(x_\alpha)| \leq k$, then the mappings $H: h(\lim\underline{X}) \to \lim h\underline{X}$ is the homeormorphism.

Proof. The space $\lim\underline{X}$ is $\aleph_m$-compact *[13]*. Thus, $\underline{X}$ is S-system *[2.12.e]*. Apply Theorem 2.5.

2.18. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well-ordered inverse system such that $hI(X_\alpha) < \aleph_m$, $\alpha \in A$, $cf(A) > \omega_m$ and $f_{\alpha\beta}$ open onto mappings with $|f^{-1}_{\alpha\beta}(x_\alpha)| \leq k$. If the projections f_α are onto mappings anfd if X_α are completelly Hausdorff, then the spaces $h(\lim\underline{X})$ and $\lim\underline{X}$ are homeomorphic.

Proof. Apply Theorems 2.5.,2.11. and Example 2.12.f)

2.19 THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse f-system of the completelly Hausdorff spaces and open onto mappings $f_{\alpha\beta}$ such that $|f^{-1}_{\alpha\beta}(x_\alpha)| \leq k$. If the projections f_α, $\alpha \in A$, are onto, then the spaces $h(\lim\underline{X})$ and $\lim h\underline{X}$ are homeomorphic.

Proof. Apply theorems 2.5.,2.11., and Example 2.12 g).

2.20. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a δ-directed inverse system of completelly Hausdorff spaces such that $f_{\alpha\beta}$ are open onto $|f^{-1}_{\alpha\beta}(x_\alpha)| \leq k$. If the limit $\lim\underline{X}$ is Lindelöf and if the projections are onto, then the spacesh$(\lim\underline{X})$ and $\lim h\underline{X}$ are homeomorphic.

Proof. Apply Theorem 2.20. and Example 2.12.g1).

2.21. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be m-directed inverse system of completelly Hausdorff spaces such that $c(x_\alpha) \leq m$ and that $f_{\alpha\beta}$, f_α are open onto mappings with $|f_{\alpha\beta}(x_\alpha)| \leq k$. If $\lim\underline{X}$ is normal, then $h(\lim\underline{X})$ and $\lim h\underline{X}$ are homeomorphic.

Proof. Apply Theorems 2.5.,2.11. and Example 2.12.g2).

We close this Section by some theorems concerning the inverse systems with closed irreducible bonding mappings.

A mapping $f:X \to Y$ is called an irreducible mapping if for each non-empty open set $U \subset X$ the set $f^{\#}(U) = \{y: y\in Y \wedge f^{-1}(y) \subset U\}$ is non-empty. Clearly, if is closed irreducible, then $f^{\#}(U)$ is open and non-empty.

2.22. LEMMA. Each closed irreducible mapping $f:X \to Y$ is p-mapping.

Proof. Let U be a p-cover of Y (see the definition after LEmma

1.4.) such that there exists a finite subfamily $U_1 = \{U_1,\dots,U_k\}$ such that $\cup\{ClU_i; i=1,\dots,k\}=Y$. Suppose that the cover $f^{-1}(U)$ is not p-cover. This means that $U=X-\cup\{Clf^{-1}(U_i); i=1,\dots,k\}$ is non-empty open set. Since f is closed and irreducible we infer that $f^{\#}(U)$ is non-empty and open. Moreover, $f^{\#}(U)\subset X-\cup\{ClU_i; i=1,\dots,k\}=\emptyset$. The contraction $f^{\#}(U)\neq\emptyset \wedge f^{\#}(U) = \emptyset$ completes the proof.

2.23. corollary. Each closed irreducible mapping $f: X \to Y$ has a unique extension $kf: kX \to kY$.

Proof. Apply Lemma 2.22. and TRheorem 2.5. of *[26]*.

Since hX is the quotient space of kX we have as in the proof of Theorem 1. in *[5]* the following lemma.

2.24. LEMMA. Each closed irreducible mapping $f: X \to Y$ has a unique extension $hf: hX \to hY$.

2.25. LEMMA. If $f: X \to Y$ is closed irreducible, then for each $y \in kY-Y$ there exists a single point $x \in kX-X$ such that $hf(x) = y$.

Proof. Suppose that there exist two disjoint point $x_1, x_2 \in hX-X$ such that $f(x_1) = f(x_2) = y$. By virtue of the construction of kf *[26:208]* it follows that $f^{-1}(V)\in x_1$ and $f^{-1}(v)\in x_2$ for each $V\in y$. This means that $f^{-1}(V)\cap U = W$ is non-empty and open for each $U\in x_1$. Since f is closed and irreducible the set $f^{\#}(W)$ is open and non-empty. Clearly, $V\cap f^{\#}(W)\neq\emptyset$. Thus, $f^{\#}(U_1)\in y, f^{\#}(U_2)\in y$ i.e. $f^{\#}(U_1)\cap f^{\#}(U_2)\neq\emptyset$ since y is the ultrafilter. The proff is completed.

2.26. COROLLARY. Lemma 2.25. holds for the mapping $hf: hX \to hY$.

Proof. Trivial since hX is the quotient space of kX.

2.27. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with irreducible bonding mappings $f_{\alpha\beta}$. Then the projections $f_\alpha: \lim\underline{X} \to X_\alpha, \alpha \in A$, are irreducible.

Proof. Let U be any open subset of $\lim\underline{X}$. We prove that for each fixed $\alpha \in A$ the projection f_α is irreducible. It suffices to prove that $f^{\#}(U)$ is non-empty. Let x be any point of U. From the definition of a base in $\lim\underline{X}$ we infer that there is a $\beta\in A$ and open set $U_\beta \subset X_\beta$ such that $x\in f^{-1}_\beta(U_\beta) \subset U$. Let $\gamma \geq \alpha,\beta$ and let $U_\gamma=f^{-1}_{\alpha\beta}(U_\alpha)$. The set $f^{\#}_{\alpha\gamma}(U_\gamma)$ is non-empty since $f_{\alpha\gamma}$ is irreducible. Clearly, this means that $f^{\#}_\alpha(U)$ is non-empty. The proof is completed.

2.28. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an S-system with onto projections. The projections $f_\alpha:\lim\underline{X} \to X_\alpha, \alpha\in A$, are closed if $f_{\alpha\beta}$ are closed.

Proof. Necessity. If F_β is is a closed subset of X_β. then from the $f_{\alpha\beta}(F_\beta) = f_\alpha f_\beta^{-1}(F_\beta)$ and from the cosedness of f_α it follows that $f_{\alpha\beta}(F_\beta)$ is closed. This means that $f_{\alpha\beta}$ is closed.

Sufficiency. Let us prove that f_α is closed. Let U be any open set about $f^{-1}_\alpha(x_\alpha)$. Since $\underline{X}$ is S-system, there is a $\beta \geq \alpha$ such that $f^{-1}_{\alpha\beta}(x_\alpha) \cap \mathrm{Cl}f_\beta(\lim\underline{X}-U) = \emptyset$. Since $f_{\alpha\beta}$ is closed, there is open U_α about x_α such that $f^{-1}_{\alpha\beta}(x_\alpha) \subset f^{-1}_{\alpha\beta}(U_\alpha) \subset X_\beta-\mathrm{Cl}f_\beta(\lim\underline{X}-U)$. Clearly, $f^{-1}_\alpha(U_\alpha) \subset U$. The proof is completed.

2.29. LEMMA. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an S-system of completelly Hausdorff spaces and closed irreducible onto mappings $f_{\alpha\beta}$. Then there exists the inverse system $h\underline{X} = \{hX_\alpha, hf_{\alpha\beta}, A\}$.

2.30. LEMMA. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse S-system with closed p-perfect onto mappings $f_{\alpha\beta}$ and Hausdorff spaces X_α. Then the projections f_α, $\alpha \in A$, era closed p-perfect.

Proof. From *[26:Lemma 3.9.]* it follows that p-mapping f is p-perfect if f is absolutely closed. A mapping $f:X \to Y$ is absolutely closed *[26.Lemma 3.8]* if f(B) is closed for every regulary closed $B \subset X$ and if for each free ultrafilter $U \in kX-X$ and each $y\in Y$ there is $u\in U$ such that $f^{-1}(y)\cap \mathrm{Cl}U = \emptyset$. By virtue of Theorem 2.28. it suffices to prove that for each free open ultrafilter U of $\lim\underline{X}$ and each x_α, X_α there is an $u\in U$ such that

$f^{-1}(y) \cap ClU = \emptyset$. For each $\beta \geq \alpha$ we have a set $Y_\beta = \cap\{Clf_\beta(U): u \in U\}$. If we suppose that Y_β is non-empty, then it has exactly one point y_β[26:208}. Since $f_{\beta\gamma}(y_\gamma) = y_\beta$, $\alpha \leq \beta \leq \gamma$, we have a point $y = (y_\beta) \in \lim \underline{X}$. By virtue of the definition of a base in $\lim \underline{X}$ it follows that each neighborhood of y intersects each $u \in U$ i.e. $y \in \cap\{ClU: u \in U\}$. This is impossible since u is free. Thus, there exist $\beta \geq \alpha$ such that y_β is empty. Then $U_\alpha = \{U_\beta : U_\beta\}$ open in X_β and $f^{-1}_\beta(U_\beta) \in U$ is free open ultrafilter on X_β[26:208]. There exists a $U_\beta \in U_\beta$ such that $f^{-1}_{\alpha\beta}(x_\alpha) \cap ClU_\beta = \emptyset$. Clearly, $f^{-1}_\alpha(x_\alpha) \cap Clf^{-1}_\beta = \emptyset$. The proof is completed since $f^{-1}_\beta(u_\beta) \in U$.

2.31. **COROLLARY.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system. If the mappings $f_{\alpha\beta}$ are closed, p-perfect and irreducible, then the condition (D) holds.

2.32. **THEOREM.** Let $\underline{X}$ be as in LEmma 2.31. Then the $\lim h\underline{X} - \lim \underline{X}$ is discrete in the subspace topology.

2.33. **THEOREM.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of completelly Hausdorff spaces and closed p-perfect irreducible onto mappings $f_{\alpha\beta}$. The mapping $H: h(\lim \underline{X}) \to \lim h\underline{X}$ is the homeomorphism if $\underline{X}$ is RS-system.

2.34. **THEOREM.** Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence with closed p-perfect irreducible onto mappings f_{nm} and countably compact completelly Hausdorff spaces X_n. Then $H: h(\lim \underline{X}) \to \lim h\underline{X}$ is a homeomorphism.

2.35. **THEOREM.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an well-ordered inverse system of completelly Hausdorff $\aleph_m$-compact spaces X_α and closed p-perfect irreducible onto mappings $f_{\alpha\beta}$. Then the mapping H is a homeomorphism.

2.36. **THEOREM.** Let $\underline{X}$ be the inverse system from the example

2.12.f). If the spaces $X_\alpha \in \underline{X}$ are completelly Hausdorff and $f_{\alpha\beta}$ closed p-perfect uirreducible onto mappings, the $h(\lim\underline{X})$ and $\lim h\underline{X}$ are homeomorphic.

2.37. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse f-system with normal limit (δ-directed inverse system with Lindelöf limit or m-directed inverse system with $c(X_\alpha) \leq m$ with open projections) and completely Hausdorff spaces X_α. If the mapping $f_{\alpha\beta}$ are closed p-perfect irreducible and onto, then H is homeomorphism.

Proof. Apply Theorem 2.33. and Example 2.12.g).

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Lončar I. Bliska kompaktifikacija inverznog limesa

SADRŽAJ

U radu se izučava bliska kompaktifikacija limesa $\lim \underline{X}$ *inverznog sistema* $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$. *Blisku kompaktifikaciju definirao je Herrman u radu [5].*

U drugom odjeljku se najprije konstruira inverzni sistem $h\underline{X} = \{hX_\alpha, hf_{\alpha\beta}, A\}$ *pa svaki inverzni sistem* $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *se otvorenim preslikavanjima* $f_{\alpha\beta}$, *pri čemu su* hX_α, $\alpha \in A$, *bliske kompaktifikacije prostora* X_α. *U lemi 2.1. je nadalje konstruirano neprekidno preslikavanje* $H: h(\lim \underline{X}) \to \lim h\underline{X}$. *Teoremi 2.3. i 2.5. su osnovni teoremi koji daju dovoljne uvjete uz koje je H-homeomorfizam. Jedan od tih uvjeta je da* $\underline{X}$ *bude S-sistem ili RS-sistem. Kako ima niz takvih sistema (Examples 2.12), to iz teorema 2.3. i 2.5. slijedi niz teorema 2.13 - 2.37.*